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Aerodynamics. — *Cases of motion in a gas with non colliding molecules.* By J. M. BURGERS. (Mededeling no. 50 uit het Laboratorium voor Aero- en Hydrodynamica der Technische Hogeschool te Delft.)

(Communicated at the meeting of May 31, 1947.)

1. *Statement of the problem.* — When the mean free path of the molecules in a gas is very large compared with the distances between the walls enclosing the gas, practically no collisions will occur between the molecules themselves. The normal mechanism which must ensure the establishment of the MAXWELLian velocity distribution then falls out, and the actual velocity distribution will be the result of the collisions with the walls. In the case of diffuse wall reflection an approximation to the MAXWELLian distribution may still be expected, but nevertheless the relation between the pressure tensor (defined by the transfer of momentum) and the thermal pressure combined with the so-called viscous stresses may deviate from that valid in an ordinary gas.

The investigation of problems of motion for extremely rarefied gases in some papers is presented as a new branch of aerodynamics, called "*superaerodynamics*" [1]). The object of the following lines is to consider a case belonging to this domain, in order to point out that when the walls between which the gas is contained, have variable velocities of an order of magnitude comparable with that of the molecular velocities, even with diffuse reflection appreciable deviations from the MAXWELLian velocity distribution may appear. In such cases a direct use of the kinetic picture becomes necessary.

In order to arrive at a simple treatment we consider a gas in the neighbourhood of a single wall, or enclosed between two parallel walls, which are supposed to move normally to their own planes. The direction of the normal is taken as x-axis. The molecules of the gas have velocity components u, v, w; the state of the gas, in particular the velocity distribution, is assumed to be a function of x and t only.

Assuming that there are no collisions between the molecules, the following law is applied for the reflection of the molecules by the walls: Take an element of the wall on the left hand side (*i.e.* the side of $-x$), and suppose that this wall has the velocity $V(t)$. At the instant t molecules with x-velocities satisfying $-\infty < u < V$ will strike upon this wall. Let the number of these molecules per unit area and in unit time be N_r (for all velocities together); and write E for the translational kinetic energy carried by them, as measured by an observer connected with the element of the

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[1]) A. F. ZAHM, Superaerodynamics, Journ. Franklin Institute **217**, p. 153, 1934.
HSUE-SHEN TSIEN, Superaerodynamics, Mechanics of Rarefied Gases, Journ. Aeron. Sciences **13**, p. 653, 1946.

wall [2]). It is assumed that all molecules are reflected, and that the translational kinetic energy carried away by them per unit area in unit time, again measured by an observer connected with the element of the wall, is equal to E. There is thus no loss of relative kinetic energy; but it is supposed that there is a re-distribution of directions and of velocities, in such a way that for the observer fixed to the element the velocity distribution of the reflected molecules is half a MAXWELLian distribution (half, because the velocities in the x-direction of the reflected molecules necessarily must be directed away from the wall).

Let $F_{\text{refl.}}(t; u, v, w)\, du\, dv\, dw$ represent the number of reflected molecules, counted per unit volume in an element of space adjacent to the wall, having velocity components between u and $u + du$, v and $v + dv$, w and $w + dw$. It is assumed that this function has the form:

$$F_{\text{refl.}} = A \left(\frac{h}{\pi}\right)^{3/2} e^{-h\{(u-V)^2+v^2+w^2\}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where v and w can take all values, while $u \geq V$. In this expression A and h are quantities depending upon the instant of reflection. They are determined by the equations:

$$\int_V^{\infty} du \int_{-\infty}^{+\infty}\int dv\, dw\, (u-V)\, F_{\text{refl.}} = \frac{A}{2\sqrt{\pi h}} = N_r \quad . \quad . \quad . \quad . \quad (2)$$

$$\frac{m}{2}\int_V^{\infty} du \int_{-\infty}^{+\infty}\int dv\, dw\, (u-V)\{(u-V)^2+v^2+w^2\}\, F_{\text{refl.}} = \frac{m}{2}\,\frac{A}{h\sqrt{\pi h}} = E \quad (3)$$

where m is the mass of a molecule.

If $V > 0$, then for an observer stationary with respect to the coordinate system, the translational kinetic energy carried away by the molecules from the wall will be greater than E, while that carried towards the wall by the arriving molecules will be less than E. The difference is equal to the product of V into the exchange of momentum between the molecules and the wall.

Equations similar to (1) — (3) can be written down for a wall on the right hand side of the gas.

2. *Velocity distribution in an arbitrary point of the space between the walls.* — We consider a group of molecules which have been reflected by the left hand wall at the instant t, and which have x-velocities between u and $u + du$, while v and w may take all values. All molecules retain their velocities so long as they do not strike the opposite wall. As the state of

[2]) In all considerations of this paper internal kinetic energy (corresponding to rotations or vibrations) is left out of account. The molecules are considered as monatomic without internal motion.

the gas is assumed to be independent of y and z, it follows that in an arbitrary element of volume with centre at x_0, observed at the instant t_0, the number of molecules per unit volume having assigned velocities will be determined by the function:

$$F(x_0, t_0; u, v, w) = a \left(\frac{\varkappa}{\pi}\right)^{3/2} e^{-\varkappa \{(u-\varphi)^2 + v^2 + w^2\}} \quad . \quad . \quad . \quad . \quad (4)$$

in which the quantities a, $\varkappa$, φ are respectively equal to the values of A, h, V pertaining to the last reflection these molecules have suffered by a wall. The instant of this last reflection and its place are dependent upon x_0, t_0 and u; hence a, $\varkappa$, φ are functions of x_0, t_0, u, or more precisely of the two variables u and $\xi = x_0 - u\,t_0$ only.

When values of x_0, t_0 and u have been given, it is always possible in an unambiguous way to find the instant of the last reflection and the position of the reflecting wall. This can be illustrated in an x, t-diagram (see fig. 1), in which the paths described by the walls are represented by

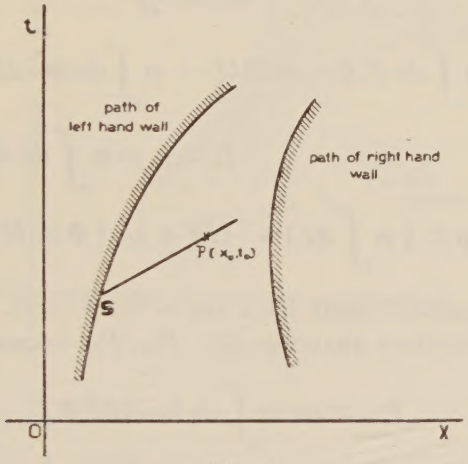


Fig. 1.

means of curves. When through a point $P(x_0, t_0)$ of this diagram we draw the line $x - u\,t = x_0 - u\,t_0$ (representing the "trace" of the molecules passing through P with the x-velocity u), and produce this line in the direction $t < t_0$, it will in general cut the path of a wall; the instant sought is determined by the point of intersection S. If the line cuts the paths of both walls, or cuts a path repeatedly, that point of intersection is to be taken for which $t_0 - t$ has the smallest value. When the line can be produced indefinitely without ever cutting the paths of the walls, we assume that a, $\varkappa$ and φ can be found from the initial conditions of the case.

As the relevant features of the problem considered here can be expressed wholly in terms of x_0, t_0 and u, it is convenient to eliminate v and w. We write:

$$\Phi(x_0, t_0; u) = \int_{-\infty}^{+\infty}\int dv\,dw\,F = a\sqrt{\frac{\varkappa}{\pi}}\, e^{-\varkappa(u-\varphi)^2} \quad . \quad . \quad . \quad . \quad (5)$$

Then:

$$\int_{-\infty}^{+\infty}\int dv\, dw\, (v^2 + w^2)\, F = \Phi/\varkappa. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

In these integrations u is kept constant, the same as x_0, t_0; hence α, $\varkappa$, φ likewise are constants [3]).

With the aid of the function Φ it is possible to write down expressions for the number N of molecules per unit volume at P; for the current NU per unit area across an element of surface through P normal to the x-axis (U being the mean velocity of flow in the x-direction); for the rate of flow of momentum per unit area across elements of surface respectively perpendicular to the three axes (J_x, J_y, J_z); for the thermal pressure p, and for the kinetic temperature T; as follows:

$$N = \int du\, \Phi; \quad NU = \int du\, u\, \Phi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

$$\left.\begin{aligned} J_x = m\int du\, u^2\, \Phi = mNU^2 + m\int du\, (u-U)^2\, \Phi \\ J_y = J_z = m\int du\, \Phi/2\varkappa \end{aligned}\right\} \quad . \quad . \quad . \quad (8)$$

$$p = \tfrac{1}{3} m \int du\, \{(u-U)^2 + 1/\varkappa\}\, \Phi = NkT. \quad . \quad . \quad . \quad (9)$$

In all these integrals the limits for u are $-\infty$, $+\infty$ [4]).

We can also introduce quantities P_{xx}, P_{yy}, P_{zz} respectively defined by:

$$\left.\begin{aligned} P_{xx} = p - m\int du\, (u-U)^2\, \Phi \\ P_{yy} = P_{zz} = p - J_y = p - J_z \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

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[3]) As Φ is dependent upon u and $\xi = x_0 - u_0 t$ only, we may call Φ a "line function", as in the x, t-diagram its values are associated not with points, but with segments of straight lines. A second diagram can be introduced, having ξ and u as coordinates; in such a diagram Φ becomes a *"point function"*, its value being associated with the position of a point.

The transformation from x, t to ξ, u (with $\xi = x - ut$) is a contact transformation. Straight lines in the x, t-plane become points in the ξ, u-plane; and inversely points in the x, t-plane are represented by straight lines in the ξ, u-plane. The curved path of a wall in the x, t-plane can be considered as a series of points; to each of these points there will correspond a straight line in the ξ, u-plane. The envelope of these lines is associated with the curved path, as every point of it is the image of a tangent to the original path in the x, t-plane.

[4]) In the present example there are no "mixed" components J_{xy} etc. in the tensor of momentum transfer, in consequence of the symmetry assumed with respect to the y- and z-directions. Such components will appear, *e.g.*, when the walls have velocity components *in* their own plane.

In the case of an ordinary gas these quantities represent viscous stresses, and would be given by [5]):

$$P_{xx} = \tfrac{4}{3}\mu(\partial U/\partial x); \quad P_{yy} = P_{zz} = -\tfrac{2}{3}\mu(\partial U/\partial x).$$

In the case of a gas with non colliding molecules, however, the coefficient of viscosity μ loses its ordinary significance.

3. *Expression of the conditions at a wall in terms of Φ.* — We again consider the wall on the left hand side. A diagram of the traces of the molecules arriving at this wall at a given instant, and of the reflected molecules going away, is represented in fig. 2. For the arriving molecules $-\infty < u < V$; for the outgoing ones $V < u < +\infty$.

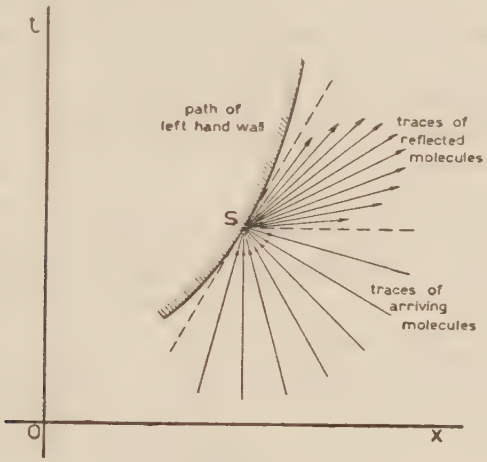


Fig. 2.

For all outgoing molecules we can write:

$$\Phi_{\text{refl.}} = A\sqrt{\frac{h}{\pi}}\,e^{-h(u-V)^2} \qquad (\text{with } u \geqslant V) \quad . \quad . \quad . \quad . \quad (11)$$

where A, h, V are independent of u, and are exclusively dependent upon the point S marking the instant of reflection on the path of the wall in the diagram.

For the arriving molecules we write:

$$\Phi_{\text{arr.}} = \alpha\sqrt{\frac{\varkappa}{\pi}}\,e^{-\varkappa(u-\varphi)^2} \qquad (\text{with } u \leqslant V) \quad . \quad . \quad . \quad . \quad (12)$$

where α, $\varkappa$, φ are determined by the circumstances of the preceding reflections or by the initial conditions. As the instant of the preceding reflection will be different for groups with different values of u, we must consider

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[5]) Compare *e.g.* J. H. JEANS, The Dynamical Theory of Gases (Cambridge, 1916) p. 242; S. CHAPMAN and T. G. COWLING, The Mathematical Theory of Non-uniform Gases (Cambridge, 1939), p. 123.

a, $\varkappa$, φ as functions of u in (12). Keeping this in mind we can write the equations determining A and h as follows:

$$\frac{A}{2\sqrt{\pi h}} = \int_{-\infty}^{V} du\,(V-u)\,a\sqrt{\frac{\varkappa}{\pi}}\,e^{-\varkappa(u-\bar{v})^2} \quad . \quad . \quad . \quad . \quad . \quad (13)$$

$$\frac{A}{h\sqrt{\pi h}} = \int_{-\infty}^{V} du\,(V-u)\left\{(V-u)^2 + \frac{1}{\varkappa}\right\} a\sqrt{\frac{\varkappa}{\pi}}\,e^{-\varkappa(u-\bar{v})^2}. \quad . \quad . \quad (14)$$

Similar equations can be written down for reflection at the wall on the right hand side.

4. *Example.* — A simple case is obtained when there is only one wall, say on the left hand side, moving with a variable velocity $V(t)$, which is supposed to be *decreasing*, and which should be very large in comparison to the velocities of the molecules in their original state. To simplify still further we will assume that all molecules originally were at rest. The original density of the molecules will be written n_0; this can be a function of x (being zero *e.g.* for sufficiently large negative values of x), but we will here consider a domain of values of x where n_0 is a constant.

In this case there is reflection only of non moving molecules by the

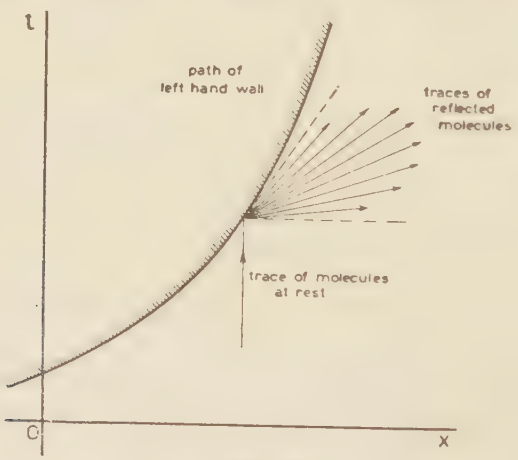


Fig. 3.

moving wall (compare fig. 3), and it is easily to be seen that eqs. (13) and (14) reduce to:

$$A/2\sqrt{\pi h} = n_0 V; \quad A/h\sqrt{\pi h} = n_0 V^3,$$

from which:

$$A = 2\sqrt{2\pi}\cdot n_0; \quad h = 2/V^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

Hence in this case A becomes a constant, whereas h is a function of the instant of reflection.

The momentum imparted by the wall to the reflected molecules and the kinetic energy carried away by these molecules (as measured by a stationary observer) respectively are given by:

$$m \int_V^\infty du\,(u-V)\,u\,A \sqrt{\frac{h}{\pi}}\, e^{-h(u-V)^2} = c\,n_0\,m\,V^2\,;$$

$$\frac{m}{2} \int_V^\infty du\,(u-V)\left(u^2 + \frac{1}{h}\right) A \sqrt{\frac{h}{\pi}}\, e^{-h(u-V)^2} = c\,n_0\,m\,V^3,$$

where c is a numerical constant, being equal to $1 + \sqrt{\pi/8} = 1{,}627$. When we suppose that the wall has a mass M per unit of area and is not influenced by other forces, the retardation of its motion will be given by the equation:

$$M \cdot dV/dt = -\,c\,n_0\,m\,V^2,$$

from which:

$$V = C/t\,; \quad x_{\text{wall}} = C \ln t \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

where $C = M/c\,n_0\,m$, assuming that the constants of integration have been taken up in the values of t and x.

The distribution of velocities at a given point of space at a given instant can now be found as follows (compare fig. 4): the point x_0 and the instant

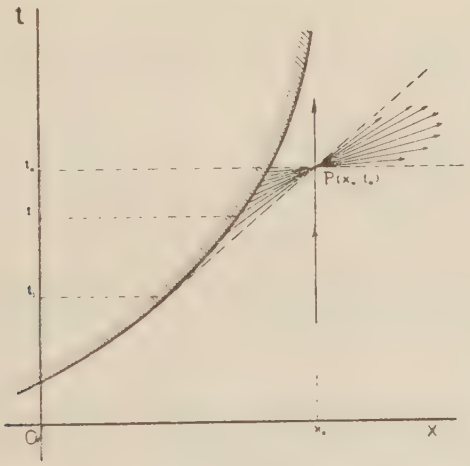


Fig. 4.

t_0 are represented by the point $P(x_0, t_0)$ of the diagram; the traces of the molecules passing through P must be: either a line parallel to the t-axis, representing the molecules at rest, or a set of lines starting from points of the path of the wall, all comprised between the horizontal line through P and the tangent from P to the path of the wall. We write t for the instant

corresponding to the starting point of a trace from the path of the wall; t_1 for the starting point of the tangent; then we shall have:

$$t_1 \leqslant t \leqslant t_0,$$

where t_1 is determined by the equation:

$$(t_0/t_1) - ln\,(t_0/t_1) = (x_0 - C\,ln\,t_0)/C + 1. \quad . \quad . \quad . \quad . \quad (17)$$

In the expression:

$$\Phi\,du = a \sqrt{\frac{\varkappa}{\pi}}\, e^{-\varkappa(u-\varphi)^2}\,du$$

we then have to substitute, corresponding to any value of t situated between the limits mentioned above:

$$\left.\begin{aligned} a = A = 2\sqrt{2\pi}\cdot n_0\,; \quad \varkappa = 2/V^2 = 2t^2/C^2\,; \quad \varphi = V = C/t \\ u = (x_0 - C\,ln\,t)/(t_0 - t) \end{aligned}\right\} . \quad . \quad (18)$$

The course of Φ as a function of u has been represented in fig. 5

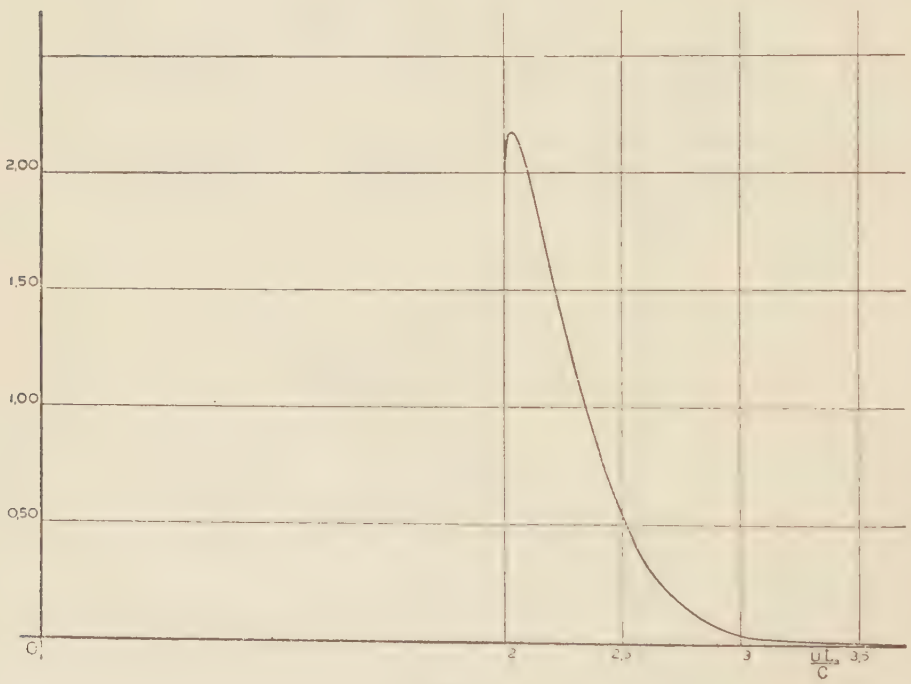


Fig. 5.

(abscissae: $u\,t_0/C$; ordinates: $\Phi C/n_0\,t_0$). Apart from the molecules which still are at rest (density n_0 per unit volume; not represented in the diagram, as this would require an integrable infinity at $u\,t_0/C = 0$), there are no molecules with velocities smaller than $u_1 = (x_0 - C\,ln\,t_1)/(t_0 - t_1)$, corresponding to the tangent line in fig. 4. Fig. 5 has been constructed for

the case: $t_1 = t_0/2$, corresponding to $(x_0 - C\,ln\,t_0)/C = 1 - ln\,2 = 0{,}307$; $u_1 t_0/C = 2{,}00$. In this case Φ has the value $2{,}00\ n_0 t_0/C$ for $u\,t_0/C = 2{,}00$; Φ passes through a maximum of approximately $2{,}18\,n_0\,t_0/C$ for $u\,t_0/C \cong 2{,}03$.

Numerical integration gives roughly:

$$\int_{u_1}^{\infty} du\ \Phi = 0{,}80\ n_0\,; \qquad \int_{u_1}^{\infty} du\ u\ \Phi = 1{,}80\ n_0\ C/t_0.$$

Hence the average velocity of the moving molecules is approximately $2{,}25\ C/t_0$; this is in good accord with:

$$\int_{u_1}^{\infty} du\ u^2\ \Phi = 4{,}07\ n_0\ (C/t_0)^2.$$

The mean velocity taken over moving molecules and molecules at rest together (total density: $N = 0{,}80\ n_0 + n_0 = 1{,}80\ n_0$) comes out as $U \cong 1{,}00\ C/t_0$.

We further have:

$$m \int_0^{\infty} du\ (u - U)^2\ \Phi = 2{,}27\ n_0\ m\ (C/t_0)^2,$$

calculated over all molecules together; and for the moving molecules (again by numerical integration):

$$m \int_{u_1}^{\infty} du\ \Phi/2\varkappa = 0{,}43\ n_0\ m\ (C/t_0)^2.$$

[The value of V corresponding to the average velocity $2{,}25\ C/t_0$ of the moving molecules would be $1{,}40\ C/t_0$; the actual average value of V is somewhat higher, *viz.* $1{,}46\ (C/t_0)$; the mean values of v^2 and w^2 for the moving molecules are each equal to $V_{av}^2/4 = 0{,}53(C/t_0)^2$; the value of the integral given above corresponds to $0{,}80\ n_0\ m$ times that amount.]

Finally: $p = 1{,}04\ n_0 m\ (C/t_0)^2$; $P_{xx} = -1{,}23\ n_0 m\ (C/t_0)^2$; $P_{yy} = P_{zz} = 0{,}61\ n_0 m\ (C/t_0)^2$.

5. *Case of two walls moving according to a parabolic law.* — When the gas is enclosed between two walls, the solution of the problem becomes much more difficult, even when the two walls move symmetrically according to some simple prescribed law. Only the first steps could be taken for a case where the velocity of the walls is small compared to that of the molecules.

We assume the motion of the left hand wall to be given by $x_1 = -a\,t^2$, that of the right hand wall by $x_2 = +a\,t^2$, both for $t < 0$, a being a constant. The gas consequently is compressed between the two walls. It

will not be possible to continue the motion until $t = 0$, and circumspection will also be necessary with regard to the initial phases.

The paths of the two walls in the x, t-plane have been indicated in fig. 6.

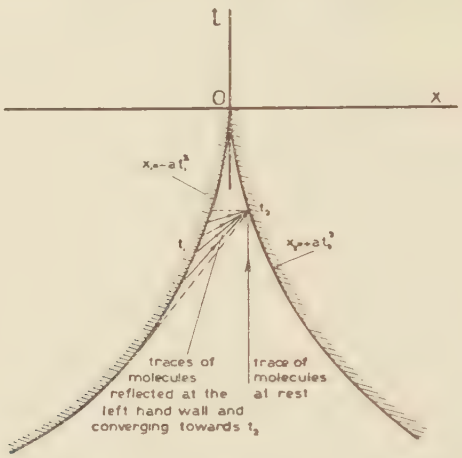


Fig. 6.

Traces of molecules striking the right hand wall at the instant t_2, in so far as these traces have started from reflections at the left hand wall, must be comprised between the horizontal line through t_2 and the tangent drawn from t_2 to the left hand parabola. There may be also traces referring to molecules which never had been in touch with a wall before the instant t_2. If again we assume that originally the molecules were at rest, the traces of the latter molecules will be parallel to the t-axis. The density of the molecules in the original state, $n_0(x)$, is assumed to be given.

Consider the molecules reflected by the left hand wall at an instant t_1. We write:

$$t_1 = t_2(1+\eta), \text{ with } 0 \leqq \eta \leqq \sqrt{2}. \quad . \quad . \quad . \quad . \quad . \quad (19)$$

The velocity of these molecules is found to be:

$$u = (-at_2) \cdot (2 + 2\eta + \eta^2)/\eta. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

Infinite velocity is obtained with $\eta = 0$; the maximum value of u, corresponding to the tangent, is obtained with $\eta = \sqrt{2}$.

In forming the equations corresponding to (13), (14) we must give attention both to the contribution to be derived from the molecules at rest, and to the contribution obtained from the molecules reflected by the left hand wall. For both groups we have $V = 2at_2$. For those of the first group the only value of u to be considered is 0, while also v and w are zero; the density $n_0(x)$ of this group is to be taken for $x = at_2^2$. For the molecules of the second group α and $\varkappa$ are (as yet unknown) functions of $t_1 = t_2(1+\eta)$, while $\varphi = -2at_1 = -2at_2(1+\eta)$. For shortness we write $-at_2 = \tau$; $-at_1 = \tau_1 = \tau(1+\eta)$. Then A and h will be the

same functions of τ as a and $\varkappa$ are of τ_1. When the integrations are expressed by means of η, we obtain:

$$\frac{A}{2\sqrt{\pi h}} = 2 n_0 \tau + \tau^2 \int_0^{\sqrt{2}} d\eta \, \frac{(2-\eta^2)(2+4\eta+\eta^2)}{\eta^3} \, a \sqrt{\frac{\varkappa}{\pi}} \, e^{-\frac{\varkappa (2-\eta^2)^2 \tau^2}{\eta^2}} \tag{21}$$

$$\left.\begin{aligned} \frac{A}{h\sqrt{\pi h}} = 8 n_0 \tau^3 + \tau^4 \int_0^{\sqrt{2}} d\eta \, \frac{(2-\eta^2)(2+4\eta+\eta^2)}{\eta^3} \cdot \\ \cdot \left\{ \frac{(2+4\eta+\eta^2)^2}{\eta^2} + \frac{1}{\varkappa \tau^2} \right\} a \sqrt{\frac{\varkappa}{\pi}} \, e^{-\frac{\varkappa (2-\eta^2)^2 \tau^2}{\eta^2}} \end{aligned}\right\} \tag{22}$$

In virtue of the symmetry of the situation, separate equations referring to the reflection at the other wall are not required.

Formulae (21), (22) represent two integral equations for the functions a and $\varkappa$. It has been attempted to solve them by means of a series development. After a number of trials it was found that these series should start with the terms:

$$a = A_0 \tau_1^{-2} \{1 + g \tau_1^{5/3} (q_1 + q_2 \ln \tau_1)\}$$
$$\varkappa = g^2 \tau_1^{4/3} \{1 + 2 g \tau_1^{5/3} (p_1 + p_2 \ln \tau_1)\},$$

where A_0, g, q_1, q_2, p_1, p_2 are constants. An approximate evaluation of the integrals then becomes possible in the case where $g \tau^{5/3}$ is very small in comparison to unity, provided the domain of integration can be divided into two parts, as follows:

$$\text{(I)}\ 0 \leqq \eta \leqq \beta g \tau^{5/3}; \qquad \text{(II)}\ \beta g \tau^{5/3} \leqq \eta \leqq \sqrt{2},$$

where β should be a number of the order 10 or 20, while still $\beta g \tau^{5/3} \ll 1$.

When the developments are worked out as far as $(g \tau^{5/3})^2$, equations are obtained which fix the values of q_1, q_2, p_1, p_2; whereas A_0 and g remain arbitrary. The process, however, is laborious and has not been wholly worked out.

The result that $\varkappa$ and h are proportional to $\tau^{4/3}$, while the volume (the space between the two walls) is proportional to τ^2, shows that with the conditions accepted (velocity of the walls small in comparison with the mean molecular velocity) the temperature of the gas in first approximation is proportional to $(volume)^{-2/3}$, as is the case with an ordinary monatomic gas following POISSON's law for an adiabatic change of state.

Zoology. — *The influence of concentration, duration of treatment and stage of development in the lithium-effect upon the development of Limnaea stagnalis.* By CHR. P. RAVEN, J. C. KLOEK *), E. J. KUIPER and D. J. DE JONG. (From the Zoological Laboratory, University of Utrecht).

(Communicated at the meeting of May 31, 1947.)

In 1942, one of the authors discovered the remarkable effects of a treatment with LiCl upon the eggs of *Limnaea stagnalis* (RAVEN, 1942). These effects are of two kinds. First, the embryos may develop into large, thin-walled vesicles, resembling the "exogastrulae" of sea urchins. As a matter of fact, the study of these structures in sections has shown that their wall consists partly of ectodermal, partly of entodermal cells; hence, we are justified in considering them as exogastrulae. In other instances, development proceeds further, but a graded series of malformations is obtained; they affect especially the head region. Dorsal displacement of the eyes, leading to synophthalmic or cyclopic embryos, splitting of one of the eyes, reduction of one or both eyes and tentacles are common malformations in these series; in extreme cases, the head is nearly entirely absent. In all these instances, merely the head region is affected; the foot and trunk are normal or only slightly disturbed. Only in a small percentage of cases, highly abnormal, teratomorphic forms appear, with serious disturbance of the relation between different parts of the body. For the sake of brevity, we shall further refer to all these cases as "head malformations".

As to the physiological aspects of the Li-action, a great variability was noted. The effects of the treatment depended on the concentration of lithium, the length of exposure time and the stage of development at which the eggs were exposed. However, the combination of these factors did not suffice to explain the diversity of the results. In some instances, where portions of different egg masses at the same stage of development were exposed simultaneously to the same solution during the same length of time, some of them showed a normal development, while in others all eggs became exogastrulae. It was clear, therefore, that individual differences in susceptibility of the egg masses played a part. Moreover, it was suspected that the procedure of the experiments, in which whole egg masses or portions of them were exposed to the solutions, might be responsible in part for the variability of the results.

Therefore, the experiments were continued in 1943 by J. C. KLOEK,

*) J. C. KLOEK died 16.IV.1947. This paper is dedicated to his memory by the other authors.

with the purpose to work out a method allowing to improve the reproducibility of the results. Although KLOEK made a great number of experiments, he did not succeed in this; the individual differences in susceptibility are, evidently, so great as to be subversive of all regulations of treatment. It became clear, therefore, that only by means of a great number of experiments, the results of which could be treated statistically, the regularities underlying the diversity of the Li-effects could be disclosed.

The experiments of KLOEK gave a key to one of these regularities which had not been detected before. They indicated that the results of the Li-treatment do not only depend on the stage at which the eggs are transferred to the Li-solution, but still more so on the stage at which the treatment is ended and the eggs are returned to tap water; with respect to this, a regular alternation of periods of increased and lessened susceptibility seemed to exist. In order to check this presumption, a number of experiments were undertaken in 1945 by F. J. KUIPER. At the same time, D. J. DE JONG examined the effects of Li-treatment in advanced stages of cleavage which had not been investigated before.

Taken together, the experiments of the 4 authors give a sufficient basis admitting a statistical treatment of the results, in order to bring to light the regularities of the Li-effect.

Material and methods.

4 Concentrations of LiCl have been used: 0.01 %, 0.005 %, 0.002 % and 0.001 % (= 0.00236 M, 0.00118 M, 0.00047 M and 0.00024 M.) The solutions were made from a stock solution of 1 % LiCl in distilled water by diluting with tap water.

Together, 437 experimental series, including a total of 9482 eggs, have been performed. Table I gives their distribution over the diverse concentrations.

TABLE I.

Concentration of LiCl-solution	Number of series	Number of eggs.
0.01 %	74	2258
0.005 %	173	3592
0.002 %	8	287
0.001 %	182	3345
	437	9482

With 0.002 % LiCl, only a small number of experiments have been made; they will not be discussed below. With the other concentrations, the number of experiments is sufficient to permit a statistical treatment.

In the first experiments, whole egg masses or portions of them were exposed to the solutions. To avoid the complications caused by the external jelly, in later experiments the latter was removed and the isolated egg

capsules were transferred to the solution. However, here a new difficulty arose: in control egg capsules, cultured in tap water, the development of the eggs is seldom entirely normal; often they show considerable abnormalities and die at the trochophore or "veliger" stage. Evidently, tap water is an inappropriate culture medium for single egg capsules. Therefore, KLOEK made a number of experiments in order to find a suitable medium.

First, the influence of the standard salt solutions according to RINGER, HOLTFRETER or LEHMANN (1937) in various dilutions was tested. They gave no improvement, as compared with tap water: in single egg capsules exposed to the solutions, the development of the eggs was often very abnormal, seldom entirely normal. The best results were obtained with LEHMANN solution diluted 4 to 32 times; HOLTFRETER solution proved to be very unsuitable.

A remarkable result was obtained, when single egg capsules were put into distilled water: the embryos developed nearly normally till a stage with pigmented eyes, then suddenly they died.

In order to test a possible protective effect of the jelly, single egg capsules were put into tap water, to which mucous substance of the tunica interna of the egg mass was added. However, this proved to have a very detrimental effect: the eggs died within 4 days. It is possible that the mucus promoted bacterial growth which was injurious to the eggs.

PH-measurements with a glass electrode (executed in coöperation with the Laboratory of Comparative Physiology of the Utrecht University [1]) showed that the pH in one day old egg masses in tap water lies between 5.49 and 5.62. The measurements were carried out in the following way: the egg mass was cut open lengthwise, then the glass electrode was put on the cut surface of the jelly and read off immediately. Since the pH of Utrecht tap water and of the above-mentioned salt solutions is much higher (7.14—8.84), that of old distilled water still slightly higher (5.88—6.09) than that of the jelly, it is possible that this accounts for the deficient development of the eggs in isolated egg capsules in these media.

These considerations led to the attempt to adjust the pH of the environment by means of a substrate of agar-agar. The pH of a layer of 2 % agar-agar, as determined with the glass electrode, was 5.65; hence, it agrees quite well with that of the jelly. Egg capsules in tap water or distilled water on a bottom of agar-agar showed a highly abnormal development, however. At last the following culture method proved to give satisfactory results: The egg capsules were placed on a bottom of agar-agar in a Petri dish, without water; this dish was closed by a cover and put in a larger one, containing water, to prevent drying up. By this method, the eggs develop normally till hatching in a large percentage of cases.

This method was employed in the later experiments. In general, an experimental series took the following course: The egg capsules of an egg

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[1]) It is a pleasure to thank Dr. H. J. VONK for his assistance in these determinations.

mass were freed from the common jelly. A number of these capsules served as controls; they were put immediately into a tube with tap water. The remaining capsules were, after determination of the stage of development of the eggs, distributed over some tubes with LiCl solutions. After a definite exposure time, these solutions were replaced by tap water; the stage of development was noted again. The tap water was renewed frequently. 24 Hours after the end of the Li-treatment, the experimental eggs and the controls were put on the agar-bottom in a Petri dish, where they continued their development. The eggs were arranged in rows on the agar and could be followed in their development individually by placing the Petri dish under a binocular microscope.

Stages of development.

In order to indicate exactly the stage of development of the eggs at the beginning and the end of the Li-treatment, a series of normal stages, covering the first part of the cleavage period, was composed. The stages 1—19 have been described and pictured by RAVEN (1946). In short, they are the following:

Stage 1:	First cleavage.	Beginning of cleavage furrow at animal pole.
2.	„ „	Beginning of cl. f. at vegetative pole.
3:	„ „	Cl. furrow cut half way through egg.
4:	„ „	Cl. f. cut entirely through egg. Cells rounded.
5:	2-cell stage.	Beginning flattening of blastomeres.
6:	„ „	Further flattening of blastomeres.
7:	„ „	Flattening of blastomeres completed.
8:	„ „	Lenticular cleavage cavity.
9:	„ „	Wide cleavage cavity.
10:	Second cleavage.	Beginning of cleavage furrow.
11:	„ „	Cells rounded.
12:	4-cell stage.	Beginning flattening of blastomeres.
13:	„ „	Beginning of cleavage cavity.
14: }		Wide central cleavage cavity with periodic
15: }	„ „	expulsion of fluid.
16:	Third cleavage.	Formation of 1st micromeres.
17:	„ „	Cells rounded.
18:	8-cell stage.	Flattening of blastomeres.
19:	„ „	Wide cleavage cavity.

For the purpose of the present investigation, the following stages have been added:

Stage 20:	Formation of 2d micromeres; 12 cells.	
21:	Division of 1st micromeres; 16 cells.	
22:	16-cell stage.	Flattening of cells.
23:	„ „	Wide cleavage cavity.
24:	Formation of 3d micromeres; 20 cells.	
25:	Division of 2d micromeres; 24 cells.	

In many instances, the regularities come out more distinctly in combining these stages into groups; then, stages 1—9 are taken together as 2-cell stage; stages 10—15 as 4-cell stage; stages 16—19 as 8-cell stage, the following stages are indicated as "morula".

Results.

1. *Exogastrulation.*

On an average, in the experimental series treated with 0.01 % LiCl 41 % of the eggs developed into exogastrulae; with 0.005 % LiCl, only 16 %, with 0.001 % LiCl 35 % of exogastrulae developed.

With 0.01 % LiCl, an exposure time of 1 hour suffices to give the maximum percentage of exogastrulae. With 0.005 % LiCl, a treatment of 1 hour gives a low percentage of exogastrulae; the number of exogastrulae rises with increasing length of exposure time; a maximum is reached at 7 hours of treatment. When the duration of the treatment exceeds 12 hours, the percentage of exogastrulae drops; at the same time, the number of embryos showing unspecific abnormalities or early death increases. With 0.001 % LiCl, as a rule no exogastrulae develop when the exposure time is less than 6 hours; the maximum is reached with 24 hours of treatment.

As to the influence of the stage of development, this is most evident in the series with 0.01 % LiCl. Fig. 1 summarizes the relation between stages of a. beginning b. ending of treatment and percentage of exogastrulae. Both graphs show a distinct maximum with a value of nearly 100 % exogastrulae:

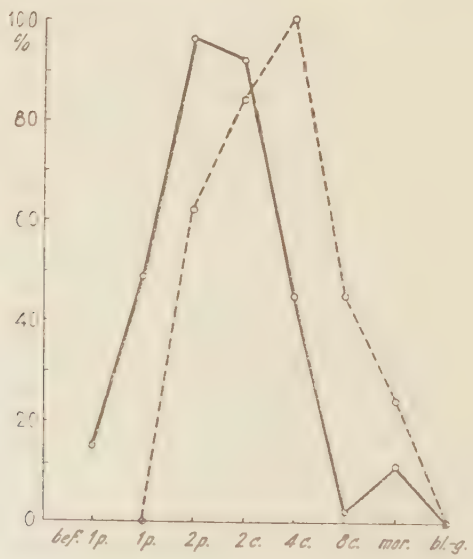


Fig. 1. *Limnaea stagnalis.* 0.01 % LiCl. Percentage of exogastrulae (ordinates) in dependence of stage of development (abscissae) at which treatment is begun (full line) and ended (broken line). *bef. 1 p* = before formation of 1st polar body; *1 p, 2 p* = 1 or 2 polar bodies present; *2c, 4c* etc. = 2-cell, 4-cell stage etc.; *mor* = morula; *bl — g* = blastula and gastrula stages.

for the beginning of treatment it lies at the stages: 2 polar bodies to 2-cell stage: for the ending of treatment there is an even sharper maximum (100 % exogastrulation) at the 4-cell stage.

These graphs point to the presence of a distinct maximum of sensibility for the production of exogastrulae at the 2- to 4-cell stage. Additional evidence for this is furnished by the other series which permit at the same time to delimit more sharply this period of sensibility.

Both in 0.005 % and 0.001 % LiCl. no exogastrulae developed when the Li-treatment was ended in stages 1—6: in 0.001 % LiCl. no exogastrulae developed when treatment was begun after stage 22: in 0.005 % LiCl. with suboptimal exposure times (1—2½ hours) no exogastrulae developed when the eggs were exposed after stage 12—13. We may conclude from these figures that there is a maximum of sensibility for the production of exogastrulae between stages 7—12 (i.e. shortly before and during the second cleavage) and a decreased sensibility until stage 22 (16 cell stage) or, making allowance for the fact that the effective concentration of Li in the eggs will be reached in 0.001 % LiCl only after at least 6 hours, somewhat later: after this stage, in no case exogastrulae have been produced.

2. *Head malformations.*

Head malformations occur less frequently than exogastrulae. On an average, the series with 0.001 % LiCl yielded 7 % of head malformations, those with 0.005 % and 0.01 % LiCl only 4 %.

With 0.01 % LiCl, a treatment of 1 hour produced the maximum percentage of head malformations; longer exposure times gave lower values. With 0.005 %, a treatment of 1 hour suffices to give some head malformations; the percentage rises till an exposure time of 6 hours, then drops; with exposure times exceeding 8 hours no head malformations were produced. With 0.001 %, a treatment during at least 5 hours is needed to produce head malformations.

As is also the case with the exogastrulae, the influence of the stage of development is most evident in the 0.01 % LiCl series. Fig. 2 shows that

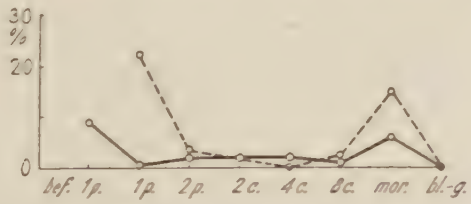


Fig. 2. *Limnaea stagnalis*. 0.01 % LiCl. Percentage of head malformations in dependence of stage of development at which treatment is begun and ended. Explanation cf. fig. 1.

there are 2 periods of maximum sensibility: 1°. immediately after laying (beginning of treatment before 1st polar body, ending between 1st and 2d polar body formation), 2°. at late cleavage stages (beginning and ending

of treatment at "morula" stage, i.e. after the 8-cell stage). Between the maxima, there is a period of low sensibility, in which nearly no head malformations are produced.

In the series with 0.005 % and 0.001 % LiCl, the first of these maxima does not appear. Evidently, when the eggs are put into these solutions immediately after laying, the effective concentration of Li within the eggs is not reached before the first period of sensibility has already ended. On the other hand, the second period of sensibility also in these series is clearly visible. Fig. 3 shows this for 0.001 % LiCl with regard to the stage of beginning of treatment. Up to the 2-cell stage, head malformations are extremely rare; at the 8-cell stage, there is a distinct rise, which continues in the next stages and leads to a maximum at stages 23—25, with 42 % of head malformations; then the line drops to zero at still older stages. No head malformations have been obtained as yet when the eggs were put into LiCl-solutions after stage 25. On the other hand, when the treat-



Fig. 3. *Limnaea stagnalis*. 0.001 % LiCl. Percentage of head malformations in dependence of stage of beginning of treatment. *St. 19/22* = stage 19—22; *e. blast* = early blastula. Further explanation cf. fig. 1.

ment was ended at stages 1—7, also no head malformations developed. Making allowance for the time needed to reach the effective concentration of Li in the eggs (5 hours in 0.001 % LiCl), we may conclude that there is a first period of sensibility immediately after laying, which is of short duration; it is followed by a period of minimum sensibility. From stage 7 on sensibility rises again slowly to a maximum lying at least 5 hours after stages 23—25, then there is a rather sudden drop.

Periodicity.

As has been mentioned above, the results of KLOEK's experiments pointed to a regular alternation of periods of increased and lessened susceptibility of the eggs. In his series with 0.005 % LiCl, head malformations occurred when the eggs had been taken from the LiCl solution and transferred to tap water at stages 7—10, 14—15 or 19, whereas no head malformations developed when the treatment was ended in stages 1—6, 11—13 and 16—18. Since the stages 7—10, 14—15 and 19 represent comparable phases

of the 2-, 4- and 8-cell stage, respectively, namely, the stages with wide cleavage cavity preceding the next divisions, it was suspected that this regularity had a real significance. As the number of cases was rather small (69 series including 771 eggs, yielding 42 head malformations), the experiments of KUIPER were intended to check this possibility. Their results did not corroborate, however, those of KLOEK, and so the experiments seemed to prove that these findings had been only accidental.

In summarizing the results of the series with 0.001 % LiCl of the 4 authors, however, a same regularity appeared. As Table II shows, it does not only concern the head malformations, but the exogastrulae as well.

TABLE II.

Number of exogastrulae and head malformations in dependence of stage of ending of treatment with 0.001 % LiCl.

Stage	Number of series	Number of eggs	Exogastrulae		Head malformations	
			Number	%	Number	%
2	1	10	—	—	—	—
3	3	31	—	—	—	—
4	3	33	—	—	—	—
5	3	32	—	—	—	—
6	4	46	—	—	—	—
7	3	32	—	—	—	—
8	8	88	—	—	1	1
9	3	28	5	18	1	4
10	10	104	4	4	3	3
11	—	—	—	—	—	—
12	3	30	—	—	—	—
13	1	16	—	—	—	—
14	3	36	—	—	—	—
15	3	26	22	85	—	—
16	1	16	2	13	—	—
17	3	26	8	31	2	8
18	5	65	—	—	—	—
19	9	209	20	10	1	0.5
	66	828				

The total number of exogastrulae and head malformations produced is only small; this may be due to a suboptimal exposure time, and, as regards the head malformations, to the general reduction of sensibility during these stages (cf. fig. 3). However, the table seems to indicate an increased susceptibility of the eggs at stages 8—10, 15—17 and 19. In general, this corresponds to the above-mentioned results, though there is a slight shift in the stages as compared with these. However, still the numbers of cases in the various groups are small, and it is difficult to come at a definite conclusion. We can only say that the alternation of more and less sensible periods in correspondence to the phases of cleavage is indicated without being definitely proved.

Discussion.

The Li-effects in *Limnaea* are of 2 kinds; in some cases the eggs develop to exogastrulae, in other cases various head malformations are produced; in many series both abnormalities may arise side by side. The question arises as to what factors decide which of these abnormalities appears. Since the disturbance of development in the case of exogastrulae is much more serious and shows itself earlier in development, it seemed reasonable to suppose that the disturbing influence has been stronger in this case; one might expect that exogastrulae would develop especially with higher concentrations of LiCl and longer exposure times. As a matter of fact, the series with 0.01 % LiCl yielded, on an average, the highest percentage of exogastrulae, whereas head malformations were most numerous in the series with 0.001 % LiCl. On the other hand, within each of these groups the duration of treatment seemed to have no differential effect. Both with regard to exogastrulae and head malformations with 0.01 % LiCl 1 hour of treatment suffices to give the maximum percentage; with 0.005 %, the maximum is reached at 6—7 hours exposure time; and with 0.001 %, 5—6 hours of treatment at least are needed to obtain the Li-effects.

It is evident, therefore, that the length of the exposure time is no factor in deciding between the 2 divergent ways of development. Another circumstance proves to be of paramount importance, however: the stages during which the eggs are exposed to the Li-solutions. For exogastrulation, on the one hand, and head malformations, on the other, there appear to exist different periods of sensibility. The maximum of sensibility for the production of exogastrulae lies shortly before and during the second cleavage. It must be emphasized that this phase of development appears to be a "critical" stage in many respects: in eggs developing in distilled water or urea solutions, development comes to a standstill in just this phase (RAVEN & KLOMP, 1946). It is possible that the permeability of the eggs is increased at this moment which would explain the high susceptibility of the eggs to the composition of the medium.

On the other hand, for the production of head malformations there are 2 periods of increased sensibility: one immediately after laying, the other reaching its maximum at least 5 hours after the 24-cell stage; between these maxima, there is a period of minimum sensibility.

During the latter period, another regularity is indicated though not definitely proved: the alternation of phases with increased and lessened sensibility in correspondence to the phases of cleavage. When the eggs are taken from the LiCl solution and transferred to tap water at stages with wide cleavage cavity preceding the next division, both exogastrulae and head malformations are produced, whereas this is not the case when the treatment is ended in the intermediate phases of cleavage. At first, it seems rather queer that the moment of ending of the Li-treatment should have such an important effect. One might think that the Li-treatment would act in preparing the way for some reaction occurring only at the moment of

transfer to tap water, comparable to the activation of the sea urchin egg upon transfer to sea water after treatment with butyric acid according to LOEB. On second thought, the explanation is, probably, much more simple. When the eggs with their capsules are put into the Li solution, Li-ions will permeate through the capsule membrane; in his way, the concentration of Li in the capsule fluid will gradually rise. The egg will take up these ions from the capsule fluid; in view of the low external concentrations of LiCl which are effective in bringing about serious disturbances of development, it seems probable that an elective accumulation of these ions in the eggs takes place. It will take some time before the really "effective" concentration of Li is reached within the egg. Upon returning to tap water, the reversed phenomena will occur; the Li is slowly washed out of the egg capsule fluid, the egg will thereupon give off the accumulated Li-ions partly or entirely. Hence, it is clear that, especially with suboptimal exposure times, the highest concentration of Li within the eggs will be attained immediately before or even some time after the moment of return of the egg capsules to tap water. Especially during periods of generally reduced sensibility (corresponding, probably, to a raised threshold value of the effective concentration) this moment will, therefore, be of much importance; slight differences in threshold value with the stages of development will influence the results markedly. Hence, if the observed effect is real, it points to differences in sensibility coinciding with the phases of cleavage.

The fact that head malformations are induced most easily at the 24-cell stage and later is very important. As has been emphasized previously (RAVEN, 1942), these malformations are not due to local losses of cells, but to an alteration in the pattern of determination of the embryo. It is, therefore, highly significant that these alterations can be induced still at so late a stage in these "mosaic" eggs; evidently, the pattern of determination has not yet been laid down irrevocably at this stage. This approaches these Mollusks to the forms with "regulative" development, and raises the suspicion that the determination of development proceeds in both groups along the same lines.

Summary.

1. Eggs of *Limnaea stagnalis* have been exposed to solutions of LiCl in 4 concentrations, varying between 0.01 % and 0.001 % (0.00236 M and 0.00024 M). The results of 437 experimental series, including 9482 eggs, are taken together.

2. There is a distinct maximum of sensibility for the production of exogastrulae shortly before and during the second cleavage.

3. With regard to the production of head malformations, a first period of sensibility exists immediately after laying; it is followed by a period of minimum sensibility; then, the sensibility rises slowly to a maximum lying some hours after the 24-cell stage.

4. The results indicate that in the period of generally reduced sen-

sibility during the first 3 cleavages the eggs show an alternation of phases of increased and lessened sensibility corresponding to definite phases of cleavage.

5. It is evident that the pattern of determination of the organs of the head has not yet been laid down irrevocably at the 24-cell stage in *Limnaea*.

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Physics. — *Recovery and recrystallization viewed as processes of dissolution and movement of dislocations.* II. By W. G. BURGERS. (Laboratorium voor Physische Scheikunde der Technische Hoogeschool, Delft.) (Communicated by Prof. J. M. BURGERS.)

(Communicated at the meeting of March 29, 1947.)

II *Block-structure of the crystalline state.*

Starting from the assumption [LENNARD JONES (13)] that an "ideal" lattice represents the condition of minimal free energy for a crystal, we must conclude that even in the case of a pure element or compound, every crystalline testpiece, independently of its being uni- or polycrystalline, undeformed or coldworked, represents a "metastable" state of thermodynamical equilibrium. The structural differences between these states are merely gradual and not essential. In what follows we shall consider this point somewhat more in detail.

II, 1. *Single crystal.*

According to numerous observations, every "real" crystal, apart from "macroscopic" irregularities ["lineage structure" of BUERGER (14) [5]], has a certain "mosaic" structure, consisting of ideally regular lattice blocks [or lamellae, according to GRAF (16)] with dimensions of the order of magnitude of 0.1—1 micron, the blocks, however, including angles varying from perhaps seconds to minutes of arc. Their presence follows partly from measurements of the intensity of diffracted X-rays [DARWIN (17); EWALD and RENNINGER (18); DEHLINGER and GISEN (19)], partly from microscopic observations of the natural or etched surface of crystals [see in particular GRAF (16)]. Also the "structure-sensitive" character of many physical and mechanical properties [SMEKAL (20)] leads to the same conclusion. Finally the often considerable influence of minute quantities of foreign atoms on the properties of pure metals seems to find a plausible explanation on the assumption that such atoms are preferably "adsorbed" at the boundaries of the lattice blocks and in some way or other exert here their remarkable influence [BRAGG (21)].

As to the "structure" of the block-boundaries, suggestions have been made by various authors [J. M. BURGERS (22); BRAGG (23)]. It is now generally assumed that the deviations of the atoms from their normal positions in these transition layers, which necessarily must occur with regard to the positions of the atoms in both adjoining blocks, are as small as possible. Fig. 4 shows a schematic picture given by J. M. BURGERS (22): here the "fit" between two blocks which include a small angle α, is brought about by a succession of "edge-dislocations" (TAYLOR-dislocations), *all of the same sign*, lying at equal distances h, determined by $tg\ \alpha = \lambda_0/h$, where

[5] In this connection recent observations by LACOMBE and BEAUJARD (15) of corrosion patterns on aluminium crystals, prepared by recrystallisation, are of great interest.

λ_0 is the lattice constant. [It may be remembered that, by removing the atoms over a certain area of a lattice plane, as indicated in fig. 3 (which is

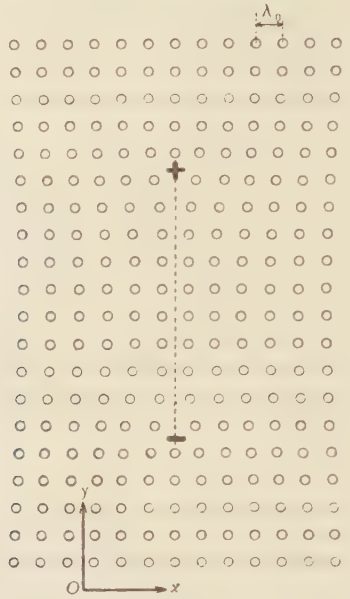


Fig. 3. Lattice region (supposed to extend to infinity in the direction of the z-axis, perpendicular to the plane of the drawing) with two edge-dislocations (TAYLOR-dislocations), one positive (above) and one negative (below). The dislocations are obtained by removing the atoms over a certain area of the lattice perpendicular to the x-axis. The dislocation lines extend at + and —, parallel to the z-axis (after J. M. BURGERS (22)).

supposed to extend to infinity in a direction perpendicular to the plane of the drawing), two edge- or TAYLOR-dislocations are created, one of which (indicated by +) is called positive, the other (—) negative] [6]).

It is, of course, a most important question whether for a definite substance, a pure metal for example, its crystals are always characterized by a definite mosaic structure. This question is closely related to the not yet solved problem of crystal growth. On the one hand it is certain that the degree of imperfection of crystals can be largely influenced by conditions of growth, at least as to their (semi-) macroscopic faults. Moreover it follows from measurements of RENNINGER (18) with rocksalt that also the "size" of the blockstructure, which governs the intensity of reflected X-rays, varies for artificial and natural crystals. ADDINK (18a), from measurements of the specific gravity, concludes that crystals of the alkali-

[6]) In the case, shown in fig. 4, the two blocks are rotated with regard to each other about an axis lying *in* the plane of their common boundary (perpendicular to the plane of the drawing). As is indicated in J. M. BURGERS' paper, the boundary between two blocks, which are rotated with regard to each other about an axis *perpendicular* to this boundary, can be realized by a succession of dislocations of a different type (so-called "screw-dislocations").

chlorides, prepared by melting, are "incomplete" compared with those from a solution. With regard to our subject, it is in this connection of special

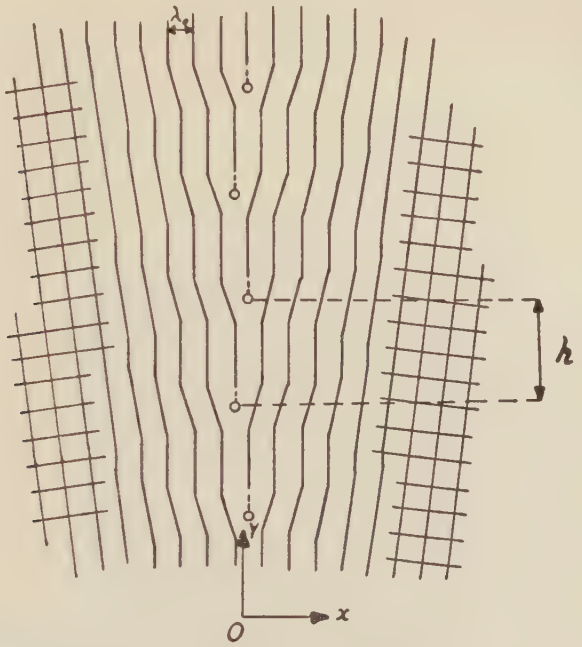


Fig. 4. Schematic picture af a transition surface between two lattice domains, formed by a set of parellel dislocation lines of edge type, all situated in the plane $x = 0$ (after J. M. BURGERS (22)).

interest that measurements by DEHLINGER and GISEN (19) with aluminium have established that crystals formed by recrystallization have a more pronounced mosaic (*viz.* smaller blocks and probably larger angular deviations between them) than crystals obtained from the melt.

On the other hand according to GRAF (16) all crystals, independently of their way of preparation, possess a structure composed of lamellae, the thickness of which varies between narrow limits. In this connection GRAF mentions two papers by HERLINGER (24), according to which at the surface of a growing lattice-block "dislocated" atoms can only be stabilized so long as the block is smaller than a minimum size; whereas for larger blocks the probability of assuming irregular positions increases to such extent that further growth becomes less probable than formation of a new block. The limiting size is estimated as approximately 0.1 micron. A similar process of this nature might lead to a block- or lamellar structure with a size of blocks (lamellae), determined by the lattice forces and thus characteristic for a given lattice. According to M. BORN (24a) a perfect single crystal, owing to the asymmetry of the vibrations of atoms, could not be larger than about a thousand repeat distances in any direction, since at distances greater than this the vibrating atoms would get completely out of step. The figure of 1000 is obtained as a rough mean value of the reciprocal of the

DEBYE temperature multiplied by the thermal expansion coefficient for many substances [7]).

II, 2. *Polycrystal* (crystal aggregate).

In an annealed polycrystalline test-piece the "block-system" is somewhat more complicated in character. Here, superimposed as it were on the mosaic structure inside the individual crystallites, is a system of intercrystalline boundaries, where much larger angular deviations between the lattices of adjacent blocks have to be bridged. Also these "true" grain boundaries most probably are not "amorphous", but possess a definite structure, dependent on the relative orientation of adjacent grains. This conception finds experimental support in experiments by CHALMERS (28) on the strength of double-crystals of tin; by SNOEK (29) on the intercrystalline oxydation of polycrystalline nickeliron, with- and without preferential orientation of the crystallites; and by LACOMBE and BEAUJARD (30) on the corrosion of high-purity aluminium. In these latter experiments, the chemical reagent ($HNO_3 + HCl + HF$), applied after electropolishing, produced, apart from corrosion figures inside the individual crystallites, boundary lines between neighbouring crystallites which were clearer developed the larger their difference in lattice orientation.

On the basis of such experimental evidence it seems justified to assume [J. M. BURGERS (22); W. L. BRAGG (23); LENNARD JONES (13)] that also

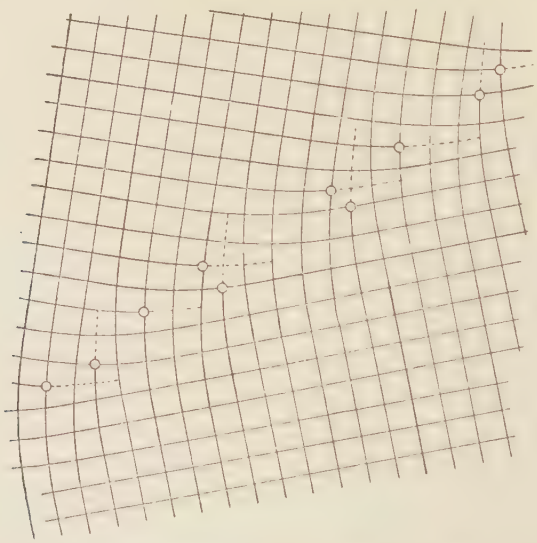


Fig. 5. Schematic picture of a transition layer ("grain boundary") between two lattice blocks, including a large angle. The layer is built up of *two* sets of dislocation lines of the edge type (compare fig. 4) (after J. M. BURGERS).

[7]) As is well known, ZWICKY (25) has postulated the existence of a stable mosaic-structure on energetic grounds. Later criticism of his conceptions by OROWAN (26) and BUERGER (27) have shown that his conclusions cannot be maintained.

the grain boundary constitutes the best possible fit between adjacent grains, affecting a minimal number of lattice planes in the transition region. As pointed out in (22), a transition surface between lattice blocks including an arbitrary angle might be built up in an analogous way, as is shown in fig. 4, now, however, introducing a double set of dislocation lines [8]). Fig. 5, for which I am indebted to J. M. BURGERS, shows this schematically.

The conception of a grain boundary, built up of dislocations, is furthermore illustrated in a very striking way by BRAGG's experiments with soap bubbles (31), as fig. 6 may show.

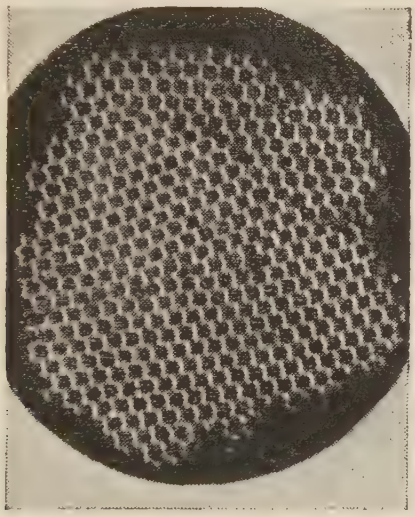


Fig. 6. "Intercrystalline boundary", as shown by floating soap bubbles (after W. L. BRAGG (31)).

II, 3. *Cold-worked state.*

The experimental fact, first observed by VAN ARKEL (32) and by DAVEY (33), that the DEBYE–SCHERRER pattern of a cold-worked metal differs from that of the annealed state at most by a slight broadening or a decreased intensity of the interference lines [9]), is proof that even here by far the major part of the atoms form still coherent lattice regions (lattice blocks). So once more, according to the current view [DEHLINGER (3); KORNFELD (2); BRAGG (21); (1), §§ 52—54] the resulting structure, even in the most severely worked condition, is conceived as consisting of lattice-blocks, held together by systems of dislocated atoms. The main difference between the cold-worked and the annealed state consists in the circumstance that cold-working, as a consequence of alternation and mutual hindrance of

[8]) As remarked in (22), in this case a certain relation must be fulfilled between the spacings of the two sets and the angle of inclination of the transition plane, in order that at great distances from this plane the lattices shall be free from stress.

[9]) We do not consider here the changes due to the formation of preferential orientations of the crystallites.

various active glide-combinations, has caused a slight elastic stressing and bending of the blocks, as is evident from the above-mentioned changes in line-width and intensity of the DEBYE–SCHERRER lines [10]). Simultaneously the essential characteristic feature of the annealed polycrystalline state, namely the presence of *groups* of nearly parallel lattice blocks *inside* the original grains, which, taken together, include much larger angles with adjoining groups (grains), is more or less effaced.

As WOOD (35) pointed out, an estimate of the minimum size of the blocks may be obtained by assuming that the broadening of the DEBYE–SCHERRER lines is wholly due to their *smallness*. In this way sizes of the order of 0.1 μ are calculated for most metals (in aluminium, where the broadening is very slight, this figure comes out as approx. 1 μ). DEHLINGER and KOCHENDÖRFER (36), taking into account that part of the line-broadening is due to the stressed state of the blocks [VAN ARKEL and BURGERS (37); MEGAW, LIPSON and STOKES (38)], give the size of mosaic-blocks in rolled copper as 0.4 μ.

In fig. 7 we give a schematic picture of the deformed state, in which the

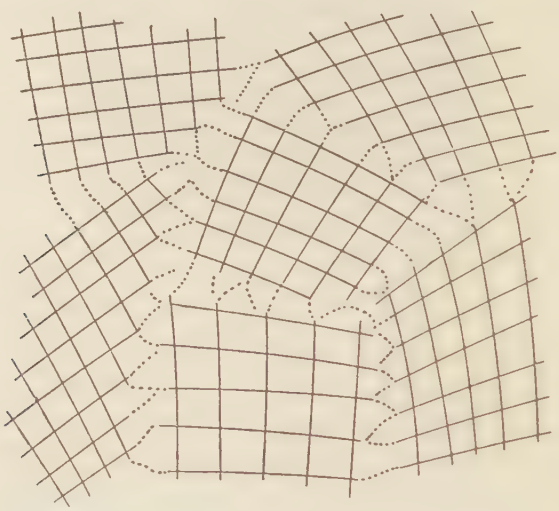


Fig. 7. Schematic representation of the cold-worked state: elastically stressed and bent "mosaic-blocks" connected by transition layers of dislocated atoms (taken from (1)).

stressed and bent state of the blocks is largely exaggerated. The transition layers have only been indicated by dots. We may consider them as built up of a complicated system of various types of dislocations. To give a more

[10]) The exceptional case of aluminium, where line-broadening after cold-work is practically absent or in any case much less pronounced then with most other metals, might be understood if an alternation of the active glide-planes were more easily brought about in this metal than in the others [DEHLINGER (3) p. 755; see also references in (1), § 41, § 47]. Experiments with sheared aluminium crystals support this view [BURGERS and LEBBINK (34)].

detailed picture is, considering all the different possibilities [see for example the cases treated in (22)], not well possible.

II, 4. *"Twodimensional" block-structure.*

For the case however, that we limit ourselves to a "twodimensional" aggregate [11]) and take into account only the presence of dislocations of the edge type, as shown in fig. 4, we can say for certain that the "structure" of the layer is essentially dependent on the angle of misfit between adjoining blocks: there, where large angles have to be bridged, a concentration of dislocations of the same sign (see fig. 3) is unavoidable (fig. 4 and 5). The "density" of the concentration of either positive or negative dislocations increases generally speaking with this angle [12]).

It must not be left out of sight that also lattice blocks which are in parallel or nearly parallel positions can be separated by systems of dislocations; when this is the case these systems must contain an equal (or nearly equal) number of dislocations of both signs. These may alternate either "individually" or in groups. Some arrangements, which can be envisaged, have been indicated in a purely schematic way in fig. 8 [13]).

The conception of the cold-worked state as built up of elastically stressed lattice-blocks, connected by transition layers of dislocated atoms, is, for the "two-dimensional" structure considered here, to some extent confirmed by the amount of energy taken up in the course of the deformation process. As pointed out in (1) (§§ 55, 56), and by BRAGG (40) and others, the elastic energy in the blocks is only a small fraction (of the order of 1 %) of the total energy of cold-work. This latter is, according to the measurements of TAYLOR and QUINNEY (41) for copper maximally about 10 calories ($\sim 4.10^8$ erg) per cm^3. If we put the energy of a dislocation line (as defined in fig. 3) at about 10^8 electron volt ($\sim 10^{-4}$ erg) per cm [SEITZ and READ (42); KOEHLER (43)], then the number of dislocation-lines per cm^2 is of the order 10^{12}. If all these were arranged along lines 0.2 micron (2.10^{-5} cm) apart (that is, at the sides of mosaic blocks of this width), this number would amount to 2.10^7 dislocation lines per cm, or about 1 dislocation line per $5A^0$: this comes near to one dislocation line per every two and a half atomic plane. This is, as BRAGG (40) remarks, equivalent to

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[11]) "Two-dimensional" is here taken in this sense that the pattern extends without change of structure in a direction perpendicular to the plane considered in the drawings.

[12]) At least up to a certain extent: in the cubic system for example, due to symmetry relations, the angle $45° - \alpha$ may be equivalent to $45° + \alpha$.

[13]) One might, of course, introduce the assumption of dislocations *inside* the blocks: in a stressed block, which, taken as a whole, is *not* bent, we could have an equal number of positive and negative dislocations; and an excess of either positive or negative in an elastically bent part. In how far this is justified depends to some extent on what one considers as a "block" [compare our fig. 11; also KOCHENDÖRFER (39) pp. 119 ff.]. In any case we do not think this additional assumption could affect our reasoning in the following paragraphs in an essential way.

an interface between two-dimensional closed packed arrangements, meeting at 30°, which is the largest angle at which they can meet (since 60° is

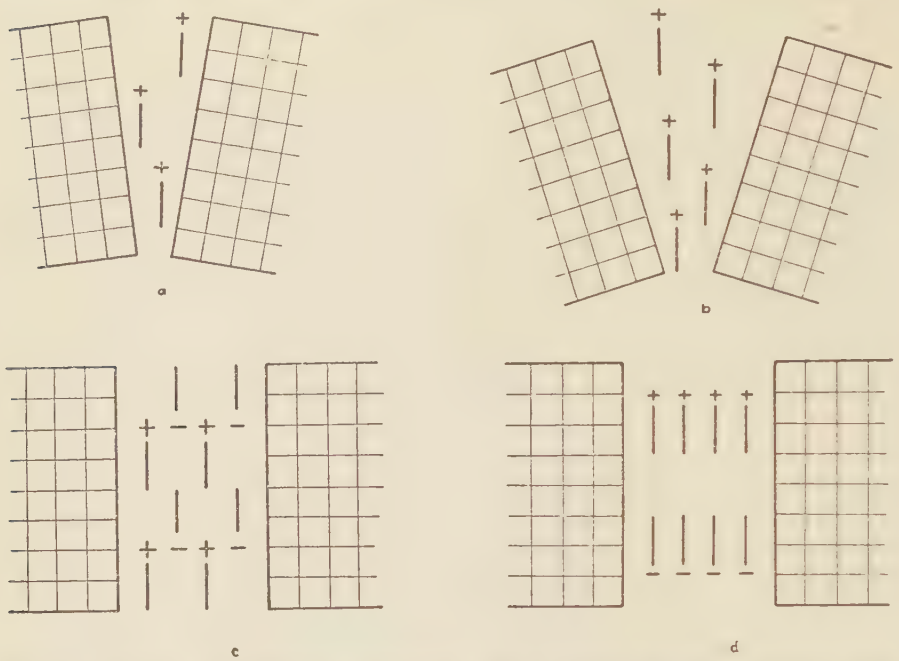


Fig. 8. Schematic representation of transition layers between neighbouring lattice blocks, built up of systems of dislocations of edge-type (TAYLOR-dislocations). Here and in figures 10 and 11 such dislocations have been indicated by $\overset{+}{|}$ (positive dislocation) and $|_{-}$ (negative dislocation): the | represents the "direction" of the layer of atoms removed from the lattice (compare fig. 3). The direction of "easy mobility" of the dislocations lies in the plane of the drawing perpendicular to the direction of the removed layer.
a, b: adjoining blocks non-parallel: surplus of dislocations of same sign.
c, d: adjoining blocks parallel: equal numbers of + and — dislocations.

equivalent to 0° owing to symmetry). In this simplified picture, this actually represents the maximum of crowding of dislocations, in agreement with the maximal degree of cold-work of the metal [14]).

II, 5. *Stability of the block structure.*

In the preceding considerations we have left aside the question in how far a deformed block-structure of the type we have in mind will be stable (or, rather, metastable). With regard to this problem only a few indications can be found in the current literature. It may be useful to recapitulate the basic conceptions which should be kept in mind in discussing this subject. The cohesive forces in the lattice are derived from the electric and other short range forces which neighbouring atoms exert upon each other.

[14]) In (40), this conclusion is reached in a somewhat different way.

Leaving aside the normal thermal vibration, every atom is in equilibrium under the forces acting upon it; in other words the forces upon every atom must have zero resultant. In a perfectly regular lattice the system of forces exhibits exact periodicity. Hence when we imagine a cut to be made in the lattice by an element of surface extending over a whole number of periods, the forces exerted by the atoms lying on one side of this element upon the atoms lying on the other side, will be zero, provided the system is not acted upon by exterior forces. In that case we say that the lattice is unstressed. Every misfit, however, produces some irregularity, which will have its influence both upon the positions of neighbouring atoms and upon the forces experienced by them. This influence theoretically can extend over infinite distances, although in most cases the presence of irregularities of more or less opposite character has a compensating effect, which materially reduces the action radius of a single irregularity. When we now again imagine a cut to be made in the lattice by means of a small element of surface, extending over a whole number of periods, the forces exerted by the atoms lying on one side of the element upon the atoms lying on the other side, may differ from zero, so that we can say that there are stresses in the lattice.

When the lattice irregularities are of a well defined character, as is the case with dislocations, we may consider those dislocations as the seats of forces, producing the observed stresses in the lattice. Some authors, like KORNFELD (2), therefore use the expression that a system of elastically stressed blocks is kept in equilibrium by forces which have their seat in the transition layers (where the dislocations are to be found). In the same connection BRAGG (21) speaks of a "dynamic stability by the boundaries" and compares the cold-worked state to a foamstructure [compare also BENEDICKS (44)].

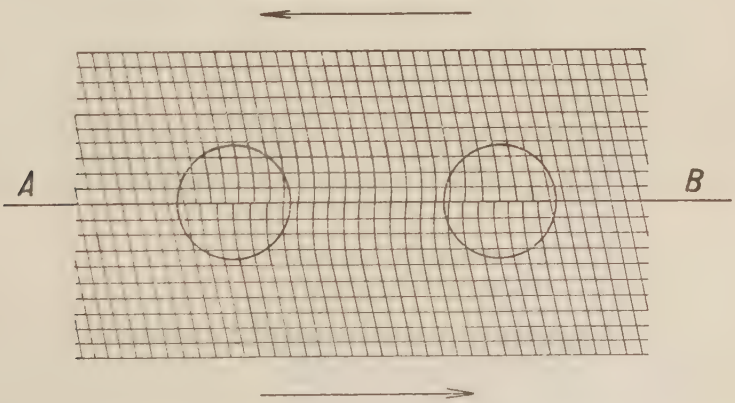


Fig. 9. Pair of dislocations, one positive (at the left) and one negative (at the right), formed as the result of a local glide jump (after OROWAN (45). In the "abridged" way of representation, described in fig. 8, such a pair of dislocations ("Verhakung" as called by DEHLINGER) is indicated with $\begin{array}{|c|}\hline + \\ \hline \end{array}$.

Now the concept of stability enters into the picture when we see the possibility that a number of atoms in the neighbourhood of a dislocation may be able to find more than one position of equilibrium without being obliged to pass through a high energy barrier. In such a case a shift from one position of equilibrium to a neighbouring one may initiate a displacement of the irregularity, and in particular the displacement of a dislocation line. As soon as such possibilities are taken in view a degree of stability can be defined, depending upon the energy which must be communicated to certain atoms in order to initiate a change in the configuration of the system of dislocations.

DEHLINGER (3) has been the first to attack this question in a quantitative way. His considerations are limited to a series of "Verhakungen", a "Verhakung" being approximately equivalent to the combination of a positive and a negative TAYLOR-dislocation, as can be formed in a perfect lattice by a so-called glide jump, in the way indicated in figure 9, due to OROWAN (45).

In what follows we shall schematize a TAYLOR-dislocation by means of a plus or minus sign, accompanied by a vertical dash,

$$+ \quad \text{or} \quad |$$
$$| \qquad -$$

the dash indicating the "removed" lattice plane (compare fig. 3). As is well known, the direction of easy mobility of such a dislocation lies perpendicular to the removed lattice plane, thus in the case indicated in the horizontal direction, in the plane of the paper. A "Verhakung" will be indicated by $\begin{matrix} + & | \\ | & - \end{matrix}$.

According to DEHLINGER's calculation [15]) an isolated "Verhakung" is unstable: there is a strong tendency for the occurrence of such shifts that the two dislocations of opposite sign move towards each other and mutually "neutralize" as soon as they have come together. Following TAYLOR (46) we may consider this as a consequence of a certain "attraction" between dislocations of opposite sign. Now the important feature pointed out by DEHLINGER is that in the case of a large number of "Verhakungen" lying in a single row [16]), the system will become stable when the density (that is,

$$\begin{matrix} & | \\ + & - \\ | & \end{matrix} \qquad \begin{matrix} & | \\ + & - \\ | & \end{matrix} \qquad \begin{matrix} & | \\ + & - \\ | & \end{matrix} \qquad \begin{matrix} & | \\ + & - \\ | & \end{matrix}$$

the number of "Verhakungen" per unit length) surpasses a certain value. By stability is meant that a positive threshold energy must be introduced

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[15]) I am indebted to Mr. F. R. N. NABARRO (Bristol) for a closer elucidation of DEHLINGER's conclusions.

[16]) Such series may be created if a large number of glide jumps of the type shown in fig. 9 all took place at different points along the same glide plane [cf. KOCHENDÖRFER (39), loc. cit.].

into the system, before dislocations can be dissolved, whereas in the case of a single "Verhakung" (one pair of dislocations only) the thermal energy of the atoms is sufficient to bring about mutual approach and neutralization of the two dislocations.

The "critical" density of the series depends upon the elastic constants of the metal; for a common metal as copper or silver DEHLINGER estimates it to be reached for one "Verhakung" per every 5 or 10 atoms. For larger densities the threshold energy ("activation energy for dissolution") increases, which means that the system of dislocations becomes more difficult to dissolve and thus is more stable.

Wo do not know of similar calculations for more complicated sets of dislocations. For example the question has not been considered whether an arrangement as indicated in fig. 10*a*, where every horizontal row carries an alternating system of positive and negative dislocations, which thus could neutralize each other, is stable in DEHLINGER's sense.

In the calculations referred to it has been assumed that the lattice is not affected by exterior forces. We can, however, also put the question of stability for a lattice which is stressed by the application of exterior forces. Such cases have been treated by TAYLOR (46), who has considered the stability of certain two-dimensional arrangements, likewise containing an equal number of positive and negative dislocations of edge-type, in the



Fig. 10. Two-dimensional arrangements of alternating positive and negative TAYLOR-dislocations. The direction of "easy mobility" ("direction of slip") is supposed to run horizontally in the figure; the dislocations can be formed by slip parallel to this direction.
a. positive and negative dislocations alternate along the same "direction of slip".
b. positive and negative dislocations lie on alternating "directions of slip".

case when an exterior shearing stress acts in a direction parallel to the direction of easy mobility. An example of such an arrangement is shown in fig. 10*b*, which differs from that depicted in fig. 10*a*, by the fact that in fig. 10*b* dislocations of opposite sign lie on alternating rows. TAYLOR has shown that in the absence of exterior stress the system of fig. 10*b* is stable. If, however, an external shear stress acts parallel to the glide plane, a displacement of *all* the positive with regard to *all* the negative dislocations is brought about. Starting from the arrangement of fig. 10*b*, every value of the shear stress which lies below a certain maximum value, gives a new stable equilibrium with a definite displacement of the positive with regard to the negative dislocations. As soon as the stress surpasses a certain value, the two sets of opposite sign migrate steadily in opposite directions (until they are stopped by some obstacle, for example a transition layer between two mosaic blocks). The magnitude of this critical stress increases with the "density" of the dislocation lines per unit surface.

Finally BRAGG (23) has considered the *mobility of a series of dislocation lines of the same sign* which form the boundary between two mosaic blocks as shown in fig. 4 (also in fig. 8*a* and 8*b*). Such a series can be moved by a relatively small force acting in a direction perpendicular to itself (thus horizontal in the figure), which is the direction of "easy-mobility" of each dislocation separately. In this process the individual atoms in the transition layer suffer only slight displacements. The resultant effect is equivalent to the growth of one block at the cost of the other, the disappearing block suffering as it where a rotation about an axis perpendicular to the plane of the drawing, so that its orientation gradually coincides with that of the growing blocks [17]).

(To be continued.)

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[17]) In (13) LENNARD JONES considers the movement of two adjoining lattice blocks, arranged in a similar way as shown in fig. 4, in a direction *parallel* to the transition plane (vertical direction in the drawing).

Mathematics. — *Eine Bemerkung über das Mass in Strukturen.* By J. RIDDER. (Communicated by Prof. W. VAN DER WOUDE.)

(Communicated at the meeting of May 31, 1947.)

Die Bemerkung, um welche es sich hier handelt, ist die Konstatierung, dass die in den Theoremen *A* und *B* enthaltene Bedingung notwendig und hinreichend ist für die Messbarkeit der abgeschlossenen Somen sowohl beim beschränkt- wie beim total-additiven Mass.

I. *Beschränkt additives Mass.*

§ 1 [1]). *Eine Struktur S* sei aufgebaut aus Elementen, *Somen* genannt, die im folgenden mit kleinen Buchstaben angedeutet werden, und den folgenden Axiomen genügen:

Axiom 1°: $\alpha)\ a \subset a;\ \beta)$ aus $a \subset b$ und $b \subset c$ folgt $a \subset c$.

Definition. $a = b$, falls $a \subset b$ und $b \subset a$.

Definition. Ein Soma ab wird *Produkt* des Somenpaares a, b genannt, falls:

$\alpha)\ ab \subset a;\ \beta)\ ab \subset b;\ \gamma)$ aus $c \subset a$ und $c \subset b$ immer folgt $c \subset ab$.

Axiom 2°: Für jedes Somenpaar a, b gibt es ein Produkt ab.

Definition. Ein Soma $a + b$ wird *Summe* der Somen a, b genannt, falls: $\alpha)\ a \subset a + b;\ \beta)\ b \subset a + b;\ \gamma)$ aus $a \subset c$ und $b \subset c$ immer folgt $a + b \subset c$.

Axiom 3°: Für jedes Somenpaar a, b gibt es eine Summe $a + b$.

Axiom 4°: Es gibt ein Soma 0 mit $0 \subset a$ für jedes Soma $a \in S$.

Axiom 5°: Es gibt ein Soma 1 mit $a \subset 1$ für jedes Soma $a \in S$ [1a]).

Axiom 6°: $ac + bc = (a + b)c$.

Definition. Ist $a \subset b$, so wird durch $b - a$ angedeutet jedes Soma x, das den Bedingungen genügt:

$$ax = 0, \quad a + x = b.$$

Axiom 7°: Zu jedem Paar von Somen a, b, mit $a \subset b$, gibt es ein Soma $b - a$.

Eine derartige Struktur ist eine BOOLEsche Algebra.

§ 2 [1]). Für jedes Soma $x \in S$ sei *eine (reellwertige) Massfunktion* $m^\circ(x)$ definiert.

Definition. Ein Soma a heisse *$m^\circ(x)$-messbar oder messbar in bezug auf* $m^\circ(x)$, wenn für jedes Soma w, mit $m^\circ(w)$ endlich,

$$m^\circ(w) = m^\circ(wa) + m^\circ(w - wa)$$

ist.

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[1]) Siehe J. RIDDER, Acta math. **73** (1941), S. 131—173.

[1a]) Auch ohne Annahme dieses Axioms behalten die nachfolgenden Sätze und Theoreme ihre Gültigkeit.

Massaxiom Ib. *Es gibt ein Soma w mit $m^\circ(w)$ endlich und $\neq 0$.*

Massaxiom II. *Das leere Soma 0 ist $m^\circ(x)$-messbar.*

Massaxiom III. *Aus $a \subset b$ folgt immer $m^\circ(a) \leqq m^\circ(b)$.*

Jedes dieser drei Massaxiome ist von den übrigen unabhängig.

Satz. *Die $m^\circ(x)$-messbaren Somen bilden einen Körper $K(m^\circ)$*; das soll heissen: für je zwei $m^\circ(x)$-messbare Somen, a und b, sind ihre Summe und ihr Produkt $m^\circ(x)$-messbar, und, falls $a \subset b$, auch $b - a$.

§ 3[2]). *Topologische Axiome:*

T.A. I. $a \subset$ *die abgeschlossene Hülle* $\bar{a}$; hierbei ist $\bar{a}$ ein dem Soma a eindeutig zugeordnetes Soma der betrachteten Struktur S.

T.A. II. $\bar{0} = 0$.

T.A. III. *aus* $a \subset b$ *folgt* $\bar{a} \subset \bar{b}$.

T.A. IV. $\bar{\bar{a}} = \bar{a}$.

Eine Struktur S, welche die topologischen Axiome I — IV erfüllt, nennen wir *eine T^2-Struktur.*

Definition. $\mathfrak{B}$ sei der kleinste Somenkörper, welche alle abgeschlossenen Somen umfasst.

Theorem A. *Notwendig und hinreichend, damit in einer T^2-Struktur $\mathfrak{B} \subset K(m^\circ)$ sei, ist, dass die Funktion $m^\circ(x)$ (§ 2) folgende Bedingung erfüllt:*

aus $\bar{x} \cdot y = 0$ und $m^\circ(x+y)$ endlich folgt immer $m^\circ(x+y) = m^\circ(x) + m^\circ(y)$.

Beweis[3]). Die Bedingung ist notwendig. Aus $\mathfrak{B} \subset K(m^\circ)$ und T.A. IV folgt, dass für jedes Soma x $\bar{x}$ $m^\circ(x)$-messbar ist.

Somit ist für $m^\circ(z)$ endlich:

$$m^\circ(z) = m^\circ(z \cdot \bar{x}) + m^\circ(z - z \cdot \bar{x});$$

speziell $z = x + y$ liefert

$$m^\circ(x+y) = m^\circ[(x+y) \cdot \bar{x}] + m^\circ[(x+y) - (x+y) \cdot \bar{x}],$$

oder, wegen $\bar{x} \cdot y = 0$ und T.A. I,

$$m^\circ(x+y) = m^\circ(x) + m^\circ(y).$$

Die Bedingung ist hinreichend. Es genügt zu beweisen, dass jedes abgeschlossene Soma x (d.h. mit $x = \bar{x}$) $m^\circ(x)$-messbar ist. z sei ein willkürliches Soma mit $m^\circ(z)$ endlich. Dann ist wegen T.A. III

$\overline{xz} \subset \bar{x} = x$, also auch $\overline{xz} \cdot z \subset xz$, und dadurch $\overline{xz} \cdot (z - xz) = 0$.

Die Bedingung gibt somit

$$m^\circ[xz + (z - xz)] = m^\circ(xz) + m^\circ(z - xz),$$

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[2]) Siehe J. RIDDER, Verhand. Ned. Akad. v. Wetensch., Amsterdam, Sect. 1, 18, n°. 4 (1944), 43 Seiten.

[3]) Das topologische Axiom II wird beim Beweise nicht benutzt.

oder

$$m^\circ(z) = m^\circ(xz) + m^\circ(z - xz).$$

Daraus folgt, dass x m°-messbar ist.

§ 3bis. Natürlich bleibt das Theorem A güultig für *äussere* Massfunktionen, welche neben den Massaxiomen I^b, II und III erfüllen das

Massaxiom V [4]. *Für jedes Soma a mit $m^\circ(a)$ endlich ist $m^\circ(a)$ gleich der unteren Grenze der Werte $m^\circ(w)$ für alle $m^\circ(x)$-messbaren Somen (w) mit $a \subset w$.*

§ 3ter. Ebenso bleibt das Theorem A gültig für eine *innere* Massfunktion $m^\circ(x)$, welche ausser den Axiomen I^b, II und III noch die folgenden Axiome erfüllt:

Massaxiom IV. *Zu jedem Soma a mit $m^\circ(a)$ endlich gibt es ein $m^\circ(x)$-messbares Soma b mit $a \subset b$ und $m^\circ(b)$ ebenfalls endlich.*

Massaxiom V'. *Für jedes Soma a mit $m^\circ(a)$ endlich ist $m^\circ(a)$ gleich der oberen Grenze der Werte $m^\circ(w)$ für alle $m^\circ(x)$-messbaren Somen (w) mit $w \subset a$* [5].

II. *Total-additives Mass.*

§ 4 [1]. **Definition.** Ein Soma $\sum_{j=1}^{\infty} a_j$ wird *Summe der abzählbar unendlichen Klasse* $a_1, a_2, \ldots, a_j, \ldots$ genannt, falls: α) jedes $a_j \subset \sum_{j=1}^{\infty} a_j$; β) aus $a_j \subset b$ $(j = 1, 2, \ldots)$ folgt $\sum_{j=1}^{\infty} a_j \subset b$.

Im folgenden betrachten wir eine Struktur S^*, welche die Axiome 1°, 2°, 4°, 5°, 6°, 7° erfüllt, und ausserdem das

Axiom $\overline{3}^\circ$. Zu jedem Somenpaar a, b gibt es eine Summe $a + b$; für jede abzählbar unendliche Klasse von Somen: $a_1, a_2, \ldots, a_j, \ldots$ gibt es eine Summe $\sum_{j=1}^{\infty} a_j$.

Definition. Ein Soma $\prod_{j=1}^{\infty} a_j$ wird *Produkt der abzählbar unendlichen Klasse* $a_1, a_2, \ldots, a_j, \ldots$ genannt, falls: α) $\prod_{j=1}^{\infty} a_j \subset$ jedes a_j; β) aus $b \subset a_j$ $(j = 1, 2, \ldots)$ folgt $b \subset \prod_{j=1}^{\infty} a_j$.

In S^* existiert für jede abzählbare Klasse von Somen ein Produkt.

Definition. Ein Körper von Somen heisse *σ-Körper*, wenn die Summe

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[4]) Jedes der Axiome I^b, III und V ist von den übrigen unabhängig; Axiom II folgt aus den Axiomen I^b und V. Siehe RIDDER, loc. cit. 1).

[5]) Das Massaxiom II ist eine Folgerung der Massaxiome I^b und IV. Jedes der Massaxiome I^b, III, IV und V' ist von den übrigen unabhängig. Siehe RIDDER, loc. cit. 1).

von abzählbar vielen seiner Somen immer zu ihm gehört; er heisse *δ-Körper*, wenn das gleiche von dem Produkt von abzählbar vielen seiner Somen gilt.

Fordern wir von der Somenfunktion $m^\circ(x)$ die Massaxiome Ib, II, III des § 2 und daneben das

Massaxiom VI. *Die Summe von abzählbar vielen, $m^\circ(x)$-messbaren Somen ist $m^\circ(x)$-messbar* [6]).

Dann gilt der

Satz. *Der Körper $\mathfrak{R}(m^\circ)$ der $m^\circ(x)$-messbaren Somen ist sowohl σ-Körper wie δ-Körper.*

§ 5. Betrachten wir eine Struktur S^*, welche somit dieselben Strukturaxiome wie in § 4 erfüllt. und daneben den topologischen Axiomen I bis IV genügt; nennen wir eine derartige Struktur *eine t²Struktur.*

Definition. $\mathfrak{B}^*$ sei der kleinste σ-Körper von Somen, welche alle abgeschlossenen Somen umfasst.

Theorem B. *Notwendig und hinreichend, damit in einer t²-Struktur $\mathfrak{B}^* \subset \mathfrak{R}(m^\circ)$ sei, ist, dass die Funktion $m^\circ(x)$ (§ 4) folgende Bedingung erfüllt:*

aus $\bar{x} \cdot y = 0$ und $m^\circ(x+y)$ endlich folgt immer $m^\circ(x+y) = m^\circ(x) + m^\circ(y)$.

Der Beweis ist wörtlich derselbe wie der des Theorems *A*. —

Da aus den drei ersten Carathéodoryschen Axiomen [7]) für eine Massfunktion sich die Gültigkeit unserer Massaxiome Ib, II, III und VI ableiten lässt, erhält man *einen Spezialfall des Theorems B bei Benutzung der drei ersten Carathéodoryschen Massaxiome* [8]).

§ 5bis. Das Theorem B bleibt gültig bei Hinzufügung der drei folgenden Massaxiome, wodurch $m^\circ(x)$ zu *einer regulären äusseren Massfunktion* wird.

Massaxiom V. (Siehe § 3bis).

Massaxiom VII. *Für jede Folge von Somen $a_1 \subset a_2 \ldots \subset a_n \ldots$ mit $\mathfrak{a} = \sum_{j=1}^{\infty} a_j$ und $m^\circ(\mathfrak{a})$ endlich ist*

$$\lim_{j \to \infty} m^\circ(a_j) = m^\circ(\mathfrak{a}).$$

Massaxiom VIII. *Sind für eine Somenfunktion $m(x)$ die einander fremden Somen $a_1, \ldots, a_n, \ldots$ $m^\circ(x)$-messbar und ist $m\left(\sum_{j=1}^{\infty} a_j\right) = \infty$, so ist auch $\sum_{j=1}^{\infty} m^\circ(a_j) = \infty$.*

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[6]) Jedes der Axiome Ib, II, III und VI ist von den übrigen unabhängig.

[7]) Siehe C. CARATHÉODORY, *Vorles. über reelle F.*, Leipzig-Berlin, 1e oder 2e Aufl., §§ 235—268.

[8]) Für den Fall eines Raumes, in welchem die topologischen Axiome I bis IV erfüllt sind, und die Massfunktion für jede Teilmenge endlich ist, wurde dieser Spezialfall von J. ALBUQUERQUE, Portugaliae Math. 3 (1943), S. 258—262, abgeleitet.

Wir bemerken nebenbei, dass die Axiome Ib, III, V, VII und VIII voneinander unabhängig sind, und dass die Axiome II und VI sich aus den vorigen ableiten lassen; auch dass das Axiomensystem Ib, III, V, VII, VIII schon in einer Struktur, welche nur die Strukturaxiome $1°$, $2°$, $\overline{3}°$, $4°$, $5°$, $6°$, $7°$ erfüllt, aequivalent ist mit dem um das vierte Axiom [7]) verringerte Axiomensystem der Carathéodoryschen regulären äusseren Massfunktionen [1]).

§ 5ter. Ebenso bleibt das Theorem B gültig für *eine reguläre innere Massfunktion,* welche ausser den Axiomen Ib, II, III und VI noch folgende Massaxiome erfüllen soll:

Massaxiom IV. (Siehe § 3ter).

Massaxiom V′. (Siehe § 3ter).

Massaxiom VII′. *Für jede Folge von Somen* $b_1 \supset b_2 \ldots \supset b_n \ldots$ *mit* $b = \prod_{j=1}^{\infty} b_j$ *und mit* $m^\circ(b_1)$ *endlich ist*

$$\lim_{j \to \infty} m^\circ(b_j) = m^\circ(b).$$

Massaxiom VIII. (Siehe § 5bis).

Die Axiome Ib, III, IV, V′, VII′, VIII sind voneinander unabhängig; die Axiome II und VI folgen aus den vorigen [1]).

Mathematics. — *On the theory of linear integral equations.* VIIIa. By A. C. ZAANEN. (Communicated by Prof. W. VAN DER WOUDE.)

(Communicated at the meeting of April 26, 1947.)

§ 4. *The system with iterated kernel.*

Defining the linear transformations B_p in $[L_2]^n$ by $B_1 = A$, $B_p = A H B_{p-1}$ $(p = 2, 3, \ldots)$, it is not difficult to see that the iterated transformations K^p and $\widetilde{K}^p$ are given by $K^p = B_p H$ and $\widetilde{K}^p = H^{1/2} B_p H^{1/2}$, while moreover $H B_p H = H B_p^* H$ for $p = 1, 2, \ldots$. These transformations stand therefore in the same relation to each other as the original transformations K and $\widetilde{K}$. Since $\widetilde{K}^p$ has the sequence λ_k^p $(k = 1, 2, \ldots)$ of characteristic values $\neq 0$ with the orthonormal sequence $\{\Psi_k\} = H^{1/2}\{\psi_k\}$ of characteristic elements, it follows that K^p has also the sequence λ_k^p of characteristic values $\neq 0$ with the H-orthonormal sequence $\{\psi_k\}$ of characteristic elements. Defining the matrix-kernels $\| K_{ij}^{(p)}(x, y) \|$ $(p \geqq 1)$ by

$$K_{ij}^{(1)}(x, y) = K_{ij}(x, y),$$

$$K_{ij}^{(p)}(x, y) = \Sigma_{q=1}^{n} \int_{\Delta} K_{iq}(x, z)\, K_{qj}^{(p-1)}(z, y)\, dz,$$

the Theorems 2—10 hold therefore for the system of integral equations with matrix-kernel $\| K_{ij}^{(p)}(x, y) \|$, replacing everywhere λ_k by λ_k^p. In particular

$$\Sigma_{j=1}^{n} \int_{\Delta} K_{ij}^{(p)}(x, y)\, f^j(y)\, dy \backsim \Sigma_k \lambda_k^p a_k \psi_k^i(x) + q_{(p)}^i(x) \quad (i = 1, \ldots, n),$$

where $H\{q_{(p)}\} = \{0\}$, and

$$K_{ij}^{(p)}(x, y) - q_{ij}^{(p)}(x, y) \backsim \Sigma_k \lambda_k^p \psi_k^i(x)\, \overline{\chi_k^j(y)} \qquad (i, j = 1, \ldots, n),$$

where $\Sigma_{r=1}^{n} h_{ir}(x)\, q_{rj}^{(p)}(x, y) = 0$.

We shall show now that, for $p \geqq 2$, all functions $q_{(p)}^i(x)$ and $q_{ij}^{(p)}(x, y)$ vanish (For the proof that this is not necessarily true for $p = 1$, even in the simple case that $n = 1$, we refer to III, § 3).

Theorem 11. *For $p \geqq 2$, we have*

$$\Sigma_{j=1}^{n} \int_{\Delta} K_{ij}^{(p)}(x, y)\, f^j(y)\, dy \backsim \Sigma_k \lambda_k^p a_k \psi_k^i(x) \qquad (i = 1, \ldots, n), \quad (13)$$

$$K_{ij}^{(p)}(x, y) \backsim \Sigma_k \lambda_k^p \psi_k^i(x)\, \overline{\chi_k^j(y)} \qquad (i, j = 1, \ldots, n). \quad . \quad . \quad (14)$$

Proof. The first part of the theorem follows from I, Theorem 15. To prove the second part, we observe that, by SCHWARZ's inequality,

$$K_{qj}(z, y) - p_{qj}(z, y) \backsim \Sigma_k \lambda_k \psi_k^q(z)\, \overline{\chi_k^j(y)}$$

(cf. (8)) implies

$$\Sigma_{q=1}^n \int_\Delta K_{iq}(x, z)\, K_{qj}(z, y)\, dz - \Sigma_{q=1}^n \int_\Delta K_{iq}(x, z)\, p_{qj}(z, y)\, dz \backsim$$

$$\Sigma_k \lambda_k\, \overline{\chi_k^j(y)}\, (\Sigma_{q=1}^n \int_\Delta K_{iq}(x, z)\, \psi_k^q(z)\, dz),$$

or, since by (10)

$$\Sigma_{q=1}^n \int_\Delta K_{iq}(x, z)\, p_{qj}(z, y)\, dz = \Sigma_{q, r=1}^n \int_\Delta A_{ir}(x, z)\, h_{rq}(z)\, p_{qj}(z, y)\, dz = 0,$$

$$K_{ij}^{(2)}(x, y) \backsim \Sigma_k \lambda_k^2\, \psi_k^i(x)\, \overline{\chi_k^j(y)}.$$

The proof for $p > 2$ follows easily by induction.

Theorem 12. *For $p \geqq 2$ we have*

$$\Sigma_{i=1}^n \int_\Delta K_{ii}^{(p)}(x, x)\, dx = \Sigma_k \lambda_k^p.$$

Proof. Denoting the matrix-kernel, corresponding with the transformation B_p, by $\| B_{ij}^{(p)}(x, y) \|$, we have

$$\Sigma_{i=1}^n K_{ii}^{(p)}(x, x) = \Sigma_{i, j=1}^n B_{ij}^{(p)}(x, x)\, h_{ji}(x) =$$

$$\Sigma_{i, j, r=1}^n h_{ri}^{(1/2)}(x)\, B_{ij}^{(p)}(x, x)\, h_{jr}^{(1/2)}(x) = \Sigma_{r=1}^n \widetilde{K}_{rr}^{(p)}(x, x),$$

hence (cf. VII, Theorem 14)

$$\Sigma_{i=1}^n \int_\Delta K_{ii}^{(p)}(x, x)\, dx = \Sigma_{i=1}^n \int_\Delta \widetilde{K}_{ii}^{(p)}(x, x)\, dx = \Sigma_k \lambda_k^p.$$

§ 5. *The case that all $A_{ij}(x, y)$ are continuous in mean and all $h_{ij}(x)$ are continuous.*

We shall suppose now that $A_{ij}(x, y)$ $(i, j = 1, \ldots, n)$ is continuous in mean in $\Delta \times \Delta$, that is, $\int_\Delta |A_{ij}(x, y)|^2\, dy$ is finite for every $x \in \Delta$, $\int_\Delta |A_{ij}(x, y)|^2\, dx$ is finite for every $y \in \Delta$, and

$$\lim_{x_2 \to x_1} \int_\Delta |A_{ij}(x_2, y) - A_{ij}(x_1, y)|^2\, dy =$$

$$\lim_{y_2 \to y_1} \int_\Delta |A_{ij}(x, y_2) - A_{ij}(x, y_1)|^2\, dx = 0,$$

We observe that these conditions imply that $\int_{\Delta} |A_{ij}(x,y)|^2\,dy$ is a continuous function of x in Δ, and $\int_{\Delta} |A_{ij}(x,y)|^2\,dx$ a continuous function of y in Δ. Furthermore we shall suppose that $h_{ij}(x)$ $(i,j=1,\ldots,n)$ is continuous in Δ, so that, by Lemma 1, 3°., $h_{ij}^{(1/2)}(x)$ $(i,j=1,\ldots,n)$ is also continuous in Δ. It is not difficult to see that, under these conditions, $K_{ij}(x,y)$ $(i,j=1,\ldots,n)$ and $\widetilde{K}_{ij}(x,y)$ $(i,j=1,\ldots,n)$ are also continuous in mean in $\Delta\times\Delta$. Finally we observe that for every $\{f\}\in[L_2]^n$ the functions $g^i(x)=\sum_{j=1}^n \int_{\Delta} K_{ij}(x,y)\,f^j(y)\,dy$ $(i=1,\ldots,n)$ are continuous in Δ, since

$$\left|\int_{\Delta} K_{ij}(x_2,y)\,f^j(y)\,dy-\int_{\Delta} K_{ij}(x_1,y)\,f^j(y)\,dy\right|\leqq \left(\int_{\Delta}|K_{ij}(x_2,y)-K_{ij}(x_1,y)|^2\,dy\right)^{1/2}\cdot\|f^j\|.$$

As a consequence, all characteristic functionsets $\psi_k^i(x)$ $(i=1,\ldots,n)$ consist of continuous functions.

Theorem 13 (*Expansion Theorem*). *If*

$$a_k=(\{f\},\{\chi_k\})=\sum_{i=1}^n \int_{\Delta} f^i(x)\,\overline{\chi_k^i(x)}\,dx$$

for an arbitrary $\{f\}\in[L_2]^n$, *then*

$$\sum_{j=1}^n \int_{\Delta} K_{ij}(x,y)\,f^j(y)\,dy=\sum_k \lambda_k a_k \psi_k^i(x)+\widetilde{p}^i(x) \qquad (i=1,\ldots,n), \tag{15}$$

uniformly in Δ, *where* $\{\widetilde{p}\}=\{\widetilde{p}^1(x),\ldots,\widetilde{p}^n(x)\}$ *consists of continuous functions, and satisfies*

$$\sum_{j=1}^n h_{ij}(x)\,\widetilde{p}^j(x)=0$$

for every $x\in\Delta$.

Proof. First of all we prove that $\sum_k |a_k|^2$ converges. This follows from

$$a_k=(\{f\},\{\chi_k\})=(\{f\},H^{1/2}\{\Psi_k\})=(H^{1/2}\{f\},\{\Psi_k\}),$$

hence, since the system $\{\Psi_k\}$ is orthonormal, by BESSEL's inequality,

$$\sum_k |a_k|^2\leqq\|H^{1/2}\{f\}\|^2=(H\{f\},\{f\})\leqq\|H\|\cdot\|\{f\}\|^2.$$

Furthermore, we see that the matrix-kernel $\|D_{ij}(x,y)\|$, corresponding with the transformation $D=AH^{1/2}$, is also continuous in mean, so that $\int_{\Delta}|D_{ij}(x,y)|^2\,dy$ is a continuous function of x in Δ for $i,j=1,\ldots,n$. Since, consequently, for $i=1,\ldots,n$, and for every $x\in\Delta$,

$$\{d_i\}=\{d_i^1(y),\ldots,d_i^n(y)\},\ \text{where}\ d_i^j(y)=D_{ij}(x,y),$$

belongs to $[L_2]^n$, the inequality (11) holds now for every $x\in\Delta$, which shows

that there exists a constant M, such that $\Sigma_k \lambda_k^2 |\psi_k^i(x)|^2 \leqq M$ $(i=1,\ldots,n)$. Hence, if $\varepsilon > 0$ is given, for k_1 and k_2 sufficiently large,

$$|\Sigma_{k_1}^{k_2} \lambda_k a_k \psi_k^i(x)| \leqq (\Sigma_k \lambda_k^2 |\psi_k^i(x)|^2)^{1/2} (\Sigma_{k_1}^{k_2} |a_k|^2)^{1/2} < \varepsilon$$

for every $x \in \triangle$. The series $\Sigma_k \lambda_k a_k \psi_k^i(x)$ of continuous functions converges, therefore, uniformly in $\triangle$, to a continuous sumfunction. Since, by what we have already remarked, $\Sigma_{j=1}^n \int_{\triangle} K_{ij}(x,y) f^j(y)\, dy$ is also continuous, the same holds for the difference $\widetilde{p}^i(x)$. But $\widetilde{p}^i(x)$ is, as follows from (5), almost everywhere in $\triangle$ equal to the function $p^i(x)$ occurring in that formula, hence $\Sigma_{j=1}^n h_{ij}(x) \widetilde{p}^j(x) = 0$ almost everywhere in $\triangle$. The functions $h_{ij}(x)$ $(i,j=1,\ldots,n)$ and $\widetilde{p}^j(x)$ $(j=1,\ldots,n)$ being however continuous, we have $\Sigma_{j=1}^n h_{ij}(x) \widetilde{p}^j(x) = 0$ for every $x \in \triangle$.

Theorem 14 (*Expansion Theorem for the iterated kernels*). *For $p \geqq 2$ we have*

$$K_{ij}^{(p)}(x,y) = \Sigma_k \lambda_k^p \psi_k^i(x) \overline{\chi_k^j(y)} \qquad (i,j=1,\ldots,n),$$

uniformly in $\triangle \times \triangle$.

Proof. The Hermitian matrix-kernel $\|\widetilde{K}_{ij}(x,y)\|$ is continuous in mean in $\triangle \times \triangle$; hence, by the Remark in VII, § 4,

$$\widetilde{K}_{qr}^{(2)}(x,y) = \Sigma_k \lambda_k^2 \Psi_k^q(x) \overline{\Psi_k^r(y)} \qquad (q,r=1,\ldots,n),$$

uniformly in $\triangle \times \triangle$. Then also, for $i,j=1,\ldots,n$,

$$\Sigma_{q,r=1}^n h_{iq}^{(1/2)}(x) \widetilde{K}_{qr}^{(2)}(x,y) h_{rj}^{(1/2)}(y) = \Sigma_{q,r=1}^n h_{iq}^{(1/2)}(x) \widetilde{K}_{qr}^{(2)}(x,y) \overline{h_{jr}^{(1/2)}(y)}$$
$$= \Sigma_k \lambda_k^2 \chi_k^i(x) \overline{\chi_k^j(y)},$$

uniformly in $\triangle \times \triangle$. This shows in particular that, for $i=1,\ldots,n$, the series $\Sigma_k \lambda_k^2 |\chi_k^i(x)|^2$ converges uniformly in $\triangle$.

In the proof of Theorem 13 we have already seen that

$$\Sigma_k \lambda_k^2 |\psi_k^i(x)|^2 \leqq M \quad (i=1,\ldots,n),$$

where M does not depend on x; hence, if $\varepsilon > 0$ is given,

$$\Sigma_{k=N} |\lambda_k^2 \psi_k^i(x) \overline{\chi_k^j(y)}| \leqq M^{1/2} (\Sigma_{k=N} \lambda_k^2 |\chi_k^j(y)|^2)^{1/2} < \varepsilon$$

for sufficiently large N. The series $\Sigma_k \lambda_k^2 \psi_k^i(x) \overline{\chi_k^j(y)}$ of continuous functions converges, therefore, uniformly in $\triangle \times \triangle$, to a continuous sumfunction, which is, by (14), equal to $K_{ij}^{(2)}(x,y)$ almost everywhere in $\triangle \times \triangle$. Since however, as may easily be seen, $K_{ij}^{(p)}(x,y)$ is, for $p \geqq 2$, continuous in $\triangle \times \triangle$, equality holds everywhere in $\triangle \times \triangle$.

The proof for $p > 2$ follows immediately by induction.

§ 6. *Comparison with the results of* J. E. WILKINS.

In this paragraph we shall compare our results with those of WILKINS. As we have already remarked in § 1, our hypotheses *a.* and *c.* on $h_{ij}(x)$ and $A_{ij}(x,y)$ are weaker than the corresponding hypotheses in WILKINS' paper. It is only in our § 5, where all $h_{ij}(x)$ are continuous and all $A_{ij}(x,y)$ continuous in mean, that our and WILKINS' hypotheses are comparable.

We shall first consider some of WILKINS' results which are proved by us under the weaker hypotheses *a.* and *c.* WILKINS' Theorem 2.2 is the first part of our Theorem 2. WILKINS' Theorem 2.3, stating that if $\chi_1,\ldots,\chi_k$ are linearly independent solutions of $K\chi = AH\chi = \lambda\chi$, then $H\chi,\ldots,H\chi_k$ are linearly independent solutions of $K^*\psi = \lambda\psi$, is contained in the first part of the proof of I, Theorem 12, where the same is proved for general symmetrisable transformations T (not necessarily of the form $T = AH$), if only $Hf = 0$ implies $Tf = 0$. WILKINS' Theorem 2.5 states that if $a_k = (\{f\}, \{\chi_k\}) = (H\{f\}, \{\psi_k\}) = 0$ for $k = 1, 2, \ldots$, then $(H\{f\}, \{g\}) = 0$ for every $\{g\} = K\{g_1\}$, where $\{g_1\} = \{g_1^1(x),\ldots,g_1^n(x)\}$ consists of continuous functions. This continuity is a superfluous condition, and the theorem in question is the latter part of our Corollary to Theorem 5.

We do not find WILKINS' Theorem 3.1 about the zero's of the FREDHOLM determinant $D(\lambda)$ in the present paper.

In his paper WILKINS considers a class of functionsets called by him the class L. This class consists of all functionsets $\{f\} = K\{g\}$, where $\{g\} = \{g^1(x),\ldots,g^n(x)\}$ consists of continuous functions. The first part of his Theorem 5.1 is our formula (6), but his hypothesis that $\{f\} \in L$ is superfluous. The second part of his Theorem 5.1, stating that if $\{f\} \in L$, then $(H\{f\}, \{f\}) = \Sigma_k |a_k|^2$, where $a_k = (\{f\}, \{\chi_k\})$, is an easy consequence of (6). Here it is necessary that $\{f\} = K\{g\}$, but not that $\{g\}$ consists of continuous functions. WILKINS' Theorem 5.2 and Theorem 5.3 together form essentially our Theorem 6. Here again it is not necessary that $\{f\} \in L$.

We shall next compare WILKINS' expansion theorems with ours in § 5. We recall that the hypotheses about $h_{ij}(x)$ and $A_{ij}(x,y)$ are now equivalent. WILKINS' Theorem 4.1 is our Theorem 13 and his Theorem 4.2 is an immediate consequence. In his Theorem 4.1 it is necessary that $\{f\} = K\{g\}$, and in his Theorem 4.2 that $\{f\} = K^2\{g\}$, but in both cases it is not necessary that $\{g\}$ consists of continuous functions.

Finally we shall say a few words about WILKINS' Theorem 7.1. In the terminology of HILBERT space, this theorem runs as follows:

Let H be a bounded, self-adjoint transformation of positive type, and let the self-adjoint transformation M be such that $M^2 = H$. Let, furthermore, the linear, completely continuous transformation A satisfy $MAM = MA^*M$. Denoting the characteristic values $\neq 0$ of $K = AH$ by λ_k and a corresponding H-orthonormal system of characteristic

elements by ψ_k $(k=1,2,\ldots)$, *we have for every element* f *of the* HILBERT *space*

$$g = AMf = \Sigma_k (Hg, \psi_k)\psi_k + p,$$

where $Mp = 0$.

This theorem may easily be proved by the methods of our paper I, using the following facts:

A. $(Hg, \psi_k) = (HAMf, \psi_k) = (MAMf, M\psi_k) = (f, MAH\psi_k) = = \lambda_k (f, M\psi_k)$,

B. The self-adjoint transformation MAM has the same sequence λ_k of characteristic values $\neq 0$ as $K = AH$ with the corresponding orthonormal sequence $M\psi_k$ of characteristic elements (Proof as in I, Theorem 18). Hence, by a well-known theorem about self-adjoint, completely continuous transformations, $(h, M\psi_k) = 0$ $(k=1,2,\ldots)$ implies $MAMh = 0$.

Starting now with

$$f = \Sigma_k (f, M\psi_k) M\psi_k + h,$$

where f is arbitrary and $(h, M\psi_k) = 0$ $(k=1,2,\ldots)$, we find by A.

$$g = AMf = \Sigma_k \lambda_k (f, M\psi_k)\psi_k + AMh = \Sigma_k (Hg, \psi_k)\psi_k + AMh,$$

so that only $MAMh = 0$ remains to be proved. This however is a consequence of $(h, M\psi_k) = 0$ $(k=1,2,\ldots)$ and B.

Physics. — *A calculation of the viscosity and the sedimentation velocity for solutions of large chain molecules taking into account the hampered flow of the solvent through each chain molecule.* By H. C. BRINKMAN. (Laboratory N.V. De Bataafsche Petroleum Maatschappij.) (Communicated by Prof. J. M. BURGERS.)

(Communicated at the meeting of May 31, 1947.)

Summary.

In a solution of long chain molecules each polymer molecule forms a molecular cluster according to KUHN. HERMANS and KRAMERS have calculated the viscosity of such solutions on the assumption that the solvent can move freely through each molecular cluster. On the other hand HERMANS has calculated the sedimentation velocity under the assumption that the flow through each cluster is hampered. In this paper expressions are derived for the viscosity and sedimentation velocity based on a single model. According to this model each molecular cluster is represented by a porous sphere which has a certain permeability for the solvent. The specific viscosity is found to be proportional to $P^{1.81}$, where P is the degree of polymerization. This is somewhat different from STAUDINGER's law which gives proportionality with P^2 and which follows from HERMANS' and KRAMERS' theories of unhampered flow. Our results for the viscosity and the sedimentation velocity show excellent agreement with the experiments for various polymers.

§ 1. *Introduction.*

KUHN [1]) has pointed out that in a solution of longchain molecules each polymer molecule forms a molecular cluster. A calculation of the viscosity of such solutions has been given by HERMANS [2]) and by KRAMERS [3]) on the assumption that the solvent can flow freely through the molecular clusters. Both authors are led to STAUDINGER's viscosity law which is verified approximately by experiments. On the other hand HERMANS [4]) has given a calculation of the sedimentation velocity on the assumption that the flow of the solvent through the molecular clusters is hampered. This calculation likewise gives results in accordance with the experiments. HERMANS [4]) pointed out that evidently the flow through the molecular clusters is hampered, but that this hindrance has a large effect on the sedimentation velocity only and a small effect on the viscosity.

[1]) W. KUHN, Kolloid Z. **68**, 2 (1934); **76**, 258 (1936).
[2]) J. J. HERMANS, Kolloid Z. **106**, 22 (1944); Physica **10**, 777 (1943).
[3]) H. A. KRAMERS, Physica **11**, 1 (1944).
[4]) J. J. HERMANS, Rec. trav. chim. **63**, 219 (1944).

In this paper a calculation is given of the hampered flow through a molecular cluster. As a result formulae for the viscosity and the sedimentation velocity are found which support the above mentioned assumption of HERMANS and which are in excellent agreement with experiments.

DEBYE [5]) recently has given a summary of a theoretical treatment of the hampered flow through a molecular cluster. He introduces a depth of penetration of the fluid flow into the cluster. For large values of this depth his results for the viscosity approximate those of HERMANS and KRAMERS. The sedimentation constant is not mentioned by DEBYE. Apparently DEBYE's way of treatment is quite different from ours as described below. His results moreover seem to differ in some ways from ours. Unfortunately DEBYE's very short description of his theory has until now made a detailed comparison impossible.

§ 2. *A model for the hampered flow through a molecular cluster.*

In a previous paper [6]) a theory was developed describing the viscous flow of a fluid through a dense swarm of particles. In this theory the swarm of particles is represented by a porous mass. An equation describing the fluid flow through a porous mass was already proposed by DARCY in 1856 [7]):

$$\mathbf{v} = -\frac{k}{\eta} \operatorname{grad} p \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where $\mathbf{v}$ is the rate of flow through a section of unit area in the porous mass;

k is the permeability of the porous mass (i.e. a phenomenological constant determining the rate of flow);

η is the fluid viscosity;

grad p is the pressure-gradient.

It appeared, however, that (1) is an approximation. It is valid for low permeabilities as occurring in subterranean flow problems. For high values of the permeability the true equation should tend to the NAVIER–STOKES equation for unhampered flow. This last equation has the following form for stationary viscous flow:

$$\operatorname{grad} p = \eta \, \Delta \mathbf{v} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where:

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

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5) P. DEBYE, Phys. Rev. **71**, 486 (1947).

6) H. C. BRINKMAN, Applied Scient. Research A in print.

7) Cf. M. MUSKAT, The flow of homogeneous fluids through porous media (New York 1937).

We suggested to modify (1) into [6]:

$$\text{grad}\, p = -(\eta/k)\,\mathbf{v} + \eta\,\Delta\,\mathbf{v} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Equation (3) gives an approximate description of an equilibrium between the pressure gradient (*grad p*), a damping force caused by the porous mass ($\eta\, v/k$) and viscous shearing stresses ($\eta\,\Delta\, v$). For low permeabilities it tends to (1) and for high permeabilities to (2).

A calculation of the viscous force exerted on a sphere embedded in a porous mass on the basis of this equation combined with certain auxiliary assumptions regarding boundary conditions, was given in the paper mentioned in [6]). It could be used to deduce an expression for the permeability:

$$k = \frac{\varrho^2}{18}\left[3 + \frac{4}{\varphi} - 3\left(\frac{8}{\varphi} - 3\right)^{\frac{1}{2}}\right] \quad . \quad . \quad . \quad . \quad . \quad (4)$$

where φ is the volume-fraction of solid matter in the porous mass ($1 - \varphi$ is the porosity) and ϱ is the radius of a sphere defined in the following way: we consider a unit of volume of the porous mass built up of n spheres each of radius ϱ, so that:

$$\varphi = \tfrac{4}{3}\,\pi\,\varrho^3\,n \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Now the area σ of the porous mass per unit of volume is:

$$\tau = 4\,\pi\varrho^2\,n \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

We then put:

$$\varrho = \varphi/3\sigma \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

and take this as the definition of ϱ.

According to KUHN [1]) a long chain molecule is built up from N statistical elements in such a way that the orientation of each element is approximately independent of all others. Each element contains ν monomeric units of lenght l_g. Therefore the degree of polymerization P is:

$$P = \nu\, N \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

The length of a statistical element A is given by

$$A = s\, l_g \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

where s is approximately equal to $\frac{2}{3}\,\nu$ [4]).

The centres of the chain elements are distributed statistically with a probability proportional to

$$\psi(r) = r^2\, e^{-\mu^2 r^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

where $\mu^2 = 3/(2\,N\,A^2)$.

In our theory we will represent this molecular cluster by a porous sphere

of radius R. For R the mean radius of the cluster following from (10) will be substituted:

$$R = \frac{2}{\mu\sqrt{\pi}} = A\sqrt{\frac{8N}{3\pi}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

Then the permeability of the porous sphere will be derived from (4) with $\varrho = A/2$.

It should be pointed out that in consequence of (10) the porosity of the cluster is in fact variable with r. A more subtle approximation might be obtained by introducing a model with an infinite radius and a varying permeability in accordance with (10), again to be combined with (4). Instead of this more subtle model we introduced the mean values for R and for the permeability k. We are confident, however, that the results obtained will be satisfactory for both models.

§ 3. *Solution of the equations of flow.*

The mathematical problem to be solved may be stated in the following way: The equations of flow must be solved for a fluid surrounding and penetrating a porous sphere of radius R. Inside of the sphere these equations are (3) and the condition of incompressibility

$$\operatorname{div} \mathbf{v} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

Outside of the sphere we have (2) and (12).

The boundary conditions at the surface of the sphere are the following [6]): The normal and tangential viscous stresses should be continuous; likewise the normal and tangential component of the velocity should be continuous.

Infinitely far removed from the porous sphere the velocity should fulfil the following conditions:

a. For the calculation of the sedimentation the velocity of flow must represent a parallel flow, e.g. in the direction of the z-axis.

b. For the calculation of the viscosity some type of flow containing velocity gradients must be introduced. We choose:

$$v_x = -\frac{v_0}{2}x; \qquad v_y = -\frac{v_0}{2}y; \qquad v_z = v_0 z.$$

Introducing polar coordinates r and ϑ symmetrically round the z-axis the solution of the equations is found to be:

a. For the sedimentation problem:

outside of the sphere:

$$\mathbf{v} = \operatorname{grad}\left[\left\{v_0 r - \frac{e}{2} - \frac{f}{r^2}\right\}\cos\vartheta\right] + \frac{e}{r}\mathbf{i} \quad . \quad . \quad . \quad (13)$$

inside of the sphere:

$$\mathbf{v} = \operatorname{grad}\left[\left\{-\frac{g}{\lambda^2}\frac{\partial}{\partial r}\frac{\sinh \lambda r}{r} - hr\right\}\cos\vartheta\right] + g\frac{\sinh \lambda r}{r}\mathbf{i} \quad . \quad (14)$$

b. For the viscosity problem:
outside of the sphere:

$$\mathbf{v} = \operatorname{grad}\left\{\left[\frac{v_0 r^2}{4} - \frac{c}{r^3}\right]P_2(\cos\vartheta) - \frac{d}{2r}(1-\cos^2\vartheta)\right\} - \frac{d}{r^2}\cos\vartheta\,\mathbf{i} \quad (15)$$

inside of the sphere:

$$\mathbf{v} = \operatorname{grad}\left\{br^2 P_2(\cos\vartheta) - \frac{a}{\lambda^2}\frac{\partial^2}{\partial z^2}\frac{\sinh \lambda r}{r}\right\} + a\frac{\partial}{\partial z}\frac{\sinh \lambda r}{r}\mathbf{i}\,. \quad (16)$$

where $\lambda^2 = 1/k$; $\mathbf{i}$ is the unit-vector in the direction of the z-axis; a, b, c, d, e, f, g and h are constants to be determined from the boundary conditions.

We will not give the expressions for these constants which would fill some pages, but will indicate in which way the final results are to be obtained from them: For the sedimentation the viscous force acting on the porous sphere should be calculated by integration of the viscous stresses over the surface of the sphere. This force should be in equilibrium with the gravitational force. For the viscosity the energy dissipation should be calculated in the same way as in EINSTEIN's well known paper [8]).

It should be pointed out that (15) and (16) are valid for a sphere which is at rest relative to the mean flow velocity of the liquid. If this is not the case the term $\eta v/k$ (i.e. the damping force inside the sphere) will modify the results. Mostly, however, the experimental conditions will approximately fulfil our assumption.

The following expressions for the viscous force K on the sphere and for the viscosity η' of the suspension are found:

$$K = 6\pi\eta v_0 R f_1(\lambda R) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

$$\eta' = \eta\left\{1 + 2{,}5\,\frac{V_0}{V} f_2(\lambda R)\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

where: η is the viscosity of the solvent; $v_0 = \frac{4}{3} R^3 G V$; G is the number of spheres per unit of volume of the fluid; V is the total volume of the fluid;

$$f_1(\lambda R) = \frac{1 - \dfrac{11}{3}\dfrac{T}{\lambda R} + \dfrac{8}{\lambda^2 R^2} - \dfrac{8}{\lambda^3 R^3}\dfrac{T}{R^3}}{1 + \dfrac{3}{2\lambda^2 R^2} - \dfrac{27}{2\lambda^3 R^3}\dfrac{T}{R^3} + \dfrac{36}{\lambda^4 R^4} - \dfrac{36}{\lambda^5 R^5}\dfrac{T}{R^5}} \quad . \quad . \quad . \quad (17a)$$

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[8]) A. EINSTEIN, Ann. Phys. **19**, 289 (1906); **34**, 591 (1911).

$$f_2(\lambda R) = \frac{1 - \frac{3C}{\lambda R} + \frac{15}{\lambda^2 R^2} - \frac{36C}{\lambda^3 R^3} + \frac{36}{\lambda^4 R^4}}{1 + \frac{10}{\lambda^2 R^2} + \frac{90C}{\lambda^3 R^3} - \frac{30}{\lambda^4 R^4} + \frac{360C}{\lambda^5 R^5} - \frac{360}{\lambda^6 R^6}} \quad . \quad (18a)$$

$(T = tgh\, \lambda R;\ C = coth\, \lambda R)$.

§ 4. *Introduction of* KUHN's *statistics. Comparison with experiment.* In order to be able to compare (17) and (18) with experiments the various quantities occurring in these formulae should be expressed in experimentally known quantities with the help of KUHN's statistics (cf. § 2). With the help of (4), (5), (7), (8), (9) and (11) we find:

$$\lambda R = \frac{R}{\sqrt{k}} = \sqrt{\frac{8N}{3\pi}} \cdot \frac{9 + 3\sqrt{\frac{8}{\varphi} - 3}}{\frac{2}{\varphi} - 3} \quad . \quad . \quad . \quad . \quad . \quad (19)$$

$$\varphi = \frac{1}{8\sqrt{N}} \left\{ \frac{3\pi}{8} \right\}^{3/2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

$$V_0 = \frac{4\pi}{3} A^3 \left\{ \frac{8N}{3\pi} \right\}^{3/2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

Applying these expressions we find from (18):

$$\eta_{sp} = \frac{\eta^1 - \eta}{\eta} = \frac{80}{9} \sqrt{\frac{8}{3\pi}} A^3 G N^{3/2} f_2(N). \quad . \quad . \quad . \quad . \quad (22)$$

The sedimentation constant (i.e. the sedimentation velocity in a gravitational field giving unit-acceleration) is found by equating (17) to the gravitational force $M - V_0 \varrho$, where M is the molecular mass of the polymer and ϱ the density of the solvent. We find:

$$\sigma = \frac{M - V_0 \varrho}{6.10^{23} N} \cdot \frac{1}{4\sqrt{6\pi\eta A}} \frac{\sqrt{N}}{f_1(N)}. \quad . \quad . \quad . \quad . \quad . \quad (23)$$

In fig. 1 we have plotted $\sqrt{N} f_1(N)$ and $\sqrt{N} f_2(N)$ as functions of N with help of (17a), (18a), (19) and (20).

Discussion of the viscosity relation.

It should be observed that in the region $N > 30$ the function $\sqrt{N} f_2(N)$ may be approximated by a linear relation $a - bN$. Such a linear relation is also proposed by BAKER, FULLER and HEISS on experimental evidence [9]). A better comparison with experiment is obtained by comparing (22) to the modified STAUDINGER law:

$$\eta_{sp} = K c P^{\alpha} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

where K is a constant; c the concentration in gram/liter; P the degree of

[9]) W. O. BAKER, C. S. FULLER, J. H. HEISS, J. Amer. Chem. Soc. **63**, 2142 (1941).

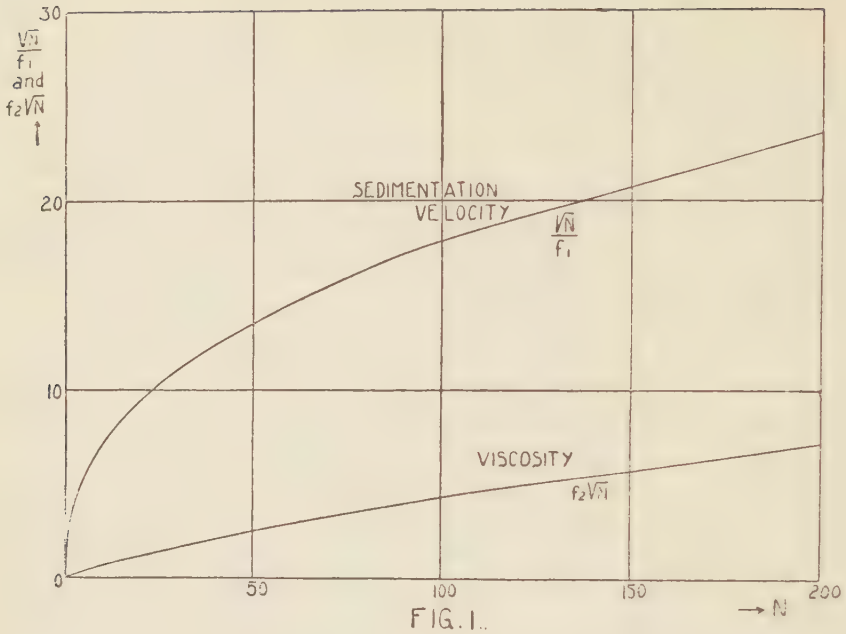

30
$\frac{\sqrt{N}}{f_1}$ and $f_2\sqrt{N}$
20
10
0
SEDIMENTATION VELOCITY
$\frac{\sqrt{N}}{f_1}$
VISCOSITY
$f_2\sqrt{N}$
50
100
150
200
N

FIG. 1.

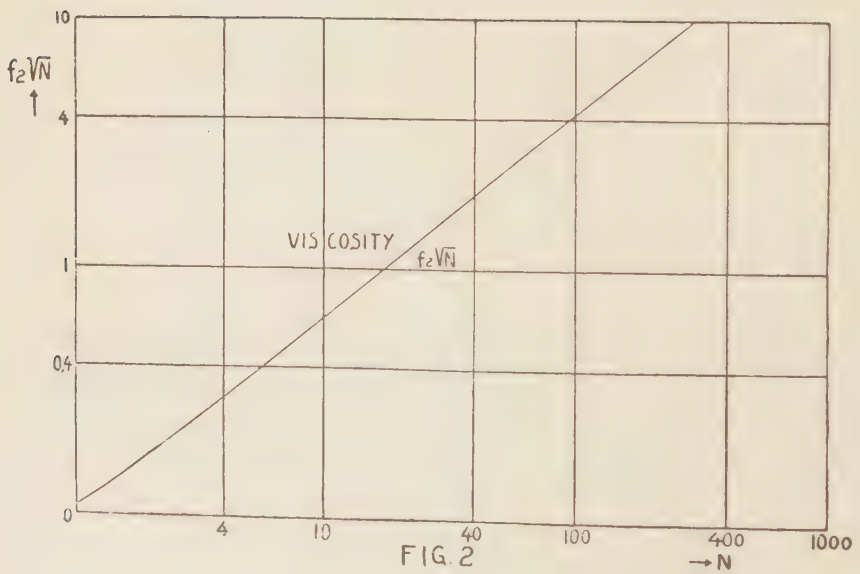

10
$f_2\sqrt{N}$
4
1
0,4
0
VISCOSITY
$f_2\sqrt{N}$
4
10
40
100
400
1000
N

FIG. 2

polymerization, and α is an exponent which is equal to 1 for the original STAUDINGER law. It should be remembered that HERMANS [2]) and KRAMERS [3]) find $\alpha = 1$ from their theories. As the factor c in (24) is proportional to P this means that η_{sp} is proportional to N^2.

In fig. 2 we have plotted the logarithm of $f_2\sqrt{N}$ against log N. The resulting curve is very nearly a straight line for $N > 3$. This means that relation (22) may be written in the form (24). The exponent is found to be:

$$\alpha = 0{,}81.$$

This agrees very well with many experimental results. We give some experimental values [10]):

polyvinylacetate	$\alpha = 0{,}79$	
polyvinylchloride	$\alpha = 0{,}76$	
polymethylmethacrylate	$\alpha = 0{,}82$	(solvent benzene)
,,	$\alpha = 0{,}76$	(solvent chloroform).

Discussion of the sedimentation-constant.

For the sedimentation constant we have compared our results to those discussed by HERMANS [4]). The values for the various constants are found in his paper.

Nitrocellulose ($A = 62$, $s = 12$, $l_g = 5{,}2$)

N	1.3	6,3	17	42	129
$10^{13}\,\sigma_{theor.}$	4,5	8,6	12,7	18,3	28,4
$10^{13}\,\sigma_{exp.}$	5,2	8,7	12,0	18	30

Cellulose-acetate ($A = 73$, $s = 15$, $l_g = 5{,}2$)

N	14	20	22	25	43	56
$10^{15}\,(\sigma)_{theor.}$	2,7	3,1	3,3	3,5	4,3	4,8
$10^{15}\,(\sigma)_{exp.}$	2,9	3,2	3,4	3,5	4,3	4,8

For nitrocellulose HERMANS' value of s was taken. For cellulose-acetate, where the various experimental quantities necessary for a comparison are known less accurately, $s = 15$ was taken instead of HERMANS' value $s = 10$. The variation of σ with N is represented excellently by our formula, while the magnitude of σ also agrees with the experimental results.

It may be concluded therefore that our substitution of a porous sphere for a molecular cluster leads to expressions for the viscosity and for the sedimentation constant which agree well with experiment.

The author is indebted to Prof. J. J. HERMANS for a discussion of the experimental material.

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10) M. FOURNIER and X. THIESSE C. R. **222**, 1437 (1946), J. H. BAVENDALE, S. BYWATER and M. G. SWAIN, J. Polym. Science **1**, 237 (1946).

Crystallography. — *Calculation of the stereographic pole figure of the cubic lattice for any given direction* [*HKL*]. II. By W. MAY. (Communicated by Prof. J. M. BURGERS.)

(Communicated at the meeting of March 29, 1947.)

5. *Explanation of the tables.*

As was remarked in section 1, in the case of the cubic face-centered lattice it is sufficient to construct the standard projections for the [1 1 0]-, [0 0 1]-, [1 1 2]-, [1 3 0]- and [1 1 1]-direction. These projections were calculated for a radius R of 10 cm with an accuracy in the coordinates of every point of 0.1 mm, which is better than can be achieved in plotting these distances.

The choice of the planes for which the coordinates are calculated is directly dependant on the fact that the projections are intended for the cubic face-centered lattice. In the following tables only those planes with $\Sigma h^2 \leq 56$ [4]) and indices all even or odd are present, but every set of indices has been reduced to the simplest form as in LAUE photographs different orders are superimposed. In the projections of SCHIEBOLD and SACHS the size of the spots gives an indication of the intensity of the X-ray reflections from the planes. It seems to us that the smallest possible spots are preferable; the important poles ({1 0 0}, {1 1 0} and {1 1 1}) can be indicated in the usual way. This procedure is also used by BARRETT in his book [2]).

In the tables [5]) are recorded the coordinates x and y in mm for every pole. In the heading of each table is given:

a. the indices of the projection direction;

b. the equations for x and y as calculated from the general equations (5) of section 4 for the special case;

c. a schematic representation of the axes (the positive part of each axis is indicated by an arrow), the indices of the pole at the centre and of the poles at the points of intersection of the axes and the reference circle.

In the tables the sets of indices of poles lying on the reference circle are preceded by !

Finally it must be remarked that the standard projection [0 0 1] cannot be calculated from the general equations (5), as the equations (4) for

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[4]) Planes with $\Sigma h^2 > 56$ seldom, if ever, appear on an ordinary LAUE photograph, at least in our experience with aluminium single crystals.

[5]) Typographical reasons made it impossible to denote negative indiecs in the usual way, .e. by a bar over the index. In the present tables the negative indices differ from the positive ones by a fatter print.

X and Y are reduced to $\frac{0}{0}$. For this special case it is assumed that: $X = k$, $Y = -h$.

The author takes here the opportunity to express his sincerest thanks to Professor dr. W. G. BURGERS for the interest and the invaluable assistance, received in many discussions.

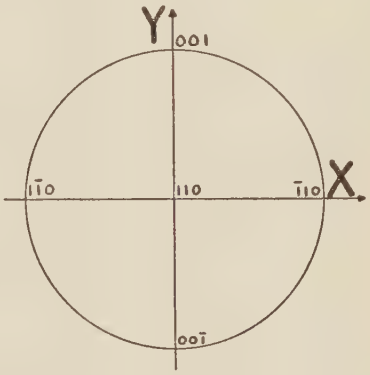
Laboratory for physical chemistry,
Technical University, Delft.

[1 1 0].

$$x = \frac{100\,(k-h)}{\sqrt{2\,\Sigma h^2 + h + k}}\ mm$$

$$y = \frac{100\,l\,\sqrt{2}}{\sqrt{2\,\Sigma h^2 + h + k}}\ mm$$

! means that the pole with this set of indices lies on the reference circle.



(h k l)	x	y	(h k l)	x	y
! 0 0 1	0.0	+ 100.0	0 1 2	+ 24.0	+ 68.0
! 0 0 1	0.0	— 100.0	0 1 2	+ 24.0	— 68.0
0 1 0	+ 41.4	0.0	1 0 2	— 24.0	+ 68.0
1 0 0	— 41.4	0.0	1 0 2	— 24.0	— 68.0
			0 2 1	+ 38.7	+ 27.4
1 1 0	0.0	0.0	0 2 1	+ 38.7	— 27.4
0 1 1	+ 33.3	+ 47.1	2 0 1	— 38.7	+ 27.4
0 1 1	+ 33.3	— 47.1	2 0 1	— 38.7	— 27.4
1 0 1	— 33.3	+ 47.1	1 2 0	+ 72.1	0.0
1 0 1	— 33.3	— 47.1	2 1 0	— 72.1	0.0
! 1 1 0	+ 100.0	0.0			
! 1 1 0	— 100.0	0.0	1 1 2	0.0	+ 51.8
			1 1 2	0.0	— 51.8
1 1 1	0.0	+ 31.8	1 2 1	+ 15.5	+ 21.9
1 1 1	0.0	— 31.8	1 2 1	+ 15.5	— 21.9
! 1 1 1	+ 81.6	+ 57.7	2 1 1	— 15.5	+ 21.9
! 1 1 1	+ 81.6	— 57.7	2 1 1	— 15.5	— 21.9
! 1 1 1	— 81.6	+ 57.7	! 1 1 2	+ 57.7	+ 81.6
! 1 1 1	— 81.6	— 57.7	! 1 1 2	+ 57.7	— 81.6
			! 1 1 2	— 57.7	+ 81.6
1 2 0	+ 16.2	0.0	! 1 1 2	— 57.7	— 81.6
2 1 0	— 16.2	0.0	1 2 1	+ 67.2	+ 31.7

(h k l)	x	y
1 2 1	+ 67.2	— 31.7
2 1 1	— 67.2	+ 31.7
2 1 1	— 67.2	— 31.7
2 2 1	0.0	+ 17.2
2 2 1	0.0	— 17.2
1 2 2	+ 13.8	+ 39.1
1 2 2	+ 13.8	— 39.1
2 1 2	— 13.8	+ 39.1
2 1 2	— 13.8	— 39.1
1 2 2	+ 57.2	+ 54.0
1 2 2	+ 57.2	— 54.0
2 1 2	— 57.2	+ 54.0
2 1 2	— 57.2	— 54.0
! 2 2 1	+ 94.3	+ 33.3
! 2 2 1	+ 94.3	— 33.3
! 2 2 1	— 94.3	+ 33.3
! 2 2 1	— 94.3	— 33.3
0 1 3	+ 18.3	+ 77.5
0 1 3	+ 18.3	— 77.5
1 0 3	— 18.3	+ 77.5
1 0 3	— 18.3	— 77.5
1 3 0	+ 23.6	0.0
3 1 0	— 23.6	0.0
0 3 1	+ 40.1	+ 18.9
0 3 1	+ 40.1	— 18.9
3 0 1	— 40.1	+ 18.9
3 0 1	— 40.1	— 18.9
1 3 0	+ 61.8	0.0
3 1 0	— 61.8	0.0
1 1 3	0.0	+ 63.4
1 1 3	0.0	— 63.4
1 3 1	+ 23.0	+ 16.3
1 3 1	+ 23.0	— 16.3
3 1 1	— 23.0	+ 16.3
3 1 1	— 23.0	— 16.3
! 1 1 3	+ 42.6	+ 90.5
! 1 1 3	+ 42.6	— 90.5
! 1 1 3	— 42.6	+ 90.5
! 1 1 3	— 42.6	— 90.5
1 3 1	+ 59.8	+ 21.1
1 3 1	+ 59.8	— 21.1
3 1 1	— 59.8	+ 21.1
3 1 1	— 59.8	— 21.1
2 3 0	+ 9.9	0.0
3 2 0	— 9.9	0.0
0 2 3	+ 28.2	+ 59.8
0 2 3	+ 28.2	— 59.8
2 0 3	— 28.2	+ 59.8
2 0 3	— 28.2	— 59.8
0 3 2	+ 37.0	+ 34.9
0 3 2	+ 37.0	— 34.9
3 0 2	— 37.0	+ 34.9
3 0 2	— 37.0	— 34.9
2 3 0	+ 82.0	0.0
3 2 0	— 82.0	0.0
2 3 1	+ 9.7	+ 13.7
2 3 1	+ 9.7	— 13.7
3 2 1	— 9.7	+ 13.7
3 2 1	— 9.7	— 13.7
1 2 3	+ 12.1	+ 51.2
1 2 3	+ 12.1	— 51.2
2 1 3	— 12.1	+ 51.2
2 1 3	— 12.1	— 51.2
1 3 2	+ 21.5	+ 30.4
1 3 2	+ 21.5	— 30.4
3 1 2	— 21.5	+ 30.4
3 1 2	— 21.5	— 30.4
1 2 3	+ 47.7	+ 67.4
1 2 3	+ 47.7	— 67.4
2 1 3	— 47.7	+ 67.4
2 1 3	— 47.7	— 67.4
1 3 2	+ 54.9	+ 38.8
1 3 2	+ 54.9	— 38.8
3 1 2	— 54.9	+ 38.8
3 1 2	— 54.9	— 38.8
2 3 1	+ 79.5	+ 22.5
2 3 1	+ 79.5	— 22.5
3 2 1	— 79.5	+ 22.5
3 2 1	— 79.5	— 22.5
3 3 1	0.0	+ 11.6
3 3 1	0.0	— 11.6
1 3 3	+ 19.7	+ 41.7
1 3 3	+ 19.7	— 41.7
3 1 3	— 19.7	+ 41.7
3 1 3	— 19.7	— 41.7
1 3 3	+ 49.0	+ 52.0
1 3 3	+ 49.0	— 52.0
3 1 3	— 49.0	+ 52.0
3 1 3	— 49.0	— 52.0
! 3 3 1	+ 97.3	+ 22.9
! 3 3 1	+ 97.3	— 22.9
! 3 3 1	— 97.3	+ 22.9
! 3 3 1	— 97.3	— 22.9
1 1 5	0.0	+ 75.6
1 1 5	0.0	— 75.6

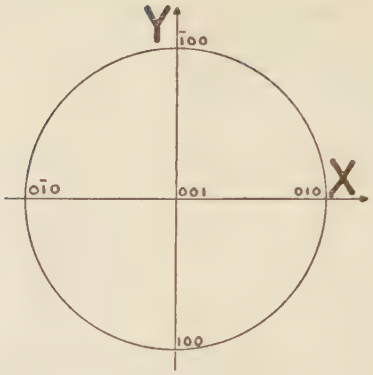
(h k l)	x	y	(h k l)	x	y
! 1 1 5	+ 27.2	+ 96.2	3 5 3	+ 11.6	— 24.6
! 1 1 5	+ 27.2	— 96.2	5 3 3	— 11.6	+ 24.6
! 1 1 5	— 27.2	+ 96.2	5 3 3	— 11.6	— 24.6
! 1 1 5	— 27.2	— 96.2	! 3 3 5	+ 64.7	+ 76.2
1 5 1	+ 30.0	+ 10.6	! 3 3 5	+ 64.7	— 76.2
1 5 1	+ 30.0	— 10.6	! 3 3 5	— 64.7	+ 76.2
5 1 1	— 30.0	+ 10.6	! 3 3 5	— 64.7	— 76.2
5 1 1	— 30.0	— 10.6	3 5 3	+ 71.0	+ 37.6
1 5 1	+ 52.9	+ 12.5	3 5 3	+ 71.0	— 37.6
1 5 1	+ 52.9	— 12.5	5 3 3	— 71.0	+ 37.6
5 1 1	— 52.9	+ 12.5	5 3 3	— 71.0	— 37.6
5 1 1	— 52.9	— 12.5			
			1 1 7	0.0	+ 81.8
3 5 1	+ 12.2	+ 8.6	1 1 7	0.0	— 81.8
3 5 1	+ 12.2	— 8.6	! 1 1 7	+ 19.8	+ 98.0
5 3 1	— 12.2	+ 8.6	! 1 1 7	+ 19.8	— 98.0
5 3 1	— 12.2	— 8.6	! 1 1 7	— 19.8	+ 98.0
1 3 5	+ 16.2	+ 57.2	! 1 1 7	— 19.8	— 98.0
1 3 5	+ 16.2	— 57.2	1 7 1	+ 33.1	+ 7.8
3 1 5	— 16.2	+ 57.2	1 7 1	+ 33.1	— 7.8
3 1 5	— 16.2	— 57.2	7 1 1	— 33.1	+ 7.8
1 5 3	+ 27.8	+ 29.5	7 1 1	— 33.1	— 7.8
1 5 3	+ 27.8	— 29.5	1 7 1	+ 49.7	+ 8.8
5 1 3	— 27.8	+ 29.5	1 7 1	+ 49.7	— 8.8
5 1 3	— 27.8	— 29.5	7 1 1	— 49.7	+ 8.8
1 3 5	+ 38.6	+ 68.2	7 1 1	— 49.7	— 8.8
1 3 5	+ 38.6	— 68.2			
3 1 5	— 38.6	+ 68.2	5 5 1	0.0	+ 7.0
3 1 5	— 38.6	— 68.2	5 5 1	0.0	— 7.0
1 5 3	+ 48.5	+ 34.3	1 5 5	+ 24.8	+ 43.9
1 5 3	+ 48.5	— 34.3	1 5 5	+ 24.8	— 43.9
5 1 3	— 48.5	+ 34.3	5 1 5	— 24.8	+ 43.9
5 1 3	— 48.5	— 34.3	5 1 5	— 24.8	— 43.9
3 5 1	+ 77.2	+ 13.6	1 5 5	+ 42.6	+ 50.1
3 5 1	+ 77.2	— 13.6	1 5 5	+ 42.6	— 50.1
5 3 1	— 77.2	+ 13.6	5 1 5	— 42.6	+ 50.1
5 3 1	— 77.2	— 13.6	5 1 5	— 42.6	— 50.1
			! 5 5 1	+ 99.0	+ 14.0
3 3 5	0.0	+ 46.3	! 5 5 1	+ 99.0	— 14.0
3 3 5	0.0	— 46.3	! 5 5 1	— 99.0	+ 14.0
3 5 3	+ 11.6	+ 24.6	! 5 5 1	— 99.0	— 14.0

[0 0 1].

$$x = \frac{100\,k}{\sqrt{\Sigma h^2 + l}}\ mm$$

$$y = \frac{-100\,h}{\sqrt{\Sigma h^2 + l}}\ mm$$

! means that the pole with this set of indices lies on the reference circle.



(h k l)	x	y	(h k l)	x	y
0 0 1	0.0	0.0	1 1 2	+ 22.5	+ 22.5
! 1 0 0	0.0	+ 100.0	1 1 2	+ 22.5	— 22.5
! 1 0 0	0.0	— 100.0	1 1 2	— 22.5	+ 22.5
! 0 1 0	+ 100.0	0.0	1 1 2	— 22.5	— 22.5
! 0 1 0	— 100.0	0.0	2 1 1	+ 29.0	+ 58.0
			2 1 1	+ 29.0	— 58.0
1 0 1	0.0	+ 41.4	2 1 1	— 29.0	+ 58.0
1 0 1	0.0	— 41.4	2 1 1	— 29.0	— 58.0
0 1 1	+ 41.4	0.0	1 2 1	+ 58.0	+ 29.0
0 1 1	— 41.4	0.0	1 2 1	+ 58.0	— 29.0
! 1 1 0	+ 70.7	+ 70.7	1 2 1	— 58.0	+ 29.0
! 1 1 0	+ 70.7	— 70.7	1 2 1	— 58.0	— 29.0
! 1 1 0	— 70.7	+ 70.7			
! 1 1 0	— 70.7	— 70.7	2 1 2	+ 20.0	+ 40.0
			2 1 2	+ 20.0	— 40.0
1 1 1	+ 36.6	+ 36.6	2 1 2	— 20.0	+ 40.0
1 1 1	+ 36.6	— 36.6	2 1 2	— 20.0	— 40.0
1 1 1	— 36.6	+ 36.6	1 2 2	+ 40.0	+ 20.0
1 1 1	— 36.6	— 36.6	1 2 2	+ 40.0	— 20.0
			1 2 2	— 40.0	+ 20.0
1 0 2	0.0	+ 23.6	1 2 2	— 40.0	— 20.0
1 0 2	0.0	— 23.6	2 2 1	+ 50.0	+ 50.0
2 0 1	0.0	+ 61.8	2 2 1	+ 50.0	— 50.0
2 0 1	0.0	— 61.8	2 2 1	— 50.0	+ 50.0
0 1 2	+ 23.6	0.0	2 2 1	— 50.0	— 50.0
0 1 2	— 23.6	0.0			
! 2 1 0	+ 44.7	+ 89.4	1 0 3	0.0	+ 16.2
! 2 1 0	+ 44.7	— 89.4	1 0 3	0.0	— 16.2
! 2 1 0	— 44.7	+ 89.4	3 0 1	0.0	+ 72.1
! 2 1 0	— 44.7	— 89.4	3 0 1	0.0	— 72.1
0 2 1	+ 61.8	0.0	0 1 3	+ 16.2	0.0
0 2 1	— 61.8	0.0	0 1 3	— 16.2	0.0
! 1 2 0	+ 89.4	+ 44.7	! 3 1 0	+ 31.6	+ 94.9
! 1 2 0	+ 89.4	— 44.7	! 3 1 0	+ 31.6	— 94.9
! 1 2 0	— 89.4	+ 44.7	! 3 1 0	— 31.6	+ 94.9
! 1 2 0	— 89.4	— 44.7	! 3 1 0	— 31.6	— 94.9

(h k l)	x	y	(h k l)	x	y
0 3 1	+ 72.1	0.0	3 2 1	— 42.2	+ 63.3
0 3 1	— 72.1	0.0	3 2 1	— 42.2	— 63.3
! 1 3 0	+ 94.9	+ 31.6	1 3 2	+ 52.2	+ 17.4
! 1 3 0	+ 94.9	— 31.6	1 3 2	+ 52.2	— 17.4
! 1 3 0	— 94.9	+ 31.6	1 3 2	— 52.2	+ 17.4
! 1 3 0	— 94.9	— 31.6	1 3 2	— 52.2	— 17.4
			2 3 1	+ 63.3	+ 42.2
1 1 3	+ 15.8	+ 15.8	2 3 1	+ 63.3	— 42.2
1 1 3	+ 15.8	— 15.8	2 3 1	— 63.3	+ 42.2
1 1 3	— 15.8	+ 15.8	2 3 1	— 63.3	— 42.2
1 1 3	— 15.8	— 15.8			
3 1 1	+ 23.2	+ 69.5	3 1 3	+ 13.6	+ 40.8
3 1 1	+ 23.2	— 69.5	3 1 3	+ 13.6	— 40.8
3 1 1	— 23.2	+ 69.5	3 1 3	— 13.6	+ 40.8
3 1 1	— 23.2	— 69.5	3 1 3	— 13.6	— 40.8
1 3 1	+ 69.5	+ 23.2	1 3 3	+ 40.8	+ 13.6
1 3 1	+ 69.5	— 23.2	1 3 3	+ 40.8	— 13.6
1 3 1	— 69.5	+ 23.2	1 3 3	— 40.8	+ 13.6
1 3 1	— 69.5	— 23.2	1 3 3	— 40.8	— 13.6
			3 3 1	+ 56.0	+ 56.0
2 0 3	0.0	+ 30.3	3 3 1	+ 56.0	— 56.0
2 0 3	0.0	— 30.3	3 3 1	— 56.0	+ 56.0
3 0 2	0.0	+ 53.5	3 3 1	— 56.0	— 56.0
3 0 2	0.0	— 53.5			
0 2 3	+ 30.3	0.0	1 1 5	+ 9.8	+ 9.8
0 2 3	— 30.3	0.0	1 1 5	+ 9.8	— 9.8
0 3 2	+ 53.5	0.0	1 1 5	— 9.8	+ 9.8
0 3 2	— 53.5	0.0	1 1 5	— 9.8	— 9.8
! 3 2 0	+ 55.5	+ 83.2	5 1 1	+ 16.1	+ 80.7
! 3 2 0	+ 55.5	— 83.2	5 1 1	+ 16.1	— 80.7
! 3 2 0	— 55.5	+ 83.2	5 1 1	— 16.1	+ 80.7
! 3 2 0	— 55.5	— 83.2	5 1 1	— 16.1	— 80.7
! 2 3 0	+ 83.2	+ 55.5	1 5 1	+ 80.7	+ 16.1
! 2 3 0	+ 83.2	— 55.5	1 5 1	+ 80.7	— 16.1
! 2 3 0	— 83.2	+ 55.5	1 5 1	— 80.7	+ 16.1
! 2 3 0	— 83.2	— 55.5	1 5 1	— 80.7	— 16.1
2 1 3	+ 14.8	+ 29.7	3 1 5	+ 9.2	+ 27.5
2 1 3	+ 14.8	— 29.7	3 1 5	+ 9.2	— 27.5
2 1 3	— 14.8	+ 29.7	3 1 5	— 9.2	+ 27.5
2 1 3	— 14.8	— 29.7	3 1 5	— 9.2	— 27.5
3 1 2	+ 17.4	+ 52.2	5 1 3	+ 11.2	+ 56.1
3 1 2	+ 17.4	— 52.2	5 1 3	+ 11.2	— 56.1
3 1 2	— 17.4	+ 52.2	5 1 3	— 11.2	+ 56.1
3 1 2	— 17.4	— 52.2	5 1 3	— 11.2	— 56.1
1 2 3	+ 29.7	+ 14.8	1 3 5	+ 27.5	+ 9.2
1 2 3	+ 29.7	— 14.8	1 3 5	+ 27.5	— 9.2
1 2 3	— 29.7	+ 14.8	1 3 5	— 27.5	+ 9.2
1 2 3	— 29.7	— 14.8	1 3 5	— 27.5	— 9.2
3 2 1	+ 42.2	+ 63.3	5 3 1	+ 43.4	+ 72.3
3 2 1	+ 42.2	— 63.3	5 3 1	+ 43.4	— 72.3

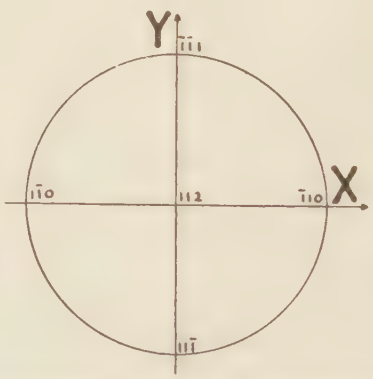
(h k l)	x	y	(h k l)	x	y
5 3 1	— 43.4	+ 72.3	1 1 7	+ 7.1	— 7.1
5 3 1	— 43.4	— 72.3	1 1 7	— 7.1	+ 7.1
1 5 3	+ 56.1	+ 11.2	1 1 7	— 7.1	— 7.1
1 5 3	+ 56.1	— 11.2	7 1 1	+ 12.3	+ 86.0
1 5 3	— 56.1	+ 11.2	7 1 1	+ 12.3	— 86.0
1 5 3	— 56.1	— 11.2	7 1 1	— 12.3	+ 86.0
3 5 1	+ 72.3	+ 43.4	7 1 1	— 12.3	— 86.0
3 5 1	+ 72.3	— 43.4	1 7 1	+ 86.0	+ 12.3
3 5 1	— 72.3	+ 43.4	1 7 1	+ 86.0	— 12.3
3 5 1	— 72.3	— 43.4	1 7 1	— 86.0	+ 12.3
			1 7 1	— 86.0	— 12.3
3 3 5	+ 26.0	+ 26.0			
3 3 5	+ 26.0	— 26.0	5 1 5	+ 8.2	+ 41.2
3 3 5	— 26.0	+ 26.0	5 1 5	+ 8.2	— 41.2
3 3 5	— 26.0	— 26.0	5 1 5	— 8.2	+ 41.2
5 3 3	+ 31.4	+ 52.3	5 1 5	— 8.2	— 41.2
5 3 3	+ 31.4	— 52.3	1 5 5	+ 41.2	+ 8.2
5 3 3	— 31.4	+ 52.3	1 5 5	+ 41.2	— 8.2
5 3 3	— 31.4	— 52.3	1 5 5	— 41.2	+ 8.2
3 5 3	+ 52.3	+ 31.4	1 5 5	— 41.2	— 8.2
3 5 3	+ 52.3	— 31.4	5 5 1	+ 61.4	+ 61.4
3 5 3	— 52.3	+ 31.4	5 5 1	+ 61.4	— 61.4
3 5 3	— 52.3	— 31.4	5 5 1	— 61.4	+ 61.4
			5 5 1	— 61.4	— 61.4
1 1 7	+ 7.1	+ 7.1			

[1 1 2].

$$x = \frac{100\,(k-h)\cdot\sqrt{3}}{\sqrt{6\Sigma h^2 + h + k + 2l}}\ mm$$

$$y = \frac{100\,(l-h-k)\cdot\sqrt{2}}{\sqrt{6\Sigma h^2 + h + k + 2l}}\ mm$$

! means that the pole with this set of indices lies on the reference circle.



(h k l)	x	y	(h k l)	x	y
0 0 1	0.0	+ 31.8	1 0 1	+ 38.8	+ 63.4
0 1 0	+ 50.2	— 41.0	0 1 1	— 38.8	+ 63.4
1 0 0	— 50.2	— 41.0	! 1 1 0	+ 100.0	0.0
			! 1 1 0	— 100.0	0.0
1 1 0	0.0	— 51.8			
0 1 1	+ 26.8	0.0	1 1 1	0.0	— 17.2
1 0 1	— 26.8	0.0	! 1 1 1	0.0	+ 100.0

(h k l)	x	y	(h k l)	x	y
! 1 1 1	0.0	— 100.0	0 3 1	+ 40.8	— 22.2
1 1 1	+ 55.5	+ 22.7	3 0 1	— 40.8	— 22.2
1 1 1	— 55.5	+ 22.7	0 3 1	+ 59.4	— 64.7
			3 0 1	— 59.4	— 64.7
0 1 2	+ 16.5	+ 13.5	1 3 0	+ 71.1	— 29.0
1 0 2	— 16.5	+ 13.5	3 1 0	— 71.1	— 29.0
1 0 2	+ 20.4	+ 50.0			
0 1 2	— 20.4	+ 50.0	1 1 3	0.0	+ 8.8
1 2 0	+ 20.4	— 50.0	1 1 3	0.0	+ 58.3
2 1 0	— 20.4	— 50.0	1 1 3	+ 24.5	+ 30.0
0 2 1	+ 36.6	— 14.9	1 1 3	— 24.5	+ 30.0
2 0 1	— 36.6	— 14.9	1 3 1	+ 24.5	— 30.0
! 2 0 1	+ 63.2	+ 77.5	3 1 1	— 24.5	— 30.0
! 0 2 1	— 63.2	+ 77.5	1 3 1	+ 34.2	— 69.8
! 0 2 1	+ 63.2	— 77.5	3 1 1	— 34.2	— 69.8
! 2 0 1	— 63.2	— 77.5	1 3 1	+ 57.1	— 11.7
1 2 0	+ 80.2	— 21.8	3 1 1	— 57.1	— 11.7
2 1 0	— 80.2	— 21.8	! 3 1 1	+ 85.3	+ 52.2
			! 1 3 1	— 85.3	+ 52.2
1 1 2	0.0	0.0	! 1 3 1	+ 85.3	— 52.2
1 1 2	0.0	+ 70.7	! 3 1 1	— 85.3	— 52.2
1 2 1	+ 15.7	— 25.7			
2 1 1	— 15.7	— 25.7	2 3 0	+ 12.5	— 51.1
1 2 1	+ 24.7	— 80.8	3 2 0	— 12.5	— 51.1
2 1 1	— 24.7	— 80.8	0 2 3	+ 20.6	+ 8.4
1 1 2	+ 34.6	+ 28.3	2 0 3	— 20.6	+ 8.4
1 1 2	— 34.6	+ 28.3	2 0 3	+ 27.0	+ 55.1
1 2 1	+ 57.7	0.0	0 2 3	— 27.0	+ 55.1
2 1 1	— 57.7	0.0	0 3 2	+ 32.8	— 8.9
2 1 1	+ 74.2	+ 40.4	3 0 2	— 32.8	— 8.9
1 2 1	— 74.2	+ 40.4	3 0 2	+ 52.9	+ 71.9
			0 3 2	- - 52.9	+ 71.9
2 2 1	0.0	— 31.8	2 3 0	+ 88.1	— 14.4
2 2 1	0.0	— 75.6	3 2 0	— 88.1	— 14.4
1 2 2	+ 12.1	— 9.9			
2 1 2	— 12.1	— 9.9	1 2 3	+ 9.5	0.0
2 1 2	+ 20.7	+ 84.7	2 1 3	— 9.5	0.0
1 2 2	— 20.7	+ 84.7	2 3 1	+ 10.7	— 35.0
1 2 2	+ 42.1	+ 11.5	3 2 1	— 10.7	— 35.0
2 1 2	— 42.1	+ 11.5	2 1 3	+ 14.2	+ 69.8
2 1 2	+ 50.2	+ 41.0	1 2 3	— 14.2	+ 69.8
1 2 2	— 50.2	+ 41.0	2 3 1	+ 14.2	— 69.8
2 2 1	+ 74.1	+ 15.1	3 2 1	— 14.2	— 69.8
2 2 1	— 74.1	+ 15.1	1 3 2	+ 20.2	— 16.5
			3 1 2	— 20.2	— 16.5
0 1 3	+ 11.7	+ 19.2	1 2 3	+ 32.1	+ 17.5
1 0 3	— 11.7	+ 19.2	2 1 3	— 32.1	+ 17.5
1 0 3	+ 13.6	+ 44.4	2 1 3	+ 36.7	+ 39.9
0 1 3	— 13.6	+ 44.4	1 2 3	— 36.7	+ 39.9
1 3 0	+ 29.5	— 48.2	! 3 1 2	+ 37.8	+ 92.6
3 1 0	— 29.5	— 48.2	! 1 3 2	— 37.8	+ 92.6

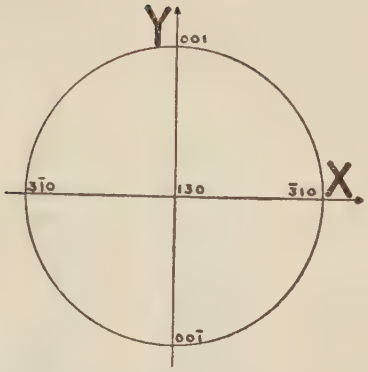
(h k l)	x	y
! 1 3 2	+ 37.8	— 92.6
! 3 1 2	— 37.8	— 92.6
1 3 2	+ 45.7	0.0
3 1 2	— 45.7	0.0
3 1 2	+ 62.1	+ 50.7
1 3 2	— 62.1	+ 50.7
2 3 1	+ 71.2	0.0
3 2 1	— 71.2	0.0
3 2 1	+ 85.2	+ 27.8
2 3 1	— 85.2	+ 27.8
3 3 1	0.0	— 37.9
3 3 1	0.0	— 67.4
1 3 3	+ 16.8	— 6.8
3 1 3	— 16.8	— 6.8
3 1 3	+ 27.3	+ 78.1
1 3 3	— 27.3	+ 78.1
1 3 3	+ 37.1	+ 7.6
3 1 3	— 37.1	+ 7.6
3 1 3	+ 47.2	+ 48.2
1 3 3	— 47.2	+ 48.2
3 3 1	+ 82.0	+ 11.2
3 3 1	— 82.0	+ 11.2
1 1 5	0.0	+ 17.2
1 1 5	0.0	+ 47.8
1 1 5	+ 15.2	+ 31.1
1 1 5	— 15.2	+ 31.1
1 5 1	+ 33.4	— 34.1
5 1 1	— 33.4	— 34.1
1 5 1	+ 41.4	— 59.2
5 1 1	— 41.4	— 59.2
1 5 1	+ 55.5	— 22.7
5 1 1	— 55.5	— 22.7
1 5 1	+ 70.6	— 48.0
5 1 1	— 70.6	— 48.0
1 3 5	+ 12.2	+ 5.0
3 1 5	— 12.2	+ 5.0
3 5 1	+ 14.1	— 40.4
5 3 1	— 14.1	— 40.4
3 1 5	+ 16.9	+ 62.1
1 3 5	— 16.9	+ 62.1
3 5 1	+ 16.9	— 62.1
5 3 1	— 16.9	— 62.1
1 3 5	+ 26.2	+ 16.0
3 1 5	— 26.2	+ 16.0
1 5 3	+ 26.2	— 16.0
5 1 3	— 26.2	— 16.0
3 1 5	+ 30.8	+ 44.0
1 3 5	— 30.8	+ 44.0
1 5 3	+ 42.4	— 5.8
5 1 3	— 42.4	— 5.8
! 5 1 3	+ 47.8	+ 87.8
! 1 5 3	— 47.8	+ 87.8
! 1 5 3	+ 47.8	— 87.8
! 5 1 3	— 47.8	— 87.8
5 1 3	+ 63.0	+ 60.0
1 5 3	— 63.0	+ 60.0
3 5 1	+ 74.9	— 7.6
5 3 1	— 74.9	— 7.6
! 5 3 1	+ 95.6	+ 29.3
! 3 5 1	— 95.6	+ 29.3
! 3 5 1	+ 95.6	— 29.3
! 5 3 1	— 95.6	— 29.3
3 3 5	0.0	— 4.4
3 3 5	0.0	+ 77.5
3 5 3	+ 11.5	— 23.5
5 3 3	— 11.5	— 23.5
3 5 3	+ 19.2	— 86.1
5 3 3	— 19.2	— 86.1
3 3 5	+ 39.9	+ 27.1
3 3 5	— 39.9	+ 27.1
3 5 3	+ 57.6	+ 5.9
5 3 3	— 57.6	+ 5.9
5 3 3	+ 69.1	+ 35.2
3 5 3	— 69.1	+ 35.2
1 1 7	0.0	+ 21.1
1 1 7	0.0	+ 43.2
1 1 7	+ 11.0	+ 31.4
1 1 7	— 11.0	+ 31.4
1 7 1	+ 37.8	— 36.0
7 1 1	— 37.8	— 36.0
1 7 1	+ 44.2	— 54.2
7 1 1	— 44.2	— 54.2
1 7 1	+ 54.4	— 27.7
7 1 1	— 54.4	— 27.7
1 7 1	+ 64.5	— 46.1
7 1 1	— 64.5	— 46.1
5 5 1	0.0	— 43.2
5 5 1	0.0	— 61.0
1 5 5	+ 20.7	— 4.2
5 1 5	— 20.7	— 4.2
5 1 5	+ 32.2	+ 72.4
1 5 5	— 32.2	+ 72.4
1 5 5	+ 33.0	+ 4.5
5 1 5	— 33.0	+ 4.5
5 1 5	+ 44.2	+ 54.2
1 5 5	— 44.2	+ 54.2
5 5 1	+ 88.9	+ 7.3
5 5 1	— 88.9	+ 7.3

[1 3 0].

$$x = \frac{100\,(k-3h)}{\sqrt{10\,\Sigma h^2 + h + 3k}}\ mm$$

$$y = \frac{100\,l\,\sqrt{10}}{\sqrt{10\,\Sigma h^2 + h + 3k}}\ mm$$

! means that the pole with this set of indices lies on the reference circle.



(h k l)	x	y	(h k l)	x	y
1 0 0	— 72.1	0.0	1 2 1	— 6.8	— 21.4
! 0 0 1	0.0	+ 100.0	1 2 1	+ 39.2	+ 24.8
! 0 0 1	0.0	— 100.0	1 2 1	+ 39.2	— 24.8
0 1 0	+ 16.2	0.0	1 1 2	+ 41.0	+ 64.9
			1 1 2	+ 41.0	— 64.9
1 0 1	— 54.8	+ 57.8	2 1 1	+ 80.0	+ 36.2
1 0 1	— 54.8	— 57.8	2 1 1	+ 80.0	— 36.2
1 1 0	— 23.6	0.0			
0 1 1	+ 13.4	+ 42.3	2 1 2	— 34.5	+ 43.7
0 1 1	+ 13.4	— 42.3	2 1 2	— 34.5	— 43.7
1 1 0	+ 61.8	0.0	2 2 1	— 22.9	+ 18.1
			2 2 1	— 22.9	— 18.1
1 1 1	— 21.1	+ 33.4	1 2 2	— 6.1	+ 38.4
1 1 1	— 21.1	— 33.4	1 2 2	— 6.1	— 38.4
1 1 1	+ 53.5	+ 42.3	1 2 2	+ 34.5	+ 43.7
1 1 1	+ 53.5	— 42.3	1 2 2	+ 34.5	— 43.7
			2 2 1	+ 59.3	+ 23.4
2 0 1	— 66.1	+ 34.9	2 2 1	+ 59.3	— 23.4
2 0 1	— 66.1	— 34.9	2 1 2	+ 66.8	+ 60.3
2 1 0	— 41.4	0.0	2 1 2	+ 66.8	— 60.3
1 0 2	— 37.2	+ 78.4			
1 0 2	— 37.2	— 78.4	! 3 1 0	— 100.0	0.0
1 2 0	— 7.1	0.0	3 0 1	— 69.2	+ 24.3
0 1 2	+ 9.9	+ 62.8	3 0 1	— 69.2	— 24.3
0 1 2	+ 9.9	— 62.8	3 1 0	— 50.0	0.0
0 2 1	+ 15.3	+ 24.2	1 0 3	— 27.3	+ 86.2
0 2 1	+ 15.3	— 24.2	1 0 3	— 27.3	— 86.2
1 2 0	+ 41.4	0.0	1 3 0	0.0	0.0
2 1 0	+ 86.7	0.0	0 1 3	+ 7.7	+ 73.0
			0 1 3	+ 7.7	— 73.0
2 1 1	— 39.2	+ 24.8	0 3 1	+ 15.8	+ 16.6
2 1 1	— 39.2	— 24.8	0 3 1	+ 15.8	— 16.6
1 1 2	— 17.0	+ 53.8	1 3 0	+ 33.3	0.0
1 1 2	— 17.0	— 53.8	! 3 1 0	+ 100.0	0.0
1 2 1	— 6.8	+ 21.4			

(h k l)	x	y
! 3 1 1	— 95.3	+ 30.2
! 3 1 1	— 95.3	— 30.2
3 1 1	— 48.5	+ 19.2
3 1 1	— 48.5	— 19.2
1 1 3	— 13.8	+ 65.5
1 1 3	— 13.8	— 65.5
1 3 1	0.0	+ 15.4
1 3 1	0.0	— 15.4
1 1 3	+ 32.0	+ 76.0
1 1 3	+ 32.0	— 76.0
1 3 1	+ 32.5	+ 17.1
1 3 1	+ 32.5	— 17.1
! 3 1 1	+ 95.3	+ 30.2
! 3 1 1	+ 95.3	— 30.2
3 0 2	— 62.5	+ 43.9
3 0 2	— 62.5	— 43.9
2 0 3	— 44.8	+ 70.8
2 0 3	— 44.8	— 70.8
3 2 0	— 34.3	0.0
2 3 0	— 13.4	0.0
0 2 3	+ 11.5	+ 54.5
0 2 3	+ 11.5	— 54.5
0 3 2	+ 14.7	+ 31.0
0 3 2	+ 14.7	— 31.0
2 3 0	+ 48.9	0.0
3 2 0	+ 76.4	0.0
! 3 1 2	— 84.5	+ 53.5
! 3 1 2	— 84.5	— 53.5
3 1 2	— 44.9	+ 35.5
3 1 2	— 44.9	— 35.5
3 2 1	— 33.6	+ 15.2
3 2 1	— 33.6	— 15.2
2 1 3	— 29.7	+ 56.4
2 1 3	— 29.7	— 56.4
2 3 1	— 13.1	+ 13.9
2 3 1	— 13.1	— 13.9
1 2 3	— 5.3	+ 50.4
1 2 3	— 5.3	— 50.4
1 3 2	0.0	+ 29.0
1 3 2	0.0	— 29.0
1 2 3	+ 29.7	+ 56.4
1 2 3	+ 29.7	— 56.4
1 3 2	+ 30.3	+ 31.9
1 3 2	+ 30.3	— 31.9
2 3 1	+ 47.8	+ 16.8
2 3 1	+ 47.8	— 16.8
2 1 3	+ 54.6	+ 73.9
2 1 3	+ 54.6	— 73.9
3 2 1	+ 74.2	+ 21.3

(h k l)	x	y
3 2 1	+ 74.2	— 21.3
! 3 1 2	+ 84.5	+ 53.5
! 3 1 2	+ 84.5	— 53.5
! 3 1 3	— 72.5	+ 68.8
! 3 1 3	— 72.5	— 68.8
3 1 3	— 40.4	+ 48.0
3 1 3	— 40.4	— 48.0
3 3 1	— 23.3	+ 12.3
3 3 1	— 23.3	— 12.3
1 3 3	0.0	+ 39.9
1 3 3	0.0	— 39.9
1 3 3	+ 27.5	+ 43.5
1 3 3	+ 27.5	— 43.5
3 3 1	+ 60.7	+ 16.0
3 3 1	+ 60.7	— 16.0
! 3 1 3	+ 72.5	+ 68.8
! 3 1 3	+ 72.5	— 68.8
5 1 1	— 86.8	+ 17.2
5 1 1	— 86.8	— 17.2
5 1 1	— 57.3	+ 12.9
5 1 1	— 57.3	— 12.9
1 1 5	— 9.8	+ 77.4
1 1 5	— 9.8	— 77.4
1 5 1	+ 6.2	+ 9.8
1 5 1	+ 6.2	— 9.8
1 1 5	+ 21.7	+ 85.8
1 1 5	+ 21.7	— 85.8
1 5 1	+ 26.3	+ 10.4
1 5 1	+ 26.3	— 10.4
5 1 3	— 77.3	+ 45.8
5 1 3	— 77.3	— 45.8
! 3 1 5	— 53.5	+ 84.5
! 3 1 5	— 53.5	— 84.5
5 1 3	— 52.4	+ 35.5
5 1 3	— 52.4	— 35.5
5 3 1	— 36.7	+ 9.7
5 3 1	— 36.7	— 9.7
3 1 5	— 32.4	+ 64.0
3 1 5	— 32.4	— 64.0
3 5 1	— 10.9	+ 8.6
3 5 1	— 10.9	— 8.6
1 3 5	0.0	+ 55.1
1 3 5	0.0	— 55.1
1 5 3	+ 5.8	+ 27.3
1 5 3	+ 5.8	— 27.3
1 3 5	+ 22.5	+ 59.2
1 3 5	+ 22.5	— 59.2
1 5 3	+ 24.5	+ 29.0

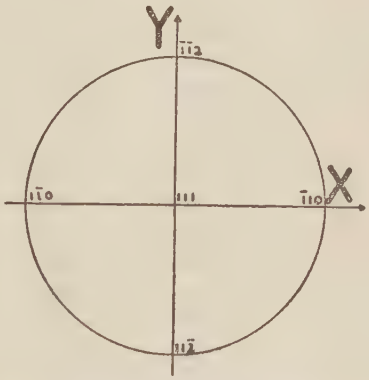
(h k l)	x	y	(h k l)	x	y
1 5 3	+ 24.5	— 29.0	7 1 1	— 61.4	+ 9.7
3 5 1	+ 45.6	+ 10.3	7 1 1	— 61.4	— 9.7
3 5 1	+ 45.6	— 10.3	1 1 7	— 7.5	+ 83.3
! 3 1 5	+ 53.5	+ 84.5	1 1 7	— 7.5	— 83.3
! 3 1 5	+ 53.5	— 84.5	1 7 1	+ 9.0	+ 7.1
5 3 1	+ 79.3	+ 13.9	1 7 1	+ 9.0	— 7.1
5 3 1	+ 79.3	— 13.9	1 1 7	+ 16.3	+ 90.0
			1 1 7	+ 16.3	— 90.0
5 3 3	— 34.5	+ 27.3	1 7 1	+ 23.5	+ 7.4
5 3 3	— 34.5	— 27.3	1 7 1	+ 23.5	— 7.4
3 3 5	— 18.3	+ 48.3			
3 3 5	— 18.3	— 48.3	5 1 5	— 65.1	+ 64.3
3 5 3	— 10.3	+ 24.5	5 1 5	— 65.1	— 64.3
3 5 3	— 10.3	— 24.5	5 1 5	— 45.8	+ 51.7
3 5 3	+ 42.8	+ 29.0	5 1 5	— 45.8	— 51.7
3 5 3	+ 42.8	— 29.0	5 5 1	— 23.5	+ 7.4
3 3 5	+ 44.9	+ 59.1	5 5 1	— 23.5	— 7.4
3 3 5	+ 44.9	— 59.1	1 5 5	+ 5.2	+ 41.0
5 3 3	+ 72.8	+ 38.4	1 5 5	+ 5.2	— 41.0
5 3 3	+ 72.8	— 38.4	1 5 5	+ 21.9	+ 43.2
			1 5 5	+ 21.9	— 43.2
7 1 1	— 82.8	+ 11.9	5 5 1	+ 61.4	+ 9.7
7 1 1	— 82.8	— 11.9	5 5 1	+ 61.4	— 9.7

[1 1 1].

$$x = \frac{50\,(k-h)\,\sqrt{6}}{\sqrt{3\,\Sigma h^2 + h + k + l}}\ mm$$

$$y = \frac{50\,(2l-h-k)\,\sqrt{2}}{\sqrt{3\,\Sigma h^2 + h + k + l}}\ mm$$

! means that the pole with this set of indices lies on the reference circle.



(h k l)	x	y	(h k l)	x	y
0 0 1	0.0	+ 51.8	! 0 1 1	— 50.0	+ 86.6
0 1 0	+ 44.8	— 25.9	! 0 1 1	+ 50.0	— 86.6
1 0 0	— 44.8	— 25.9	! 1 0 1	— 50.0	— 86.6
			! 1 1 0	+ 100.0	0.0
1 1 0	0.0	— 31.8	! 1 1 0	— 100.0	0.0
0 1 1	+ 27.5	+ 15.9			
1 0 1	— 27.5	+ 15.9	1 1 1	0.0	0.0
! 1 0 1	+ 50.0	+ 86.6	1 1 1	0.0	— 70.7

(h k l)	x	y
1 1 1	+ 61.2	+ 35.4
1 1 1	— 61.2	+ 35.4
0 1 2	+ 17.8	+ 30.9
1 0 2	— 17.8	+ 30.9
1 2 0	+ 17.8	— 30.9
2 1 0	— 17.8	— 30.9
1 0 2	+ 25.1	+ 72.6
0 1 2	— 25.1	+ 72.6
0 2 1	+ 35.6	0.0
2 0 1	— 35.6	0.0
0 2 1	+ 50.3	— 58.0
2 0 1	— 50.3	— 58.0
1 2 0	+ 75.4	— 14.5
2 1 0	— 75.4	— 14.5
1 1 2	0.0	+ 17.2
! 1 1 2	0.0	+ 100.0
! 1 1 2	0.0	— 100.0
1 2 1	+ 14.9	— 8.6
2 1 1	— 14.9	— 8.6
1 2 1	+ 19.6	— 56.6
2 1 1	— 19.6	— 56.6
1 1 2	+ 39.2	+ 45.3
1 1 2	— 39.2	+ 45.3
1 2 1	+ 58.9	+ 11.3
2 1 1	— 58.9	+ 11.3
! 2 1 1	+ 86.6	+ 50.0
! 1 2 1	— 86.6	+ 50.0
! 1 2 1	+ 86.6	— 50.0
! 2 1 1	— 86.6	— 50.0
2 2 1	0.0	— 13.9
2 2 1	0.0	— 51.8
1 2 2	+ 12.0	+ 6.9
2 1 2	— 12.0	+ 6.9
1 2 2	+ 19.8	— 79.9
2 1 2	— 19.8	— 79.9
1 2 2	+ 44.8	+ 25.9
2 1 2	— 44.8	+ 25.9
2 1 2	+ 59.3	+ 57.1
1 2 2	— 59.3	+ 57.1
2 2 1	+ 79.1	+ 22.8
2 2 1	— 79.1	+ 22.8
0 1 3	+ 12.9	+ 37.3
1 0 3	— 12.9	+ 37.3
1 0 3	+ 16.4	+ 66.2
0 1 3	— 16.4	+ 66.2
1 3 0	+ 25.8	— 29.8
3 1 0	— 25.8	— 29.8

(h k l)	x	y
0 3 1	+ 38.8	— 7.5
3 0 1	— 38.8	— 7.5
0 3 1	+ 49.1	— 47.3
3 0 1	— 49.1	— 47.3
1 3 0	+ 65.5	— 18.9
3 1 0	— 65.5	— 18.9
1 1 3	0.0	+ 26.3
1 1 3	0.0	+ 83.9
1 3 1	+ 22.8	— 13.2
3 1 1	— 22.8	— 13.2
1 1 3	+ 28.0	+ 48.5
1 1 3	— 28.0	+ 48.5
1 3 1	+ 28.0	— 48.5
3 1 1	— 28.0	— 48.5
1 3 1	+ 56.0	0.0
3 1 1	— 56.0	0.0
1 3 1	+ 72.6	— 41.9
3 1 1	— 72.6	— 41.9
2 3 0	+ 10.9	— 31.4
3 2 0	— 10.9	— 31.4
0 2 3	+ 21.8	+ 25.2
2 0 3	— 21.8	+ 25.2
0 3 2	+ 32.7	+ 6.3
3 0 2	— 32.7	+ 6.3
2 0 3	+ 33.8	+ 78.1
0 2 3	— 33.8	+ 78.1
0 3 2	+ 50.7	— 68.3
3 0 2	— 50.7	— 68.3
2 3 0	+ 84.5	— 9.8
3 2 0	— 84.5	— 9.8
1 2 3	+ 9.8	+ 17.0
2 1 3	— 9.8	+ 17.0
2 3 1	+ 9.8	— 17.0
3 2 1	— 9.8	— 17.0
2 3 1	+ 11.7	— 47.2
3 2 1	— 11.7	— 47.2
! 2 1 3	+ 18.9	+ 98.2
! 1 2 3	— 18.9	+ 98.2
! 1 2 3	+ 18.9	— 98.2
! 2 1 3	— 18.9	— 98.2
1 3 2	+ 19.6	0.0
3 1 2	— 19.6	0.0
1 3 2	+ 28.9	— 66.7
3 1 2	— 28.9	— 66.7
1 2 3	+ 35.1	+ 33.7
2 1 3	— 35.1	+ 33.7
2 1 3	+ 43.3	+ 58.4
1 2 3	— 43.3	+ 58.4

(h k l)	x	y	(h k l)	x	y
1 3 2	+ 46.7	+ 13.5	3 1 5	+ 37.0	+ 64.1
3 1 2	— 46.7	+ 13.5	1 3 5	— 37.0	+ 64.1
2 3 1	+ 72.2	+ 8.3	1 5 3	+ 37.0	— 64.1
3 2 1	— 72.2	+ 8.3	5 1 3	— 37.0	— 64.1
! 3 1 2	+ 75.6	+ 65.5	1 5 3	+ 42.6	+ 8.2
! 1 3 2	— 75.6	+ 65.5	5 1 3	— 42.6	+ 8.2
! 1 3 2	+ 75.6	— 65.5	1 5 3	+ 65.3	— 62.9
! 3 1 2	— 75.6	— 65.5	5 1 3	— 65.3	— 62.9
! 3 2 1	+ 94.5	+ 32.7	3 5 1	+ 74.0	0.0
! 2 3 1	— 94.5	+ 32.7	5 3 1	— 74.0	0.0
! 2 3 1	+ 94.5	— 32.7	3 5 1	+ 87.1	— 25.1
! 3 2 1	— 94.5	— 32.7	5 3 1	— 87.1	— 25.1
3 3 1	0.0	— 19.4	3 3 5	0.0	+ 12.7
3 3 1	0.0	— 45.1	3 3 5	0.0	— 91.5
1 3 3	+ 16.8	+ 9.7	3 5 3	+ 11.0	— 6.3
3 1 3	— 16.8	+ 9.7	5 3 3	— 11.0	— 6.3
1 3 3	+ 28.7	— 82.7	3 5 3	+ 15.0	— 60.5
3 1 3	— 28.7	— 82.7	5 3 3	— 15.0	— 60.5
1 3 3	+ 39.0	+ 22.5	3 3 5	+ 44.9	+ 43.2
3 1 3	— 39.0	+ 22.5	3 3 5	— 44.9	+ 43.2
3 1 3	+ 57.3	+ 66.2	3 5 3	+ 59.9	+ 17.3
1 3 3	— 57.3	+ 66.2	5 3 3	— 59.9	+ 17.3
3 3 1	+ 85.9	+ 16.5	5 3 3	+ 79.3	+ 45.8
3 3 1	— 85.9	+ 16.5	3 5 3	— 79.3	+ 45.8
1 1 5	0.0	+ 35.4	1 1 7	0.0	+ 39.7
1 1 5	0.0	+ 70.7	1 1 7	0.0	+ 65.1
1 1 5	+ 17.5	+ 50.5	1 1 7	+ 12.6	+ 51.1
1 1 5	— 17.5	+ 50.5	1 1 7	— 12.6	+ 51.1
1 5 1	+ 30.6	— 17.7	1 7 1	+ 34.4	— 19.9
5 1 1	— 30.6	— 17.7	7 1 1	— 34.4	— 19.9
1 5 1	+ 35.0	— 40.4	1 7 1	+ 37.9	— 36.5
5 1 1	— 35.0	— 40.4	7 1 1	— 37.9	— 36.5
1 5 1	+ 52.5	— 10.1	1 7 1	+ 50.6	— 14.6
5 1 1	— 52.5	— 10.1	7 1 1	— 50.6	— 14.6
1 5 1	+ 61.2	— 35.4	1 7 1	+ 56.4	— 32.6
5 1 1	— 61.2	— 35.4	7 1 1	— 56.4	— 32.6
1 3 5	+ 12.7	+ 22.0	5 5 1	0.0	— 24.2
3 1 5	— 12.7	+ 22.0	5 5 1	0.0	— 39.7
3 5 1	+ 12.7	— 22.0	1 5 5	+ 21.0	+ 12.1
5 3 1	— 12.7	— 22.0	5 1 5	— 21.0	+ 12.1
3 5 1	+ 14.2	— 41.0	1 5 5	+ 34.4	+ 19.9
5 3 1	— 14.2	— 41.0	5 1 5	— 34.4	+ 19.9
3 1 5	+ 21.8	+ 88.0	1 5 5	+ 36.6	— 84.6
1 3 5	— 21.8	+ 88.0	5 1 5	— 36.6	— 84.6
1 5 3	+ 25.5	0.0	5 1 5	+ 55.0	+ 74.0
5 1 3	— 25.5	0.0	1 5 5	— 55.0	+ 74.0
1 3 5	+ 28.4	+ 32.8	5 5 1	+ 91.6	+ 10.6
3 1 5	— 28.4	+ 32.8	5 5 1	— 91.6	+ 10.6

Biology. — *Mathematics of pollen diagrams.* II. By J. WESTENBERG. (Communicated by Prof. M. W. WOERDEMAN.)

(Communicated at the meeting of March 29, 1947.)

The lower extremities of the tail error lines are determined by the conditions: $n < N - n$; $n - \triangle = 0$ and $P_{n;n,N}$ = tail error value. By graphical interpolation between several values of $P_{n;n,N}$ we find

$N = 150$		
	$\frac{1}{10}$ %	$n = 4{,}9$
	$\frac{1}{2}$ %	3,7
	1 %	3,2
	2 %	2,7
	3 %	2,4
	4 %	2,2
	5 %	2,1

Similarly we find for the upper extremities ($N - n < n$; $N - n - \triangle = 0$)

$\frac{1}{10}$ %	$n = 145{,}1$
$\frac{1}{2}$ %	146,3
1 %	146,8
2 %	147,3
3 %	147,6
4 %	147,8
5 %	147,9

In testing the significance of the difference of two countings of the same species in two different strata, we have to plot these countings on the same horizontal axis in the pollen diagram (*a* and *b*), and insert their mean value in the middle between them (*c*) (fig. 5). Then we place the transparant graph over the pollendiagram, its sides covering the sides of the pollen diagram. Next we shift the graph upward or downward until the mean of the countings (*c*) is covered by the oblique line. If the position of the two counting marks (*a* and *b*) is outside the two tail error lines, we are led to conclude that it is likely, that the difference d ($= 2\triangle$) of the two countings considered, should have been significant, because the probability that a difference so great or greater should have arisen through random sampling is even less than the chosen tail error value.

In order to examine how the reliability of the method depends on the amount of work, we compute similar data for other values of N, viz. 10, 20, 50, 100, 300 and 450.

$N = 10$	Tail error	$n = \triangle$ $N - n = \triangle$	$n = 3$ $n = 7$	$n = 4$ $n = 6$	$n = 5$
	$\frac{1}{10}$ %	3,6	—	3,7	3,8
	$\frac{1}{2}$ %	3,0	—	3,3	3,4
	1 %	2,7	2,8	3,0	3,1
	2 %	2,3	2,6	2,7	2,7
	3 %	2,1	2,4	2,5	2,5
	4 %	2,0	2,3	2,4	2,4
	5 %	1,9	2,2	2,3	2,3

$N = 20$	Tail error	$n = \Delta$ $N - n = \Delta$	$n = 5$ $n = 15$	$n = 8$ $n = 12$	$n = 10$
	$\frac{1}{10}$ %	4,3	4,5	5,2	5,3
	$\frac{1}{2}$ %	3,4	3,9	4,5	4,6
	1 %	3,0	3,6	4,1	4,2
	2 %	2,6	3,2	3,6	3,7
	3 %	2,4	3,0	3,3	3,4
	4 %	2,2	2,8	3,1	3,2
	5 %	2,0	2,7	3,0	3,1

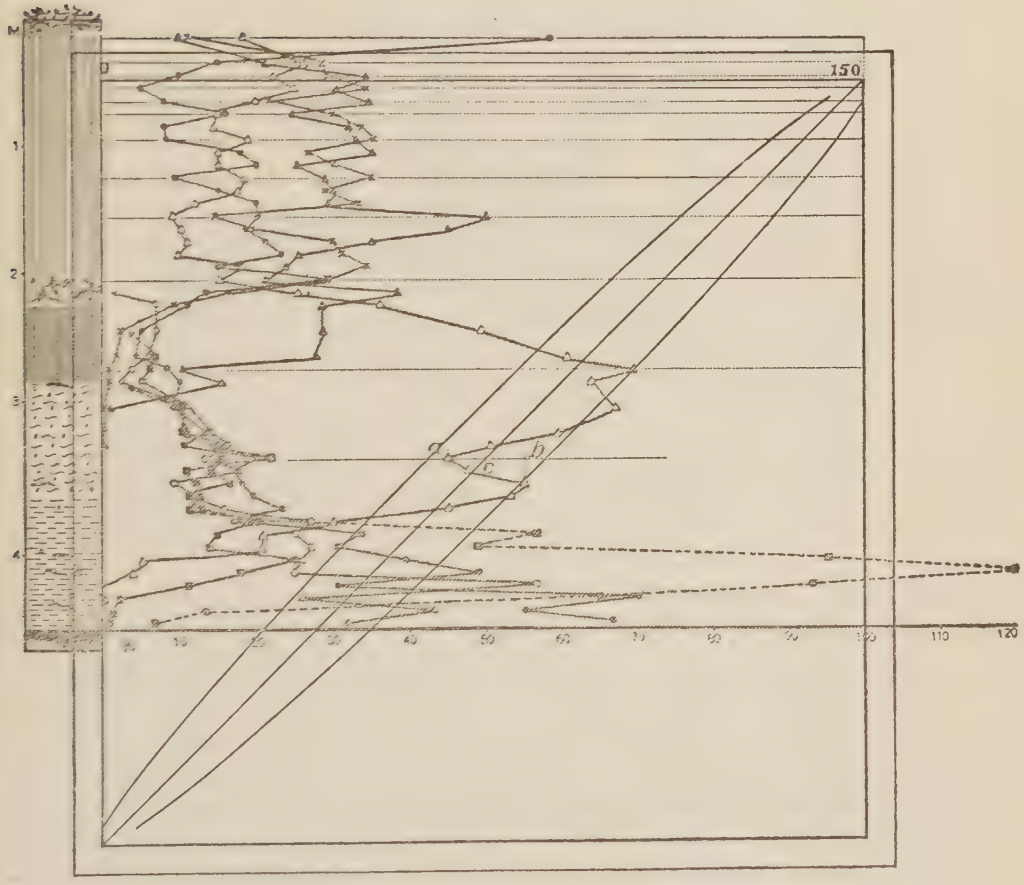


Fig. 5. Tail error line superposed on pollen diagram.

$N = 50$	Tail error	$n = \Delta$ $N - n = \Delta$	$n = 5$ $n = 45$	$n = 10$ $n = 40$	$n = 15$ $n = 35$	$n = 20$ $n = 30$	$n = 25$
	$\frac{1}{10}$ %	4,6	4,8	6,5	7,5	8,0	8,1
	$\frac{1}{2}$ %	3,6	4,2	5,6	6,3	6,7	6,9
	1 %	3,2	3,9	5,1	5,8	6,1	6,3
	2 %	2,8	3,5	4,5	5,1	5,4	5,6
	3 %	2,5	3,3	4,2	4,8	5,1	5,2
	4 %	2,3	3,1	4,0	4,5	4,7	4,9
	5 %	2,1	3,0	3,8	4,2	4,5	4,6

$N = 100$ Tail error	$n = \triangle$ $N - n = \triangle$	$n = 5$ $n = 95$	$n = 10$ $n = 90$	$n = 20$ $n = 80$	$n = 30$ $n = 70$	$n = 40$ $n = 60$	$n = 50$
$\frac{1}{10}$ %	4,8	4,9	6,8	9,1	10,4	11,1	11,4
$\frac{1}{2}$ %	3,8	4,3	5,8	7,7	8,8	9,3	9,5
1 %	3,3	4,0	5,4	7,0	8,0	8,4	8,7
2 %	2,8	3,5	4,8	6,2	7,1	7,6	7,7
3 %	2,5	3,3	4,5	5,8	6,6	7,0	7,1
4 %	2,3	3,1	4,2	5,4	6,1	6,6	6,7
5 %	2,1	3,0	4,0	5,1	5,8	6,2	6,3

$N=150$ Tail error	$n=\triangle$ $N-n=\triangle$	$n=5$ $n=145$	$n=10$ $n=140$	$n=20$ $n=130$	$n=30$ $n=120$	$n=40$ $n=110$	$n=50$ $n=100$	$n=60$ $n=90$	$n=75$
$\frac{1}{10}$ %	4,9	4,9	6,9	9,4	11,1	12,2	13,0	13,5	13,8
$\frac{1}{2}$%	3,7	4,3	5,9	8,1	9,3	10,3	11,0	11,4	11,6
1%	3,2	4,0	5,4	7,3	8,5	9,3	10,0	10,4	10.5
2%	2,7	3,6	4,9	6,5	7,6	8,3	8,9	9,2	9,4
3%	2,4	3,3	4,5	6,0	7,0	7,7	8,2	8,4	8,6
4%	2,2	3,1	4,3	5,6	6,6	7,2	7,6	7,9	8,1
5%	2,1	3,0	4,0	5,3	6,2	6,8	7,2	7,5	7,6

$N=300$ Tail error	$n=\triangle$ $N-n=\triangle$	$n=10$ $n=290$	$n=20$ $n=280$	$n=40$ $n=260$	$n=60$ $n=240$	$n=80$ $n=220$	$n=100$ $n=200$	$n=125$ $n=175$	$n=150$
$\frac{1}{10}$ %	4,9	7,0	9,8	13,3	15,5	17,2	18,2	19,1	19,4
$\frac{1}{2}$%	3,7	6,0	8,3	11,2	13,1	14,4	15,8	16,0	16.2
1%	3,2	5,5	7,6	10,1	11,9	13,1	13,9	14,5	14,7
2%	2,8	5,0	6,7	9,0	10,5	11,6	12,4	12,9	13,1
3%	2,5	4,6	6,2	8,3	9,7	10,7	11,4	11,8	12,0
4%	2,3	4,3	5,8	7,8	9,1	10,0	10,6	11,1	11,2
5%	2,1	4,1	5,5	7,3	8,6	9,4	10.0	10,4	10,5

$N=450$ Tail error	$n=\triangle$ $N-n=\triangle$	$n=15$ $n=435$	$n=30$ $n=420$	$n=60$ $n=390$	$n=90$ $n=360$	$n=120$ $n=330$	$n=150$ $n=300$	$n=180$ $n=270$	$n=225$
$\frac{1}{10}$ %	4,9	8,6	11,9	16,2	19,0	20,9	22,3	23,2	23,7
$\frac{1}{2}$%	3,8	7,3	10,1	13,5	15,9	17,6	18,7	19,4	19,8
1%	3,3	6,7	9,1	12,3	14,4	15,9	16,9	17,5	17,9
2%	2,8	6,0	8,1	10,9	12,8	14,1	15,0	15,5	15,9
3%	2,5	5,5	7,5	10,1	11,8	12,9	13,8	14,3	14,6
4%	2,3	5,2	7,0	9,4	10,9	12,1	12,8	13,4	13,6
5%	2,1	4,9	6,6	8,9	10,3	11,4	12,1	12,6	12,8

The above tables allow the construction of tail error diagrams, as described in preceding lines; in our illustrations the values of $\triangle$ pertain to a (unilateral) tail error value of $\frac{1}{2}$ %. If we draw these diagrams just as broad as that one for $N = 150$, the significant difference $d\ (= 2\triangle)$ will appear in these diagrams as $\delta = d\,\frac{150}{N}$.

As a measure for the precision of the method we choose δ for $n = N—n$.

Next we unite these diagrams into one three dimensional graph, plotting N along the third axis, as shown in fig. 6. The tail error lines are connected by a pair of smooth surfaces, the so-called tail error surfaces. The

construction is then completed by connecting the horizontal axes for $n = N - n$ by a plane [1]).

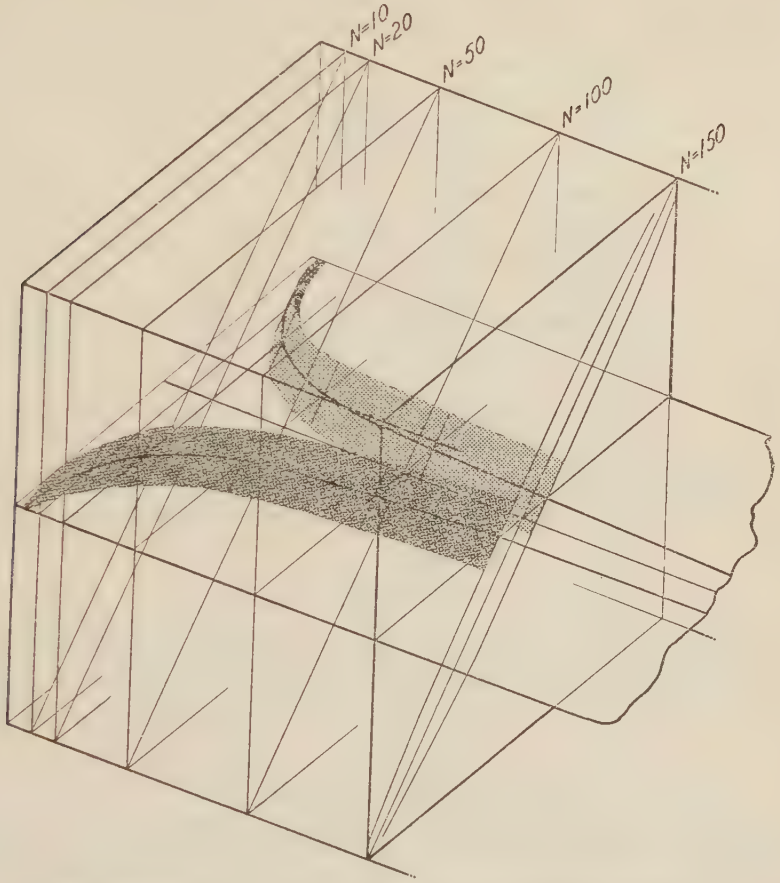


Fig. 6. Three dimensional arrangement of tail error diagrams pertaining to increasing values of N.

The section lines of this plane and the tail error surfaces demonstrate the decrease of δ with increasing N. This is depicted separately in fig. 7, the plane being extended to $N = 450$. From this figure we see, that δ decreases rapidly up to $N = 50$, still appreciably up to $N = 150$. On from $N = 150$, the decrease becomes very slight, so that we may conclude, that pollen analysts did well to choose $N = 150$.

Sometimes, however, the pollen analyst might need more precision in order to reach a conclusion. This might be the case when a series of successive strata yields samples, that suggest a slight increase or decrease of a certain species, but do not allow any decision of the kind by comparing the countings of single strata. Then the difficulty might be surmounted,

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[1]) In order to avoid a confusing complexity, fig. 6 has been simplified: the tail error surfaces have been drawn only near $n = N - n$.

by pooling the countings of two or three successive samples and comparing these with the pooled countings of another set of two or three successive

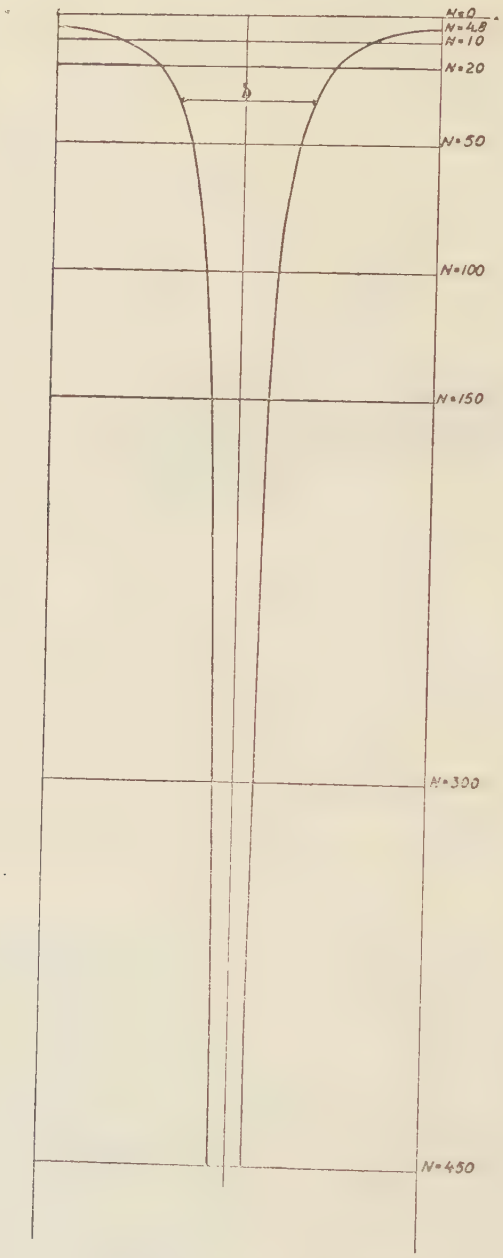


Fig. 7. Decrease of δ with increase of N.

samples. For this purpose we have to make use of the tables for $N = 300$ or $N = 450$.

As has been mentioned before, the extra pollen requires a different treatment. In the comparison of two extra pollen countings $n + \triangle$ and $n - \triangle$, the total numbers of pollen grains will amount to $N_1 + n + \triangle$ and $N_2 + n - \triangle$, when we keep to N_1 and N_2 for the marker species.

Hence the probability of $n + \triangle$ is:

$$P^*_{n+\triangle} = \frac{(N_1 + n + \triangle)!}{(n + \triangle)!\, N_1!} \cdot p^{*n+\triangle} \cdot (1 - p^*)^{N_1}$$

Similarly

$$P^*_{n-\triangle} = \frac{(N_2 + n - \triangle)!}{(n - \triangle)!\, N_2!} \cdot p^{*n-\triangle} \cdot (1 - p^*)^{N_2}$$

The probability of the simultaneous occurrence of these two events will therefore be

$$P^*_{\triangle; n, N_1, N_2, p^*} = \frac{(N_1 + n + \triangle)!\,(N_2 + n - \triangle)!}{(n + \triangle)!\,(n - \triangle)!\, N_1!\, N_2!} \cdot p^{*2n} \cdot (1 - p^*)^{N_1 + N_2}$$

Since p^* is not known, this probability can not be computed. In this formula, the powers of p^* are independent of $\triangle$, and therefore the unknown factor

$$p^{*2n} \cdot (1 - p^*)^{N_1 + N_2}$$

is the same for all possibilities with fixed values of n, N_1, N_2 and p^*. If these values are kept constant, the probability of any value of $\triangle$ occurring, is proportional to

$$F^* = \frac{(N_1 + n + \triangle)!\,(N_2 + n - \triangle)!}{(n + \triangle)!\,(n - \triangle)!\, N_1!\, N_2!}$$

The total probability of all possible values of $\triangle$, with fixed values of n, N_1, N_2 and p^* is represented by

$$\sum_{\triangle} P^*_{\triangle; n, N_1, N_2, p^*} = \frac{(N_1 + n + \triangle)!\,(N_2 + n - \triangle)!}{(n + \triangle)!\,(n - \triangle)!\, N_1!\, N_2!} \cdot p^{*2n} \cdot (1 - p^*)^{N_1 + N_2}$$

which may also be written as

$$\sum_{\triangle} P^*_{\triangle; N_1, N_2, p^*} = p^{*2n} \cdot (1 - p^*)^{N_1 + N_2} \cdot \sum_{\triangle} F^*.$$

In this summation $|\triangle| \leq n$ and $\leq N_1$ and $\leq N_2$.

The probability of a certain $\triangle$, the values of n, N_1, N_2 being fixed, is given by

$$P^*_{\triangle; n, N_1, N_2} = \frac{P^*_{\triangle; n, N_1, N_2, p^*}}{\sum_{\triangle} P^*_{\triangle; n, N_1, N_2, p^*}} = \frac{F^*}{\sum_{\triangle} F^*}$$

whatever the value of p^* might be.

In order to carry out the summation of ΣF^*, we use the following method, which was again suggested by VAN DANTZIG (private communication).

By the binomial series of NEWTON we have:

$$\sum_{0}^{\infty}{}_{k} \frac{(N_1+k)!}{N_1!\,k!} \cdot x^k \equiv (1-x)^{-(N_1+1)}$$

$$\sum_{0}^{\infty}{}_{l} \frac{(N_2+l)!}{N_2!\,l!} \cdot x^l \equiv (1-x)^{-(N_2+1)}.$$

Multiplication yields

$$\sum_{0}^{\infty}{}_{k} \sum_{0}^{\infty}{}_{l} \frac{(N_1+k)!\,(N_2+l)!}{N_1!\,N_2!\,k!\,l!} \cdot x^{k+l} \equiv (1-x)^{-(N_1+1)-(N_2+1)} \equiv$$

$$\equiv \sum_{0}^{\infty}{}_{h} \frac{(N_1+N_2+1+h)!}{(N_1+N_2+1)!\,h!} \cdot x^h.$$

Equating the coefficients of x^h, $(h = k + l)$ gives:

$$\sum_{0}^{h}{}_{k} \frac{(N_1+k)!\,(N_2+h-k)!}{N_1!\,N_2!\,k!\,(h-k)!} = \frac{(N_1+N_2+1+h)!}{h!\,(N_1+N_2+1)!}.$$

Putting $h = m_1 + m_2$, $k = m_1 + \triangle$ and $h - k = m_2 - \triangle$ we find

$$\sum_{-m_1}^{+m_2}{}_{\triangle} \frac{(N_1+m_1+\triangle)!\,(N_2+m_2-\triangle)!}{N_1!\,N_2!\,(m_1+\triangle)!\,(m_2-\triangle)!} = \frac{(N_1+N_2+m_1+m_2+1)!}{(m_1+m_2)!\,(N_1+N_2+1)!}.$$

For $m_1 = m_2 = n$, we arrive at:

$$\sum_{-n}^{+n}{}_{\triangle} \frac{(N_1+n+\triangle)!\,(N_2+n-\triangle)!}{(N_1!\,N_2!\,(n+\triangle)!\,(n-\triangle)!} = \frac{(N_1+N_2+2n+1)!}{(2n)!\,(N_1+N_2+1)!}.$$

Finally we may write

$$P^*_{\triangle;\,n,N_1,N_2} = \frac{F^*}{\sum_{\triangle} F^*} = \frac{\dfrac{(N_1+n+\triangle)!\,(N_2+n-\triangle)!}{(n+\triangle)!\,(n-\triangle)!\,N_1!\,N_2!}}{\dfrac{(N_1+N_2+2n+1)!}{(2n)!\,(N_1+N_2+1)!}}.$$

For equal values of N, we arrive at the simplified form

$$P^*_{\triangle;\,n,N} = \frac{\dfrac{(N+n+\triangle)!\,(N+n-\triangle)!}{(n+\triangle)!\,(n-\triangle)!\,(N!)^2}}{\dfrac{(2N+2n+1)!}{(2n)!\,(2N+1)!}}.$$

In order to make the method fit for use, we have to compute the values of $P^*_{\triangle;\,n,N}$ for a certain value of n and $N = 150$, varying $\triangle$ only. Next we calculate the cumulative chances of $\triangle$, and by means of graphical interpolation we find the values of $\triangle$, belonging to (unilateral) tail error values of 0,001, 0,005, 0,01 &c. The same is done for other values of n.

By graphical interpolation we also find, for which values of n, $P^*_{n;\,n,N}$

will correspond to the chosen tail error values. Thus we arrive at the following table:

Tail error	$n = \Delta$	$n = 5$	$n = 10$	$n = 20$	$n = 30$	$n = 40$	$n = 50$	$n = 60$
$\frac{1}{10}$ %	5,1	—	7,3	10,6	13,4	15,8	18,1	20,2
$\frac{1}{2}$ %	3,9	4,4	6,2	9,0	11,3	13,3	15,3	17,1
1 %	3,4	4,1	5,7	8,2	10,2	12,1	13,8	15,4
2 %	2,9	3,7	5,1	7,3	9,1	10,8	12,3	13,7
3 %	2,6	3,4	4,8	6,8	8,4	9,9	11,3	12,6
4 %	2,3	3,2	4,4	6,3	7,9	9,3	10,5	11,8
5 %	2,2	3,1	4,2	6,0	7,4	8,7	10,0	11,1

$n = 75$	$n = 90$	$n = 110$	$n = 150$	$n = 200$	$n = 300$
23,4	26,3	30,4	38,0	47,2	65,4
19,7	22,2	25,5	31,8	39,5	54,8
17,8	20,1	23,0	28,8	35,8	49,5
15,8	17,9	20,5	25,5	31,7	43,8
14,6	16,3	18,8	23,5	29,1	40,2
13,6	15,3	17,5	21,9	27,1	37,4
12,8	14,4	16,5	20,6	25,5	35,3

From this table we can proceed to the construction of a tail error diagram, to be used as previously described.

Since the extra pollen species are considered as such, because of their great variability in abundance, the pollen analyst is not likely to need more precision for the interpretation of extra pollen countings. For this reason we refrain from computing tables for the treatment of pooled extra pollen data.

Acknowledgements.

Many years ago, I took up the plan to deal with the present subject, but it was not possible for me, to go through the matter all by myself. Therefore I wish to make grateful acknowledgement to those, who enabled me to carry out my plan.

To Prof. Dr. M. J. VAN UVEN, I feel indebted for his kind advice in questions of computation.

To Prof. Dr. D. VAN DANTZIG I wish to express my thanks for criticising the manuscript and for giving the mathematical solution of the last part of the problem.

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To my friend A. J. D. VEEN, I owe the careful drawing of the illustrations.

Summary.

In stating the significance of the difference between two frequency distributions, we have to test our samples by means of statistical methods imposing certain properties on the frequency curves of the populations.

In a vast majority of cases the normal law is assumed, including two parameters, i.e. the mean of the population, and the standard deviation. In the classical treatment of data, we have to substitute the parameters by so-called statistics, these being their estimates as computed from the samples. This is tolerable as long as large samples are considered. In the case of small samples, this procedure will appreciably affect the reliability of our conclusions. In later years, this difficulty has been surmounted by designing the so-called *t*-test. Since no unknown parameters appear in the formula, this test enables us to get exact information on the significance of the difference between two small samples, taken from normal populations. The application of this method, however, is still limited by the assumption of the normal law. In many cases of counting work, as in pollen analysis, the data may be supposed to follow the binomial law. Since the standard deviation, or the initial probabilities do not appear in the formula for the distribution of the mean of two samples, we have no need for computing their estimates. A suitable treatment is then carried out by applying R. A. FISHER's binomial test, which is exact, even for small samples. The derivation of its mathematical basis is given in full, in order to attract the attention of biological workers. The test is tabulated as far as needed for pollen analytical research. The practice of the method resolves itself into a graphical application, as is exemplified with a pollen diagram. The custom of counting 150 pollen grains is justified on theoretical grounds. The current theory does not hold for extra pollen countings. In order to meet the needs, a special method is developed for a similar treatment of extra pollen data.

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Geology. — *Some data on the Muriah volcano (Java), and its leucite-bearing rocks.* By L. BOOMGAART. (Communicated by Prof. J. H. F. UMBGROVE.)

(Communicated at the meeting of May 31, 1947.)

The Muriah volcano complex, East of Semarang on the North-coast of Java, protrudes into the Java-Sea. It comprises the Muriah volcano (inactive), 1602 m, and North of it the small ashvolcano Genuk, 717 m (formerly called Tjilering). East of this Genuk volcano we find some small isolated hills; from North to South the Bako, 157 m, the Ragas, 122 m and the Tempur, 49 m.

During an investigation of the northern sector by the present author 83 rocksamples have been collected (60 of solid rocks, 23 of boulders) and a microscopical study made, especially of the leucite-bearing rocks. Of 67 samples the content of K_2O and Na_2O were chemically determined by the Laboratory of the Geological Survey at Bandung.

I. The Muriah volcano.

The whole of the northern aspect of the mountain is controlled by the imposing triangular peak of the Sutorenggo, 1604 m, which stands out on the Southside of the Tempur-cauldron. This northern crater is surrounded by steep walls from 1100—1600 m, through which the Gelis-river forces an outlet to the North. The village of Tempur, situated in the crater, lies at an altitude of 600 m.

The northern part of the Muriah is mainly built up by breccias. Flows of basalto-andesitic rocks and of leucite-bearing rocktypes lie upon the breccias, or are found as intercalations in breccias. Some tufflayers have been observed also. No definite sequence of deposition could be reconstructed.

Leucite-bearing rocks.

This type of rock has been found as flows, as boulders in superficial block-fields, in one case as a dike (Tempur-cauldron), as constituent of breccias and as boulders in tuff. The following leucite-bearing rocktypes can be distinguished.

1. Leucite-tephrites without orthoclase (sanidine), except a little groundmass-orthoclase.

2. Leucite-tephrites with orthoclase (sanidine) as phenocrysts. The leucite is mostly altered. All these samples are from the crater-area.

3. Leucite-tephrites with nephelite. Only one sample.

4. Leucite-basanites (characterized by olivine). Plagioclase as phenocryst is present in 46 % of the examined rocks. In the other specimen plagioclase is generally sporadically present.

5. Leucitite-tephrites (with little plagioclase in the groundmass). No orthoclase has been detected.

6. Leucitites (with sporadical plagioclase in the groundmass).

Leucite. The larger phenocrysts are idiomorphic, sometimes built up by a number of smaller idiomorphic individuals. Smaller crystals are more or less rounded, sometimes hardly showing a well-defined crystallographic shape. In one thin-section an irregular fragment of leucite — of phenocryst-like size — has been observed.

The outline of the leucite against the surrounding groundmass can be quite different. Larger crystals are outstanding by themselves. Or their shape is accentuated by a dark rim of dustlike particles. Smaller crystals sometimes also show a similar rim. Or the groundmass augite-crystals are systematically arranged along their outline, in this way marking the leucite. But sometimes — particularly if the groundmass is rich in feldspar — the leucite is not conspicuous at all. Inclusions of microlites then may be a help in detecting the leucite microscopically.

The larger crystals, and sometimes the smaller ones also, show polysynthetical twinning-lamellae. Alterations of various types occur, among which replacement by calcite and zeolitic material should be mentioned. The greater percentage of the examined rocks contains fresh leucite; most of the altered leucite has been observed in rocks from the Tempur cauldron-area.

Inclusions of augite, apatite and microlites occur quite frequently. In one case more or less rounded, bi-refringent speckles, consisting of accumulations of dustlike mineral particles, were visible. Both sanidine and plagioclase are present in leucite-phenocrysts.

Measurements as to the size of leucite crystals have been carried out. Only in 29 % of the examined rocks megascopically visible crystals were found. The leucite generally occurs as crystals of less than 1000 microns, and from measured sections it appeared that 85 % of the leucite crystals are smaller than 175 microns.

Plagioclase. In 64 % of the studied rocks as phenocrysts, and in the greater part as groundmass material. In one section the central part of the phenocrysts is replaced by isotropic material of low refractive index (sodalite?).

Orthoclase. In 7 % of the examined rocks as phenocryst (sanidine), and at least in half of the slides as groundmass material. The mineral character of some phenocrysts appeared to be positive. Zonal growth is absent. In a few sections sanidine phenocrysts with plagioclase crystals enclosed could be observed.

Augite. Practically in 100 % of the examined rocks, both as palegreen diopsidic phenocrysts (zonal growth) and as groundmass material. Aegirine-augite was determined in 5 sections, enstatite in only one case.

Olivine. This mineral is often partly or completely altered into a brown-red, non-pleochroitic, bi-refringent mineral, probably Fayalite. Pleochroitic Iddingsite has not been found.

From the other minerals, as hornblende, biotite, nepheline, ore and apatite, calcite, chlorite and zeolites, no particulars are to be mentioned.

K_2O-percentage.

The percentage ranges from 2—10. The average of 60 samples of solid rock shows 5.08 % K_2O and 3.66 % Na_2O.

The Seno-river leucite-tephrite exposure.

In the Seno-river, on the middle of the Eastside of the northern Muriah sector, an exposure of leucite-tephrite has been found, the average K_2O-percentage of which justifies special mentioning. The average of 4 samples is 8.06 % K_2O and 3.09 % Na_2O. In black, dense and porphyritic rock leucite crystals up to 5 mm are present. Augite and/or biotite may be megascopically visible. In one of these rocks the peculiar large leucite crystals, mentioned above, have been detected.

The best exposure of this occurrence was found in a waterfall, with the following section from top to bottom:

10 m "karren"-like tephrite.
7 m tephrite in basaltic columns.
5 m Muriah breccias
8 m tuffaceous breccias.

This columnar development of tephrite occurs also at 1200 m North of the Seno waterfall, on the righthand bank of the Gelis-river. Its mineralsgical-petrographical description corresponds with that of the waterfall, whereas its analysis shows 6.81 % K_2O and 2.13 % Na_2O.

Other rock-types.

Besides the leucite-bearing rocks, breccias and tuffs, the following rocks have been collected:

Basalts and andesites.
Syenite-porphyries (4 dikes in Tempur-cauldron).
Trachytes.
Syenite (as inclusion in a solid exposure of leucite-tephrite).
Diorite (as a boulder in one of the rivers).

The Genuk Mountain.

This heavily eroded volcano is built up by nearly flatlying, well-stratified tuffs. Pumice constituents are more abundant than in Muriah tuffs. Of minor importance are the following rock exposures:

Leucite-basanite. Flow, exposed in a river. 2.55 % K_2O and 1.98 % Na_2O.
Leucite-tephrite. Flow, exposed on trail. 2.01 % K_2O and 3.07 % Na_2O.
Breccia. Only a few exposures.
Basalto-andesitic rock, as flow and blockfields.
Sodalite-tephrite.

Plagitrachyte, as a dike of 1 m in river-canyon.

Limestone with fossil remains (foraminifera, lamellibranchiatae and gastropodes).

Quartz-diorite, as a boulder on one of the slopes.

As to the relation between the Muriah and the Genuk, in one of the rivers between both mountains the following observations have been made:

a. Where Genuk tuffs and Muriah breccias (the latter characterized by leucite-bearing components) contact each other, these tuffs appear to lie below the breccias. In one place the plane of separation dips 60 degrees South; the breccias have been pushed up against the tuffs.

b. In one place the following vertical section was observed:

Debris and weathered soil.

Tuff bank, with the same type of tuff as on the Muriah, lacking the clearly visible pumice components, characteristic for the Genuk tuffs.

Breccia with leucite-bearing components and pumice.

With these observations on hand, it seems justified to assume, that the Genuk efflata are of older age than the Muriah breccias.

Bako hill.

This hill consists of trachytic rocks. Salic constituents are plagioclase and orthoclase as phenocrysts; the latter predominant. Femic components are augite and a little biotite. One sample contained some sodalite. The average of three samples shows 6.16 % K_2O and 3.87 % Na_2O.

Ragas hill.

Consists also of trachytic rock. In one sample of a trachyte, sanidine phenocrysts in a groundmass of orthoclase and plagioclase with a little biotite were observed. (6.15 % K_2O and 4.86 % Na_2O). Another section shows phenocrysts of sanidine and plagioclase in a groundmass of the same minerals, whereas femic constituents are absent. The analysis of this sample shows 3.88 % K_2O and 3.27 % Na_2O.

Geology. — *The Muriah Volcano (Central Java) and the origin of its leucite-bearing rocks.* By R. W. VAN BEMMELEN. (Communicated by Prof. J. H. F. UMBGROVE.)

(Communicated at the meeting of May 31, 1947.)

The Muriah, or Murjo volcano, is situated on the North coast of Central Java on the labile border of the Sunda Shelf, Northwest of the oil-bearing area of Rembang (Tjepu).

A vertebrate fauna is found in its Southfoot (Patihajam), belonging, according to VON KOENIGSWALD, to the Trinil fauna, thus dating the age of the volcano as Middle-Pleistocene.

In late quaternary time the volcanic structure was domed up so that a number of radial "sector-graben" were formed (see fig. 1.)

The two most important ones are the cauldrons of Rahtawu and Tempur, which are the catchment basins respectively of the southern and the northern Gelis River.

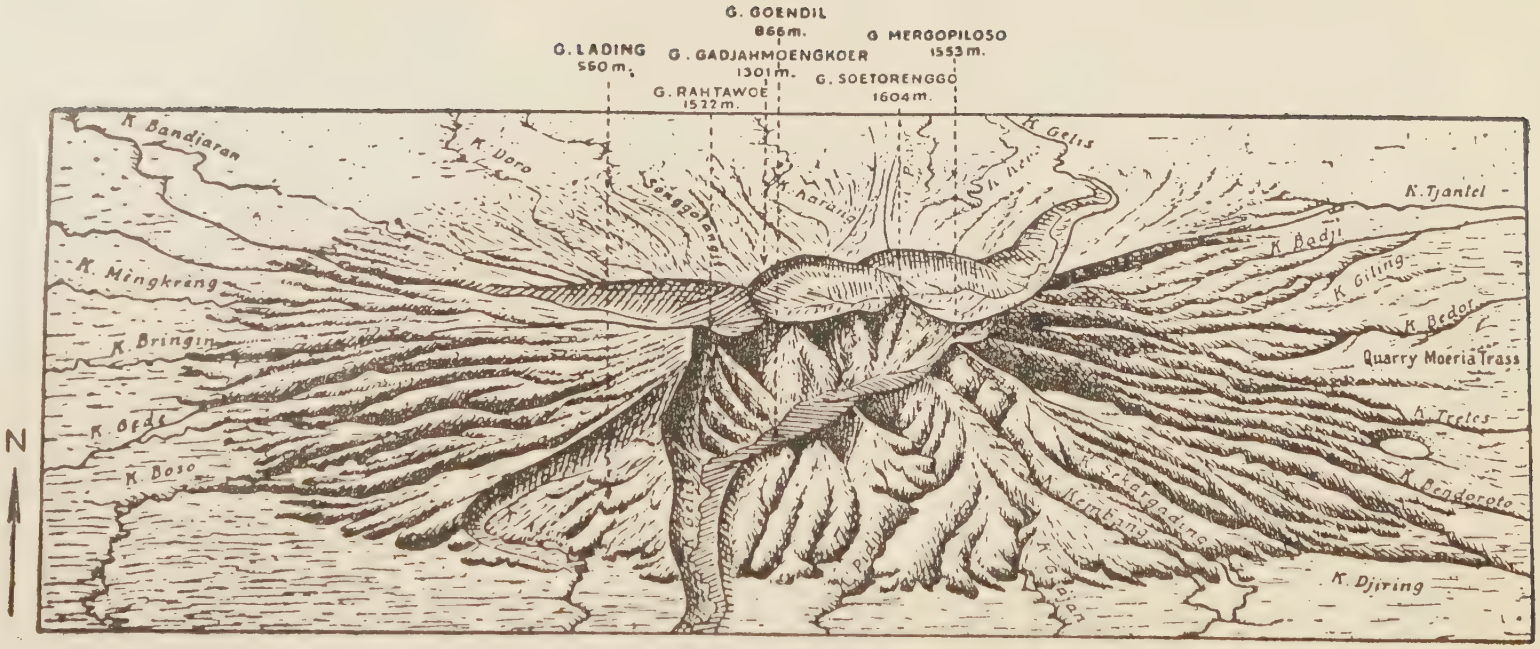
Together these cauldrons form a SSW-NNE rift across the present volcanic ruin, separated by the divide of the Sutorenggo (1604 m, highest summit of the Muriah complex). In these cauldrons the deepest parts of the Muriah are exposed.

At the North foot, the heavily eroded tuffcone of Genuk is found, which is, according to BOOMGAARD, older than the main cone of the Muriah, whilst the Bako and Ragas are younger parasitic centres of eruption.

The basement of the Muriah complex is exposed in Patihajam and in the Genuk. In the former it consists of Globerina marls and clays, presumably belonging to the Turi Beds (Plio-Pleistocene); in the latter limestones with foraminiferas, lamellibranchs and gastropods are found.

Moreover, KUIPER found in the volcanic rocks of the Rahtawu cauldron large inclusions of contactmetamorphic limestones, sandy limestones and marls, which were microscopically studied by the present author. In some samples the following fossils could be determined: *Katacycloclypeus annulatus* Mart., large microspheric and macrospheric *Lepidocyclinae*, and *Cycloclypeus* sp.; these limestones belong to the Rembang Layers, which are of Miocene age. Other samples did not contain *Lepidocyclina*, but only small Foraminifera's (*Rotalia, Operculina, Amphistegina, Globigerina*, etc.), indeterminable lamellibranchs, gastropods, echinids, algae (*Lithothamnium*), corals (*Porides*). These are assigned to the younger neogene (Pliocene) strata of the Rembang sequence.

The clastic constituents of the sandy limestones are cataclastic quartz, undulatory extinguishing felspar-fragments, tourmaline, andalusite, glaucophane, and muscovite. This is detritus derived from the old Sunda land



Isometric Diagram of the G. Moeria (Central Java).

Fig. 1. *Isometric diagram of the Muriah Volcano.*

in the North, which was the source area of the marine deposits of the Rembang geosyncline during the Neogene, and which submerged in late quaternary time, now forming the Java Sea (Sunda Shelf).

The limestones are partly marmorized, containing near the contact recrystallized patches of quartz-mosaic, porphyroblasts of basic plagioclase with broad twinning lamellae and sometimes developed as skeleton crystals intergrown with calcite, besides numerous grains of colourless pyroxene (diopside).

In some slides the direct contact between the volcanic rocks and the limestones could be studied. Near to the contact these volcanic rocks did not contain leucite. It are trachyandesites or latites with phenocrysts of colourless or pale green diopsidic augite (sometimes with polysynthetic twins), biotite (sometimes contorted crystals), basic plagioclase with zonal development, and sanidine with Karlsbad twinning. The latter mineral occurs also in a second generation, together with plagioclase as idiomorphic laths with a fluidal arrangement. The matrix is a turbid glass with pyroxene microlites. Accessory minerals are rather stout and numerous apatite prisms, magnetite and pyrite.

The abundance of biotite and orthoclase indicates a preponderance of potash over soda in the chemical composition.

These potassic trachy-andesitic rocks are separated from the limestone by a narrow chilled edge (about 1 mm thick), in which the turbid glass matrix prevails. This chillface contains further small grains and crystal-agglomerates of pyroxene. These crystals are either colourless, or they show a grading into the green colour of aegirine. Other constituents are idiomorphic laths of alkalifelspar, and rectangular or hexagonal sections of nepheline. Biotite and plagioclase, abundant in the igneous rocks further from the contact, are mostly absent in the chillface. Slender needles of apatite form an accessory mineral.

On account of the presence of aegirine-like pyroxene and nepheline phenocrysts, this chill-face might be called a nepheline phonolite, in which the soda content is presumably higher than the potash. Also in the contactmetamorphic limestones nepheline crystals were found.

The presence of sodium minerals in the chill-face and the contact-metamorphic limestones forms a contrast with the mineral associations of the Muriah volcanic rocks, in which potassium-minerals such as leucite, orthoclase, and biotite are abundant [1]).

This observation is of importance for our insight in the genesis of these Mediterranean suite of the Muriah, as will be explained below.

First the general sequence of the volcanic rocks of the Muriah complex will be given.

The oldest eruption products are common basalts and andesites, mentioned by BOOMGAARD from the Genuk in the North, belonging to the Pacific suite, which is the normal type of magma in these orogenic zones.

Then follows the Mediterranean suite chiefly consisting of leucite-basalts, basanites and tephrites, which form the bulk of the Muriah complex.

The latest eruptions are more leucocratic trachytes, occurring in the parasitic Bako and Ragas cones, further a plagitrachite dike in the Genuk cone, and syenite-porphyry dikes exposed in the Tempur cauldron.

Holocrystalline rocks are represented by a diorite boulder, and a syenite inclusion in leucite tephrite flow, both found by BOOMGAARD, and a sanidinite boulder (collected by the forester of the Muriah area in 1937 and presented to the Geological Museum). The chemical analyses 1—7 were made by the Chemical Laboratory of the Geological Survey, from a collection made in 1937 and microscopically studied by the author. (Nos 1—6 from leucite bearing rocks [2]) and No 7 appertains to the above mentioned sanidinite). Nos 8—15 are taken from IDDINGS & MORLEY (1915) [3]). See table.

The formation of this Mediterranean suite can be explained by DALY'S theory of limestone assimilation.

The contact metamorphic limestone inclusions in the Rahtawu cauldron indicate that the magmatic hearth of the volcano had apparently risen into the neogene Rembang series, which are rich in limestones. Therefore, the roof of the magma chamber was situated at that time (i.e. Lower-Middle Pleistocene) at a depth of some kilometres below the surface. In the initial stage of the volcanic activity, some flows of basalt and andesite were still issued (in the Genuk volcano), which proves that the original composition of the magma corresponded with that of the Pacific kindreds. But, thereafter, the composition was changed by limestone assimilation, which caused a desilication of the magma according to the following theoretical equation:

$$\underset{\text{limestone}}{(Mg\,Co_3)\;Ca\,Co_3} + \underset{\text{magmatic compounds}}{2\,SiO_2\,(+\,MgO)} \rightarrow \underset{\text{diopside}}{Ca\,Mg\,(SiO_3)_2\downarrow} + \underset{\text{resurgent gas.}}{CO_2\uparrow}$$

Besides diopside also basic plagioclase occurs, not only as phenocrysts, but also as porphyroblastic contact mineral. The CaO content of the phenocrysts may be partly derived from the limestone, but as contact mineral it may entirely be formed by the constituents of the marly limestone ($Ca\,CO_3$ and clay-minerals):

$$\underset{\text{limestone}}{Ca\,CO_3} + \underset{\text{clay minerals or magmatic compounds.}}{2\,SiO_2 + Al_2\,O_3} \rightarrow \underset{\text{anorthite}}{Ca\,Al_2\,Si_2O_8} + CO_2\uparrow$$

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[1]) A similar observation was made by IDDINGS and MORLEY (J. of Geol. 1915, p. 237), who found nepheline (and stilbite) only in the numerous cavities of the lavas.

[2]) The author's report with the petrographical descriptions of these rocks was lost during the occupation of the Geological Museum at Bandung by the Japanese.

[3]) J. IDDINGS and E. W. MORLEY, "Contributions to the petrography of Java and Celebes." Journal of Geol. 23, 231—245 (1915).

TABLE 1.

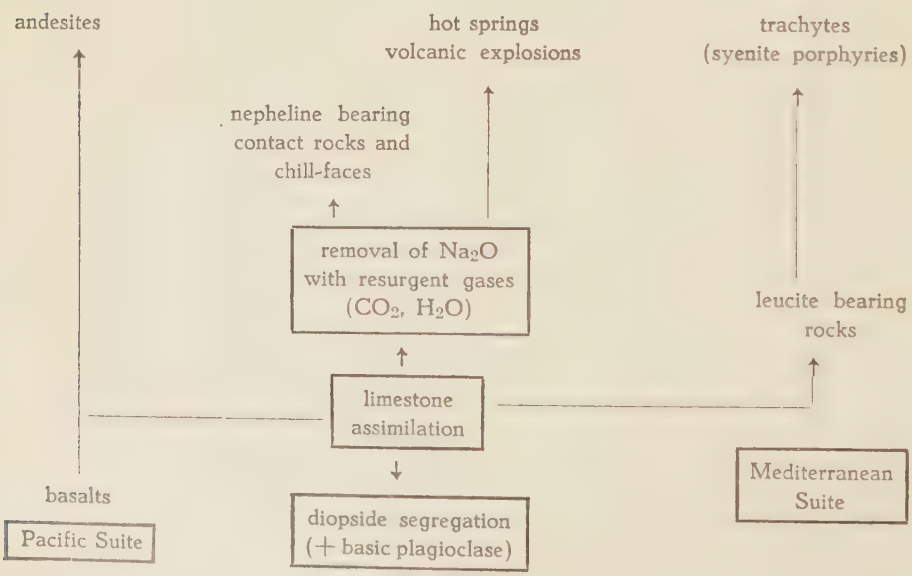
No.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
SiO_2	44.36	48.45	48.80	49.22	51.21	56.67	62.86	45.03	46.54	46.60	47.73	48.32	48.66	50.18	51.85
Al_2O_3	13.98	18.52	19.64	16.81	20.84	18.09	19.06	16.59	15.95	16.73	17.93	17.81	17.69	17.82	19.08
Fe_2O_3	3.98	3.73	2.88	5.21	3.06	3.94	1.15	4.55	5.24	4.17	4.47	4.65	4.66	4.04	4.25
FeO	5.48	4.17	3.26	4.19	1.95	2.59	0.15	6.37	5.51	4.78	4.58	4.62	4.40	3.89	2.69
MnO	0.17	0.20	0.22	0.17	0.19	0.15	0.07	0.64	0.18	0.41	0.96	0.41	1.49	0.30	0.51
MgO	10.26	2.98	1.72	4.21	1.00	1.74	0.10	3.95	4.70	4.69	4.27	3.37	3.03	2.88	1.48
CaO	12.35	8.39	6.05	9.67	5.58	5.63	1.57	11.09	10.69	10.82	9.58	9.15	6.43	7.19	5.81
Na_2O	2.64	4.47	3.98	2.63	4.85	3.92	3.88	3.53	2.28	2.62	3.62	3.14	3.93	3.29	4.46
K_2O	2.94	5.51	8.32	3.81	6.91	5.10	9.50	5.29	4.44	5.47	4.81	4.79	6.10	6.65	6.61
H_2O+	1.50	1.02	3.33	1.71	2.41	0.57	1.11	0.34	0.52	0.71	0.44	0.82	0.80	0.96	0.55
H_2O-	0.56	0.62	0.46	0.86	1.36	0.52	0.36	0.15	0.59	0.45	0.24	0.17	0.58	0.55	0.47
TiO_2	1.21	0.77	0.58	1.13	0.46	0.73	—	0.10	1.11	0.95	0.86	0.88	0.81	0.76	0.66
P_2O_5	0.93	0.68	0.26	0.44	0.04	0.33	0.04	0.96	1.18	1.50	0.52	0.82	0.79	0.76	1.23
Total	100.36	99.51	99.50	100.06	99.86	99.98	99.85	99.59	98.93	99.90	100.02	98.95	99.37	99.27	99.65
Spec. Density	2.900	2.729	2.607	2.799	2.460	2.673	2.379	Leucite shoshonite	Leucite basalt	Leucite tephrite	Biotite vicoite	Leucite tephrite	Leucite tephrite	Leucite tephrite	Leucite shoshonite

Meanwhile part of the soda of the magma escaped with the resurgent gases into the contact aureole, or by volcanic explosions, because soda has a greater affinity to carbondioxyde than potash. This escape of the soda appears from the presence of sodic minerals, such as nepheline, in the above described chill-face and in the contact metamorphic limestones.

It is conceivable that especially along the contact an upward current of resurgent gases occurs, carrying along Na_2O as a chemical compound. This marginal contact zone was suddenly cooled down during the eruption and consolidated into a chill-face, in which the sodium-felspatoid (nepheline) could crystallize, whilst also the diopsidic pyroxene phenocrysts were partly altered by a kind of autopneumatolysis into a sodium-rich, aegirine-like pyroxene. Consequently, this chill-face assumed the appearance of a sodium-rich Atlantic rock type. Such in contradistinction to the bulk of the magma, which belongs to the Mediterranean suite. In the latter the potash felspathoid (leucite) could be formed, due to the desilication and the passive enrichment of the potash content.

In the later stages of the eruption cycle more leucocratic rocks were produced (trachytes, syenite-porphyry dikes). These rocks might be interpreted as the result of progressive crystallization differentiation in the hearth.

On account of these observations and considerations it seems to be very probable, that the Mediterramean kindreds of the Muriah complex were derived from the Pacific magma type by a "pathological" side branch of differentiation caused by intensive limestone assimilation, according to the following scheme:



Oceanography. — *On the desirability of a research into certain phenomena in the region of upwelling water along the coast of South West Africa.* By MARGARETHA BRONGERSMA—SANDERS, D.Sc. (Communicated by Prof. H. BOSCHMA.)

(Communicated at the meeting of May 31, 1947.)

In some bays on the coast of South West Africa (especially in Walvis Bay) and sometimes also in the open sea outside of them a mass mortality of fish takes place periodically. Some mortality takes place every year; very great mortalities, however, occur in some years only. They occur always in the southern summer; very great mortalities occur mostly in December and especially at Christmas time.

The surroundings of Walvis Bay are further characterized by a very peculiar sediment on the sea bottom (see VON BONDE, 1928; MARCHAND, 1928; COPENHAGEN, 1934). The peculiarites of this sediment are the following: the presence of a great quantity of H_2S; a high organic content; nearly complete absence of living organisms (benthonic invertebrates, the scavengers of the seabottom are absent!), anaerobic bacteria excepted; abundance of fish remains; a very high % of skeletons of diatoms. I must stress the fact, that the peculiarities of this sediment are not caused by stagnation of the lower water layers. Besides in the Walvis Bay itself the sediment occurs also in the open sea outside it; and there is, as far as I am aware, no barrier preventing circulation whatever. The area, in which this sediment occurs, stretches nearly from Cape Cross to south of Conception Bay (21° 30′—24° 30′ S), a distance of approximately 200 miles and from the coast line 25—30 miles west (i.e., about to the 77 fathom line). On account of the absence of living organisms in the sediment the area is called the Azoic zone of the West coast. In this area there is a coastal belt between the Azoic zone and the shore extending from Pelican Point to 23° 38′ S where the bottom consists of fine grey sand, and the absence of green sulphurous mud being noteworthy.

Hypotheses put forward in recent publications (REUNING, 1925; MARCHAND, 1928; CLASSEN, 1930; COPENHAGEN, 1934) agree in the supposition, that H_2S is the cause of the mortality, although opinions differ as to the origin of the gas. Except COPENHAGEN the authors agree, that the H_2S derives from sulphur compounds of the land, and is brought to the sea by the river Kuiseb; the coast region being very rich in sulphur minerals. According to MARCHAND sulphides and sulphates are carried down as such by the Kuiseb and broken up by chemical action with the sea water setting free noxious compounds and gazes fatal to fish. The periodicity of the mortality is explained by MARCHAND by the periodical rainfall in the hinterland and consequently by the periodical emanating of water of the Kuiseb

river (the lower course of this river is choked up with sand and communication with the sea is by seepage only) on the bottom of the bay (Kuiseb underground water hypothesis). CLASSEN undependently arrived at nearly the same hypothesis. REUNING although an adherent of the Kuiseb underground water hypothesis contends, that this is not the sole cause, but that there exist beds of pyrites beneath the bay.

COPENHAGEN (1934) was the first who brought to the fore, that the sulphur compounds of the so-called azoic region are not derived from the land. According to this author the H_2S is formed by reduction of sulphates present in the watery sediment under influence of anaerobic bacteria. Periodically, i.e., in the summer months gas would escape in great quantity and would cause a mass mortality of fish.

That the H_2S of the azoic region has not been transported there by rivers, viz., that it does not originate from the sulphur compounds of the land, but that it results from the reduction of sulphates of the sea water, is certainly correct. However, I cannot agree with the supposition, that H_2S is the cause of the mortality. In my opinion the mortality is caused by noxiousness of red water of dinoflagellates; the occurrence of red water in its turn being the result of the presence of upwelling water. On account of this upwelling nutrients are available nearly all the year round in a high concentration. As a result the production of zoo/phytoplankton is excessively great. In this very eutrophic surroundings at certain times of the year a "waterbloom" occurs of some elements of the phytoplankton. MARCHAND after defending the Kuiseb underground water hypothesis adds at the end of his paper, that during occurrences of mortality the water along this area of the coast is of a blood red colour: "Now, some writers have suggested that this is also due to the flood waters from the rivers. Such, however, is not the case, for I have seen this red water and have taken some of it and examined it under the microscope when the red colour is revealed to be due to masses of a species of phosphorescent organism, viz., *Noctiluca*. Such red water occurrences are quite common on the South African Coast. "Dark water", due to masses of diatoms in the sea is also seen quite often on this part of the West Coast and I would suggest here, that sometimes minor occurrences of fish mortality are due not to the setting free of noxious compounds and gases from the bottom, but to vegetable decay of these masses of *Noctiluca* and diatoms polluting the water." (MARCHAND, 1928, p. 3).

I do not agree with MARCHAND in the following respects: In the first place not minor, but *all* periodical occurring mass mortalities in this region are caused by noxiousness of the plankton. Secondly in MARCHAND's opinion the noxious effect is pollution of the water by a great mass of dead plankton; it being indifferent if dinoflagellates are present or a mass of other dead phytoplankton organisms, e.g., diatoms. However, the mortality occurs only during occurrences of red water, and, therefore, only red water organisms are responsible.

Dinoflagellates are lovers of warmth; therefore, a mass development of dinoflagellates, and with that great outbreaks of red water will occur especially in that time of the year, when the temperature of the surface water is relatively high. The highest temperatures are to be met with in the southern summer. This is in my opinion the explanation of the fact, that the mortalities, as they are caused by great outbreaks of red water, always occur in the southern summer.

The above mentioned hypothesis has to be justified by researches in loco; this being impossible during the war, I searched for another proof, which I found by raising the following question: If in the Walvis Bay region upwelling water is the cause of the occurrence of red water and in its turn red water causes mass mortality of fish, does one find in other regions of upwelling water fish mortality caused by red water too? To answer this question I made a study of the literature on cases of mass mortality of fish (and also of invertebrates); the answer proved to be in the affirmative (BRONGERSMA, 1943; 1944; 1947). In the literature on dinoflagellates the trio upwelling water — red water — mass mortality appeared to be known already from the Californian region. However, as far as I am aware, the question whether they occur in other regions of upwelling water too, has never been put forward. In those regions of upwelling, where the phenomenon is very intense (South West Africa; Peru-Chile) red water and mass mortality occur very often; near Walvis Bay sometimes 4 or 5 times a season. In other regions of upwelling (for instance near the coast of California) these phenomena occur on rare occasions only (ALLEN, 1941).

From the above mentioned it is obvious, that there are certain regions (those peculiar regions of upwelling water), where red water and mass mortality of fish (and invertebrates) take place periodically or episodically on account of similar causes. Such regions undoubtedly have existed in the geological past too.

As has been emphasized by BRONGERSMA (1944; 1947) in certain parts of regions of upwelling the quantity of organic material reaching the sea bottom is so enormous, that anaerobic conditions originate and as a result a rather great part of it does not get oxidized [1]). This is the result of various factors: of the enormous great year production of the plankton in regions of upwelling, of the phenomena of red water and mass mortality, etc. Not only the high organic content, but all the above mentioned peculiarities of the azoic region near Walvis Bay are in last instance caused by upwelling water (for the argumentation see BRONGERSMA, 1947). It will be obvious that if these peculiarities occur combined in a fossil deposit, the chance is rather great, that this sediment originated in a region of upwelling water too. In this respect it is noteworthy, that the sediment of the azoic

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[1]) The possibility is not excluded that the peculiar sediment found by the John Murray expedition in the vicinity of Cape El Hadd, S.E. Arabia (SEYMOUR SEWELL, 1934; 1935; STUBBINGS, 1939; WISEMAN & BENNETT, 1940) owes its origin to the presence of upwelling water too.

region shows a remarkable resemblance with certain bituminous fish shales; the similarity being particularly striking with those shales that consist for a considerable part of the siliceous remains of fossil diatoms (Californian miocene Monterey shales; oligocene menelite shales of the Carpathians, etc.). These shales are considered by most oil geologists as source beds of petroleum.

To decide if the sediment of the azoic region near Walvis Bay is of importance with regard to oil geology, it must be settled if this sediment is gyttja or sapropelium. The presence of H_2S in a sediment is certainly no proof that it is sapropelium, for in the deeper layers of gyttja H_2S occurs just as well. The point is where the boundary O_2/H_2S is to be met with. It seems probable that in the sediment of the azoic region the boundary lies nearly at the surface of the sediment, whereas in certain times of the year (during heavy outbreaks of red water) it lies in the free water. A sharp boundary between gyttja and sapropelium probably does not always exist in nature. There will be transitional stages, one being nearer to gyttja, the other to sapropelium. In any case the sediment of the azoic region is not a typical gyttja; a character of the latter being the presence of benthonical invertebrates. As the latter are very scarce in this region a great part of the sediment will be converted by bacterial action only. It seems probable that the sediment of the azoic region, at least parts of it, approach to a true sapropelium. Most oil geologists are nowadays of the opinion that a marine sapropelium may originate only by stagnation of the lower water layers. If, however, the sediment of the azoic region by a further analysis (exact determination of the boundary O_2/H_2S; labile organic compounds preserved?) appears to be (near to) a sapropelium indeed, proof is furnished that a sapropelium of considerable extent may originate in the open sea without any water stagnation.

According to MARCHAND (l.c.) fish remains are very numerous in the sediment of the azoic region; when trawling in this zone fish bones are brought up by the bucketful. "The area seems to be a burialground for fish." In recent marine sediments remains of fish are usually very scarce or absent. Otoliths and isolated teeth are the only fish remains, that occur rather frequently, but the softer bones like the vertebrae and the ribs are very scarce. There are, however, fossil deposits in which fish remains occur in great quantity. SMIRNOW (1930) in dealing with the oligocene of the northern Caucasus speaks of a churchyard of fish. Besides isolated parts of fish often nearly complete skeletons are to be met with. Because such deposits were formed in the geological past, it is worth while to search for an analogon in recent time. It is, therefore, desirable to study those rare sediments in which fish remains occur in a comparatively great quantity.

For a fossilisation of vertebrates a very quick embedding after death is a first requirement. The latter is possible in two ways: either a rather thick layer of sediment is deposited directly after the death of the fish (for instance if the cause of death is a volcanic one), or the fish sinks on, and

partly or wholly sinks away in a soft sediment, in which anaerobic conditions prevail. Therefore, it seems probable, that part of the fish perished by the mass mortalities, at least as far as they sink down on the peculiar sediment of the azoic region, have a good chance to fossilize. As it has turned out by the present research that the mass mortalities as well as the peculiar sediment are caused by the presence of upwelling water, these regions are probably of importance to vertebrate paleontology. With regard to the possibility that in the sediment of the azoic region fossils occur in statu nascendi, it is desirable to study if fish remains occur in the deeper layers of the sediment and in what state of preservation.

The records of red water and mass mortality of all parts of the world summarized in BRONGERSMA (1947) will make it obvious, that it is not a mere hypothesis but a well established fact, that red water of dinoflagellates may cause mass mortality among fish and invertebrates. Opinions differ, however, greatly as to the question, how the mortality is brought about. The supposition is made by some authors, that the death of fishes and invertebrates is caused by the accumulation of toxic substances in the sea water following the decay of large quantities of plankton. A second hypothesis is, that the myriads of dinoflagellates cause asphyxiation by clogging the gills. In view of recent researches on paralytic shellfish poisoning, by which it is shown, that a powerful toxin may be produced by the living dinoflagellate plankton, this assumption probably has to be revised.

Paralytic poisoning of men by eating mussels is rather rare in Europe. On the Pacific coast of North America (upwelling water!) shellfish poisoning is a rather common occurring phenomenon. By research work started in California in 1927 it has turned out, that the agent responsible for the toxicity of the mussels is contained in the ocean water and approaches the shellfish beds more or less periodically (that is in the summer months) from offshore. There appears to be a close relation between the number of certain dinoflagellates and the poisonousness of the mussels, the annual maxima of certain species of *Gonyaulax* appearing to occur preceding and during each poison period. In 1933 the poison was isolated directly from dinoflagellate plankton.

Shellfish themselves appear to be resistant to comparatively large quantities of the poison; the same applies to other cold blooded invertebrates and vertebrates. Although the resistance of cold blooded animals is much higher, it still seems very probable, that paralytic shellfish poison or a related poison produced in the living dinoflagellate plankton is the cause of the mass mortality of fishes and invertebrates occurring periodically or episodically in various parts of the world. This conception is advocated already by SOMMER et al. (1937) with regard to the mortality of marine animals during the San Diego occurrences of red water: "Quantities of shellfish poison much larger than have been measured in the San Francisco region, quite possibly existed in the living plankton at that time, but since tests for toxin were not made this cannot be checked. Perhaps if the

numbers of *Gonyaulax catenella* and related species associated with it during the present investigation had been greater, the quantities of poison extracted from the digestive glands of the mussels might have been many times larger than the tests have indicated, and the poison, possibly of sufficient potency to have caused either metabolic disturbances in the mussels or their death."

Along the coast of California red water and mass mortality occur on rare occasions only. Outbreaks of red water are very rare (1901, 1907, 1917, 1924, 1933, 1938, 1939); whereas an attending mortality is even more rare. Therefore, it may last many years, before the hypothesis, that the mortality is brought about indeed by a poison related to paralytic shellfish poison, can be tested. In the Walvis Bay region, however, mortality occurs nearly annually. Therefore, this is the obvious place to test the above mentioned hypothesis.

The war now being over oceanographical research can be started again: and, therefore, it is possible to justify the above mentioned hypotheses by researches in loco.

Summarizing the following studies have to be made:

A. *Red water and mass mortality.* A qualitative and quantitative study of the plankton all the year round. What dinoflagellates except *Noctiluca* may be responsible for outbreaks of red water along this coast? What species of dinoflagellates may be responsible for mortality? As mortality occurs only during heavy outbreaks of red water the following must be studied: What is the extent of the red water and what is the number of dinoflagellates per litre during outbreaks being heavy enough to cause mortality? What oceanographical peculiarities (a high temperature being the result of a decrease of the upwelling?) coincide with the occurrence of red water? What oceanographical peculiarities are present in years of abnormal great outbreaks of red water and with that in the years of abnormal great mortality? Does the region of the most frequent occurrence of red water coincide with the azoic region?

B. *Sediment.* A further analysis of the sediment of the azoic region. Which of both is this sediment, gyttja or sapropelium? Labile organic compounds preserved? Cu, Ni, Va or Mo perhaps present? Does the boundary O_2/H_2S lie at some distance under the surface of the sediment or is H_2S present up to the very surface? Determination of organic carbon and nitrogen from the top layer as well as from layers at a depth of some dm under the surface.

C. *Poison.* Is the fish mortality brought about by a poison related to paralytic shellfish poison, the latter being produced by the living dinoflagellates during occurrences of red water?

Does paralytic poisoning of man by eating mussels in the southern summer occur on this coast?

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Zoology. — *On the subspecies of Python curtus Schlegel occurring in Sumatra.* By L. D. BRONGERSMA. (Communicated by Prof. H. BOSCHMA.)

(Communicated at the meeting of May 31, 1947.)

STULL (1935, p. 393) distinguishes between two subspecies of *Python curtus* Schl., one of these is recorded from Malaya, the other from Sumatra and Borneo. In a subsequent paper this author (STULL, 1938, p. 297) recognizes three subspecies, viz., *Python curtus curtus* Schl. from Sumatra, *Python curtus brongersmai* Stull from Malaya, and *Python curtus breitensteini* Steind. from Borneo. Of these only the first two will be discussed in this paper.

Python curtus brongersmai Stull is characterized by a high ventral count (168—174), by having two supraoculars on each side, and by two upper labials entering the orbit (STULL, 1938, p. 297). *Python curtus curtus* Schl. has a low ventral count (152—156), a single supraocular on each side, and the upper labials separated from the orbit by subocular scales. In a footnote STULL (1938, p. 297, note 2) mentions a specimen, which this author considers as being problematical; although from Sumatra, it agrees with *Python curtus brongersmai* (ventrals 171, two labials entering the orbit) rather than with *Python curtus curtus*.

A revisional study of the specimens of *Python curtus* in the collections of the Rijksmuseum van Natuurlijke Historie, Leiden, and of the Zoologisch Museum, Amsterdam, showed that both *Python curtus curtus* and *Python curtus brongersmai* occur in Sumatra.

Python curtus curtus Schl.

Python curtus SCHLEGEL, Dierentuin, 1872, Kruipende Dieren, p. 54, fig. (Sumatra); O'SHAUGHNESSY, Zool. Record for 1877, 1878. Rept., p. 10; HUBRECHT, Notes Leyden Mus., vol. 1, pt. 4, 1879, p. 244; BOULENGER, Proc. Zool. Soc. Lond., 1889, p. 432 (part.); BOULENGER, Fauna Brit. Ind., Rept. Batr., 1890, pp. 245, 246 (part.); BOULENGER, Cat. Sn. Brit. Mus., vol. 1, 1893, p. 89 (part.); FLOWER, Proc. Zool. Soc. Lond., 1896, p. 877 (part.); FLOWER, Proc. Zool. Soc. Lond., 1899, p. 656 (part.); BOULENGER, Rept. Batr., Vertebr. Fauna Mal. Pen., 1912, p. 109 (part.); DE ROOIJ, Rept. Indo-Austr. Arch., vol. 2, 1917, pp. 20, 28 (part.); ROBINSON and KLOSS, Journ. Fed. Mal. St. Mus., vol. 8, pt. 2, 1920, p. 301 (part.); WERNER, Arch. Naturg., vol. 87, Abt. A, pt. 7, 1921, p. 240 (part.); FLOWER, List Vertebr. Anim. Gardens Zool. Soc. Lond. 1828—1927, vol. 3, 1929, p. 159 (part.).

Python curtus breitensteini, STULL, Proc. Boston Soc. Nat. Hist., vol. 40, no. 8, 1935, p. 393 (part.).

Python curtus curtus, STULL, Occ. Papers Boston Soc. Nat. Hist., vol. 8, 1938, p. 297.

Schlegel's Python. LOVERIDGE, Reptiles Pacific World, edition Infantry Journal, 1945, p. 116, and edition Macmillan Company, 1945, p. 122.

Specimen examined:

1 ♂, juv., Sumatra, leg. S. MÜLLER, Mus. Leiden, Herp. reg. no. 3782.

This specimen is the type of the species. It is labelled "Sumatra" without a further indication as to the locality. However, it is known that MÜLLER in the years 1833—1835 collected in the western part of the island of Sumatra, in the surroundings of Padang and in the Padang Highlands (VETH, 1879, pp. 75, 77). HUBRECHT (1879, p. 245) mentions that this snake was procured "from that part of Sumatra which lies between Padang and Indrapura". I have not been able to trace the source from which HUBRECHT derived this evidence.

The scale counts of this specimen are given in table I. It further presents the following characters. Anal single; 2 preoculars and 2 postoculars. Two narrow subocular scales on each side, separating the upper labials from the orbit. The rostral and the anterior two labials are deeply pitted. Lower labials 17, the 2nd to 5th, and the 8th to 13th pitted. Loreals: on each side one large shield, 10 smaller shields and some granules. Total length 517 mm, tail 43 mm (i.e., 8 % of the total length).

TABLE I. *Python curtus curtus* Schl.

Specimen	Scale rows			Ventrals	Subcaudals	Upper labials	Supra-oculars	
	Neck	Mid-body	In front of vent				r	l
Type	50	57	32	157	$^2/_2+1+^{23}/_{23}+2+^2/_2+1$	11	1	1
MT. KABOR [1]	47	57	31	156	total : 31	11	1	1
KABA WETAN [1]	47	59	35	152	total : 31	11/10	1	1

[1] Dr. O. S. DAVIS, in litteris.

To this subspecies belong two specimens examined by STULL (1938, p. 297). This author (Dr. O. S. DAVIS, in litteris) kindly supplied me with some data concerning these specimens. One is a female from the vicinity of Mt. Kabor. This locality I presume to be the "Kabor" mentioned by ANONYMUS (1869, p. 3) as an area north east of Fort de Kock in the Padang Highlands. The other specimen is a male from Kaba Wetan, an estate (ANONYMUS, 1939, p. 167) near Kelobak (or Klobak) in the sub-district of Kepahiang (or Kepajang) in the Benkulen Residency. The scale counts of these two specimens have been incorporated in table I. In both specimens the upper labials are separated from the orbit.

Python curtus brongersmai Stull

Python curtus, BLANFORD, Proc. Zool. Soc. Lond., 1881, pp. 216, 222 (Singapore); SECRETARY, Proc. Zool. Soc. Lond., 1889, p. 393 (Malacca); BOULENGER, Proc. Zool. Soc. Lond., 1889, p. 432 (part.), pl. 44; BOULENGER, Fauna Brit. India, Rept. Batr., 1890, pp. 245, 246 (part.); BOULENGER, Cat. Sn. Brit. Mus., vol. 1, 1893, p. 89 (part.); FLOWER, Proc. Zool. Soc. Lond., 1896, p. 877 (part.); S[CLATER], List Vertebr. Anim. Gardens Zool. Soc. Lond., 9th ed., 1896, p. 607; FLOWER, Proc. Zool. Soc. Lond., 1899,

pp. 604, 656 (part.); RIDLEY, Journ. Straits Br. Roy. As. Soc., 1899, pp. 197, 207; WERNER, Zool. Jahrb., Syst., vol. 13, 1900, p. 489 (Surbo Dolok, Sumatra), p. 500 (part.); SCHENKEL, Verh. Natf. Ges. Basel, vol. 13, pt. 1, 1901, p. 154; BOULENGER, Rept. Batr., Vertebr. Fauna Mal. Pen., 1912, pp. 107, 109 (part.); DE ROOIJ, Rept. Indo-Austr. Arch., vol. 2, 1917, pp. 20, 28 (part.), figs. 8, 15 (Medan, Sumatra); BOULENGER, Journ. Fed. Mal. St. Mus., vol. 8, pt. 2, 1920, p. 289 (Siolak Daras, Korinchi Valley, Sumatra); ROBINSON and KLOSS, Journ. Fed. Mal. St. Mus., vol. 8, pt. 2, 1920, p. 301 (part.); WERNER, Arch. Naturg., vol. 87, Abt. A, pt. 7, 1921, p. 240 (part.); SWORDER, Singapore Naturalist, no. 2, 1923, p. 59; V., Tropische Natuur, vol. 13, 1924, p. 109 (part.) (Banka); VAN OORT, Verslag 's Rijks Mus. Nat. Hist. 1924—1925, 1925, p. 23; WERNER, Sitz. Ber. Ak. Wiss. Wien, Mathem. Naturw. Kl., Abt. I, vol. 134, 1925, p. 45 (Medan, Sumatra); WERNER, Miscell. Zool. Sumatr., no. 19, 1927, p. 1 (Medan, Sumatra); FLOWER, List Vertebr. Anim. Gardens Zool. Soc. London 1828—1927, vol. 3, 1929, p. 159 (part.); VAN OORT, Verslag 's Rijks Mus. Nat. Hist. 1928—1929, 1929, p. 36 (Medan, Sumatra); VAN DER MEER MOHR, Tropische Natuur, vol. 19, 1930, p. 156 (part.), fig. 2 (egg); SMITH, Bull. Raffl. Mus., no. 3, 1930, p. 39; DITMARS, Bull. New York Zool. Soc., vol. 38, pt. 5, 1935, p. 165, fig. (Sumatra); NOBLE, Copeia, 1935, no. 1, p. 1 (brooding habit); VAN DER MEER MOHR, Tropische Natuur, vol. 25, 1936, p. 23; KOPSTEIN, Bull. Raffles Mus., no. 14, 1938, p. 131; WESTERMANN, Treubia, vol. 18, 1942, p. 163 (Banka).

Aspidoboa curta, SAUVAGE, Bull. Soc. Philom. Paris, ser. 7, vol. 8, 1884, p. 143 (Sumatra).

Python curtus curtus, STULL, Proc. Boston Soc. Nat. Hist., vol. 40, no. 8, 1935, p. 393 (nec SCHLEGEL); STULL, Occ. Papers Boston Soc. Nat. Hist., vol. 8, 1938, p. 297, note 2 (problematical specimen from Sumatra).

Python curtus brongersmai STULL, Occ. Papers Boston Soc. Nat. Hist., vol. 8, 1938, p. 297 (Singapore, Kuala Lumpur), p. 298 (Malacca, Kuala Lumpur).

Specimens examined:

1 ex., Medan, Deli, Sumatra, from Zoological Gardens, Rotterdam, Mus. Leiden, Herp. reg. no. 5427.

1 ex., Sumatra, from Zoological Gardens, Rotterdam, Mus. Leiden, Herp., reg. no. 8457.

1 ex., Medan, Deli, Sumatra, leg. Dr. L. P. LE COSQUINO DE BUSSY, 21. IX. 1908, Zool. Mus. Amst. (specimen a of table II).

2 ex., Deli, leg. Dr. L. P. LE COSQUINO DE BUSSY, 1920, Zool. Mus. Amst. (specimens *b, c*).

1 ex., probably Deli, leg. Dr. L. P. LE COSQUINO DE BUSSY, Zool. Mus. Amst. (specimen *d*).

1 ex., Deli, leg. Dr. KUIPER, Zool. Mus. Amst. (specimen *e*).

1 ex., Deli, leg. E. H. KONING, Zool. Mus. Amst. (specimen *f*).

1 ex., Deli, leg. W. P. V. DE ZWART, 1919, from Zoological Gardens, Amsterdam, Zool. Mus. Amst. (specimen *g*).

2 ex., N. E. Sumatra, leg. Jhr. F. C. VAN HEURN, 1920, Zool. Mus. Amst. (specimens *h, i*).

1 ex., estate Tinjoang, Sungei Bedjangkar, Assahan, N. E. Sumatra, leg. VAN ERP, from Zoological Gardens, Amsterdam, 2. IX. 1937, Zool. Mus. Amst. (specimen *j*).

1 ex., Kuala Simpang, S.E. Atchin, Sumatra, Zool. Mus. Amst. (specimen *k*).

1 ex., Port Dickson, Negri Sembilan, Malaya, 17. I. 1946, presented by the Sublieutenants of the Royal Netherlands Navy, Mus. Leiden, Herp. reg. no. 8316.

The scale counts for the individual specimens are given in table II. The characters of this series of Sumatran specimens may be summarized in the

following notes. The series has been compared to a specimen from Port Dickson, Malaya; this specimen will be described more extensively in another paper.

Scales in 46—53 rows on neck, in 55—59 rows at mid-body, and in 32—35 rows in front of vent. Ventrals 167—175, average 171 (12 specimens); anal single; subcaudals 25—36, some of them single. Supraoculars 1 or 2, rarely 3; there are two supraoculars on each side in 7 specimens; one on each side in 3 specimens; two supraoculars on one side and one shield on the other side occur in 2 specimens. Specimen *b* has three supraoculars on the right side, of which the middle one is very narrow; on the left side this specimen has only one supraocular, which has two short incisures in its medial border, thus also pointing to a division into three shields. All specimens have 2 preoculars and 2 postoculars. Upper labials 11 or 12; the rostral and the anterior two labials deeply pitted. No suboculars, the 5th and 6th labial (6 specimens) or the 6th and 7th labial (5 specimens) entering the orbit. In specimen *f* the 5th upper labial of the right side is narrowly separated from the orbit by the 6th labial, the latter alone entering the orbit; on the left side both the 5th and 6th labials enter the orbit. In specimen *g* the 7th upper labial alone enters the orbit on the left side, narrowly separating the 6th labial from the orbit; on the right side the 6th and 7th upper labials enter the orbit. Lower labials 16—20, generally 19 (6 specimens) or 20 (5 specimens), 16 and 18 occurring only in one specimen each. Of these generally the 2nd to 5th (8 specimens) and the 13th to 17th (7 specimens) are pitted. In some specimens the 2nd to

TABLE II. *Python curtus brongersmai* Stull

Specimen	Scale rows			Ventrals	Subcaudals	Upper labials	Supraoculars	
	Neck	Mid-body	In front of vent				r	l
No. 5427	52	56	32	169	$^{1}/_{1}+2+^{25}/_{26}+.:.$	12 (6, 7)	2	2
No. 8457	50	55	33	171	$^{4}/_{4}+4+^{20}/_{20}+1$	11 (6, 7)	2	2
a	47	58	35	173	$^{5}/_{5}+5+^{20}/_{20}+6$	12 (6, 7)	2	2
b	51	57	35	171	$^{4}/_{5}+1+^{1}/_{1}+^{17}/_{18}+1$	11 (5, 6)	3	1
c	—	55	—	± 174	$^{27}/_{27}+3$	11 (5, 6)	1	1
d	—	—	—	171	$^{1}/_{1}+1+^{20}/_{20}+4$	12 (6, 7)	2	2
e	46	57	33	$173+^{2}/_{2}$	$7+^{19}/_{19}+1$	11 (5, 6)	1	2
f	52	56	32	167	$^{2}/_{2}+4+^{24}/_{24}+1$	11 (r. 6; l. 5, 6)	1	1
g	51	58	32	169	$^{28}/_{28}+1$	12 (r. 6, 7; l. 7)	2	1
h	49	57	32	167	$^{22}/_{21}+2+^{3}/_{4}+1$	11 (5, 6)	2	2
i	52	57	32	171	$^{4}/_{4}+2+^{22}/_{22}+1$	11 (5, 6)	2	2
j	49	57	33	173	$^{1}/_{1}+1+^{1}/_{1}+2+^{23}/_{23}+1$	11 (5, 6)	2	2
k	53	59	34	175	$^{29}/_{29}+1$	12 (6, 7)	1	1
WERNER, 1900	—	—	—	171	$^{31}/_{30}+1$	r. 10 (4, 5) l. 12 (5, 6)	—	—
WERNER, 1925	—	—	—	174	$^{32}/_{31}+1$	9 (4, 5)	—	—
SIOLAK DARAS [1])	—	57	—	171	total : 29	two entering orbit	1	1

[1]) Dr. O. S. DAVIS, in litteris.

4th, 2nd to 6th or 7th, as well as the 11th, 12th or 13th to 16th or the 12th to 15th are pitted. Loreals very variable. Generally there are one or two large shields, with some smaller shields and some granules; the smaller shields and granules varying from 5 to 9 or even 13 in all.

The largest Sumatran specimen examined (*k*) has a total length of 1640 mm, tail 120 mm (i.e., 7.3 % of the total length).

The notes given above show that the number of supraoculars varies from 1 to 3 in *Python curtus brongersmai*; therefore, this number cannot be used to separate this subspecies from *Python curtus curtus*. The only well marked differences between these two species consist of the number of ventrals and of the presence or absence of subocular scales. Therefore, the problematical specimen mentioned by STULL (1938, p. 297, note 2) must be referred to *Python curtus brongersmai*; it is the specimen from Siolak Daras (or Sioelak Deras) in Korinchi Valley, first recorded by BOULENGER (1920, p. 289).

A search of literature brought to light several instances in which specimens referable to *Python curtus brongersmai* Stull had been recorded from Sumatra, viz., the records by SAUVAGE (1884, p. 143), WERNER (1900, p. 489; 1925, p. 45; 1927, p. 1), DE ROOIJ (1927, p. 28, figs. 8, 15). From the island of Banka, this species was recorded for the first time by V. (1924, p. 109), while recently WESTERMANN (1942, p. 613) mentioned two specimens from this island.

It is as yet impossible to indicate sharply defined areas in Sumatra for each of these two subspecies. *Python curtus curtus* Schl. has been recorded from the Residency West Coast of Sumatra (Padang Highlands and coastal area (type?)), as well as from the Benkulen Residency (Kaba Wetan). *Python curtus brongersmai* Stull is known from North East Sumatra (Sultanate of Deli and surrounding territory; S.E. Atchin) as well as from Korinchi Valley in the Residency West Coast of Sumatra. Future collecting will have to show whether the areas inhabited by these subspecies are sharply delimited, or whether some overlapping occurs.

The native name in Malaya is Ular sawah darah (cf. BOULENGER, 1912, p. 109); in the island of Banka it is known as Ular bakas. Both V. (1924) and WESTERMANN (1942, p. 613) give notes on the belief of the Malay and Chinese in Banka, that this snake may cause leprosy.

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Physics. — *The intensities of direct and scattered X-radiation in a horizontal water sheet exposed to a cylindriform beam of soft vertical X-rays.* By R. H. DE WAARD. (X-ray department of the Medical University Clinic, Utrecht.) (Communicated by Prof. H. R. KRUYT.)

(Communicated at the meeting of May 31, 1947.)

1. *Introduction.*

Various illnesses are, nowadays, treated with X-rays. In a simple treatment some definite part of the skin is exposed to a beam of X-radiation, and it is important to know what, in such a case, the distribution of direct and scattered radiation will be in the patients body.

Distributions of this sort have often been investigated by measurements on waterphantoms. In this paper it will be shown that in several cases these distributions can also be approximated theoretically.

2. *Statement of the problem and its mathematical expression by an integral equation.*

The theoretical developments to be given in this paper refer to the arrangement represented in fig. 1. On the surface of a horizontal water

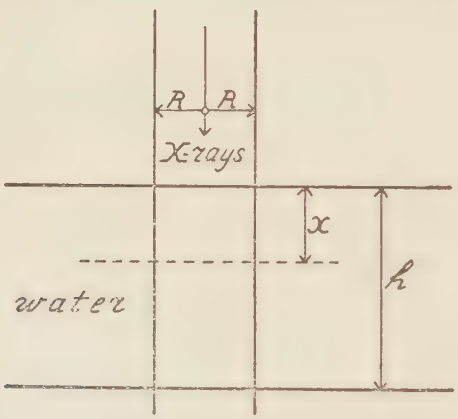


Fig. 1. Cylindriform beam of vertical X-rays falling on a horizontal watersheet of thickness h. R: radius of a horizontal cross-section of the beam.

sheet falls a cylindriform beam of vertical X-rays having all the same wavelength λ, the thickness of the sheet being denoted by h and the radius of the beam by R.

Now the intensities of direct and scattered X-radiation in the water sheet can be approximately calculated by a method given by the author in a preceding paper [1]). It was assumed in this paper that in any given level

[1]) The intensity of scattered X-radiation in medical radiography. Proc. Kon. Ned. Akad. v. Wetensch.., Amsterdam, 49, 955—966 and 1016—1024 (1946).

the intensity of scattered radiation is constant throughout the original beam and zero beyond it. In the present paper we will denote by S_x and D_x the intensities of scattered and direct radiation in a level at distance x under the water surface; D_0 is then the intensity of the incident radiation and we have

$$D_x = D_0 \, e^{-\mu x} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where

$$\mu = 2.5 \, \lambda^3 + 0.18 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

is the coefficient of enfeeblement of the X-radiation in question when propagating in water. When, moreover, we assume that the distribution of scattered radiation over various directions is given by J. J. THOMSON's well-known formula [2]) S_x will satisfy the integral equation

$$\left. \begin{aligned} S_x = \int_0^x \{D_0 \, e^{-\mu y} + S_y\} \, \Phi_{\mu R} \{\mu (x-y)\} \, dy + \\ + \int_x^h \{D_0 \, e^{-\mu y} + S_y\} \, \Phi_{\mu R} \{\mu (y-x)\} \, dy \end{aligned} \right\} \quad . \quad . \quad . \quad (3)$$

where

$$\Phi_{\mu R}(u) = \tfrac{3}{8} \times 0.18 \left[\left(1 + \frac{u^2}{2}\right) \int_0^\infty \frac{e^{-z}}{z} \, dz + \frac{u^2}{2} \, \frac{1-z}{z} \, e^{-z} \right]_{z = \sqrt{u^2 + (\mu R)^2}}^{z = u} .$$

This integral equation is not essentially different from the one treated in the preceding paper mentioned, and an approximate solution can be obtained in very much the same way.

3. *Approximate solution of the integral equation.*

It was found in the paper mentioned that the function $\Phi_{\mu R}(u)$ can be conveniently approximated by an expression of the form

$$\Psi_{\mu R}(u) = k_1 \, e^{-\gamma_1 u} + k_2 \, e^{-\gamma_2 u} \, . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

where $k_1 = 0.200$ and $\gamma_1 = 10$ whilst the constants k_2 and γ_2 are different for different values of μR and can be derived from the graphs of fig. 2. Now, when in the integral equation (3) we replace $\Phi_{\mu R}(u)$ by the expression for $\Psi_{\mu R}(u)$ we arrive at an integral equation which can be solved by elementary methods. The solution is

$$S_x = D_0 \left(\frac{m}{1-m} \, e^{-\mu x} + g_{11} \, e^{-\xi x} + g_{12} \, e^{\xi x} + g_{21} \, e^{-\eta x} + g_{22} \, e^{\eta x} \right) . \quad (5)$$

where the constants m, ξ, η are given by the formulae

$$m = \frac{2 k_1 (\mu \gamma_1)}{(\mu \gamma_1)^2 - \mu^2} + \frac{2 k_2 (\mu \gamma_2)}{(\mu \gamma_2)^2 - \mu^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

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[2]) See M. et L. DE BROGLIE, Introduction à la Physique des Rayons X et Gamma, Paris 1928, pp. 131—138.

and

$$\left.\begin{aligned} \xi &= \sqrt{\tfrac{1}{2}p - \sqrt{\tfrac{1}{4}p^2 - q}} \qquad \eta = \sqrt{\tfrac{1}{2}p + \sqrt{\tfrac{1}{4}p^2 - q}} \\ \text{with} \quad p &= (\mu\gamma_1)^2 + (\mu\gamma_2)^2 - 2k_1(\mu\gamma_1) - 2k_2(\mu\gamma_2) \\ \text{and} \quad q &= (\mu\gamma_1)^2(\mu\gamma_2)^2\left(1 - \frac{2k_1}{\mu\gamma_1} - \frac{2k_2}{\mu\gamma_2}\right) \end{aligned}\right\} \quad . \quad . \quad (7)$$

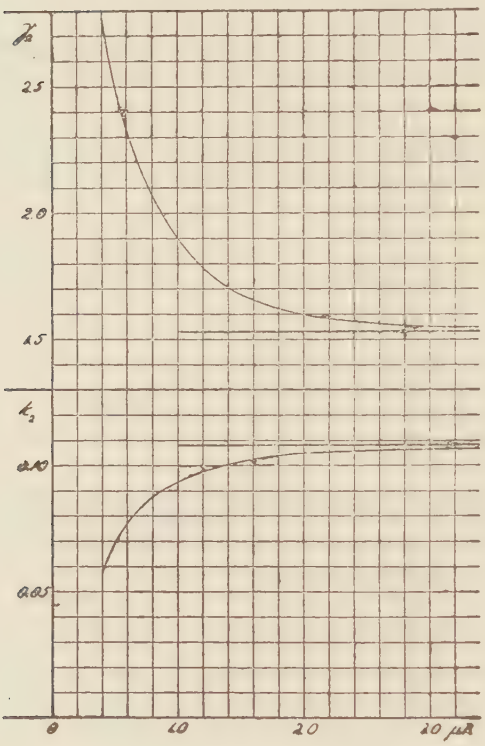


Fig. 2. Graphs showing the dependence of k_2 and γ_2 on μR.

whilst the values of the quantities g can be derived from the equations

$$\left.\begin{aligned} &\frac{1}{(1-m)e_1} + \frac{g_{11}}{a_1} + \frac{g_{12}}{a_3} + \frac{g_{21}}{b_1} + \frac{g_{22}}{b_3} = 0 \\ &\frac{1}{(1-m)e_2} + \frac{g_{11}}{a_2} + \frac{g_{12}}{a_4} + \frac{g_{21}}{b_2} + \frac{g_{22}}{b_4} = 0 \\ &\frac{e^{-\mu h}}{(1-m)e_3} + \frac{g_{11}e^{-\xi h}}{a_3} + \frac{g_{12}e^{\xi h}}{a_1} + \frac{g_{21}e^{-\eta h}}{b_3} + \frac{g_{22}e^{\eta h}}{b_1} = 0 \\ &\frac{e^{-\mu h}}{(1-m)e_4} + \frac{g_{11}e^{-\xi h}}{a_4} + \frac{g_{12}e^{\xi h}}{a_2} + \frac{g_{21}e^{-\eta h}}{b_4} + \frac{g_{22}e^{\eta h}}{b_2} = 0 \end{aligned}\right\} \quad . \quad (8)$$

where

$$\left.\begin{array}{lll} e_1 = \mu\gamma_1 - \mu & a_1 = \mu\gamma_1 - \xi & b_1 = \mu\gamma_1 - \eta \\ e_2 = \mu\gamma_2 - \mu & a_2 = \mu\gamma_2 - \xi & b_2 = \mu\gamma_2 - \eta \\ e_3 = \mu\gamma_1 + \mu & a_3 = \mu\gamma_1 + \xi & b_3 = \mu\gamma_1 + \eta \\ e_4 = \mu\gamma_2 + \mu & a_4 = \mu\gamma_2 + \xi & b_4 = \mu\gamma_2 + \eta \end{array}\right\}. \quad (9)$$

The total intensity J_x in the level at distance x under the water surface is then

$$J_x = S_x + D_x = D_0\left(\frac{e^{-\mu x}}{1-m} + g_{11}e^{-\xi x} + g_{12}e^{\xi x} + g_{21}e^{-\eta x} + g_{22}e^{\eta x}\right) \quad (10)$$

4. *Discussion of the case of thick watersheets.*

The most important case is that where the watersheet is rather thick. In this case the necessary calculations are comparatively simple. We can obtain the total intensity J_x in levels at some distance from the bottom by putting $h = \infty$ in the equations (8) and substituting the resulting values of the quantities g in (10). Now by doing so we find from the last two equations (8) that

$$g_{12} = g_{22} = 0;$$

the first two equations then give

$$g_{11} = \frac{-\frac{a_2}{e_2}\left(1 - \frac{e_2}{e_1}\frac{b_1}{b_2}\right)}{(1-m)\left(1 - \frac{a_2}{a_1}\frac{b_1}{b_2}\right)} \quad \text{and} \quad g_{21} = \frac{-\frac{b_1}{e_1} + \frac{b_1}{e_1}\frac{a_2}{a_1}}{(1-m)\left(1 - \frac{a_2}{a_1}\frac{b_1}{b_2}\right)}$$

and so, as $\frac{a_2}{a_1} \times \frac{b_1}{b_2}$ and $\frac{e_2}{e_1} \times \frac{b_1}{b_2}$ appear to be small quantities, the formula for J_x may be written

$$J_x = D_0\left(\frac{e^{-\mu x}}{1-m} + P\,e^{-\xi x} + Q\,e^{-\eta x}\right) \quad . \quad . \quad . \quad . \quad (11)$$

where

$$P = \frac{-1}{1-m} \cdot \frac{a_2}{e_2}\left\{1 - \frac{b_1}{b_2}\left(\frac{e_2}{e_1} - \frac{a_2}{a_1}\right)\right\}$$

and

$$Q = \frac{-1}{1-m}\left(\frac{b_1}{e_1} - \frac{b_1}{e_2}\frac{a_2}{a_1}\right)\left(1 + \frac{a_2}{a_1}\frac{b_1}{b_2}\right) \quad \left.\right\} \quad . \quad . \quad . \quad . \quad . \quad (12)$$

This formula will now be applied in some special cases.

Case Ia. $\mu = 0.30$, $R = 10$ cm.

Formula (2) gives the corresponding wavelength λ to 0.363 Å. From the curves of fig. 2 we can derive $k_2 = 0.1066$, $\gamma_2 = 1.55$ whilst $k_1 = 0.200$, $\gamma_1 = 10.0$. Substituting these numbers in (6) and (7) and then applying (9) and (11) we find

$$J_x = D_0\,(15.533\,e^{-0.3x} - 14.095\,e^{-0.315x} - 0.103\,e^{-2.80x}). \quad . \quad (13)$$

The relation between J_x and x given by this formula is graphically represented by the upper curve of fig. 3. In this curve is slightly indicated that

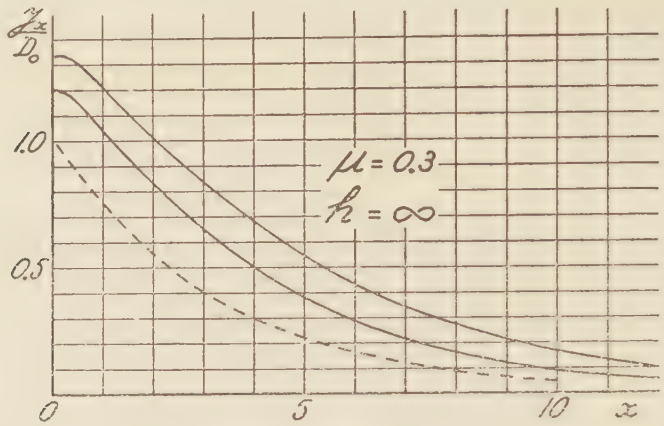


Fig. 3. Full curves: calculated values of $\frac{J_x}{D_0}$ at various depths in the watersheet.
Upper curve: $\mu = 0.3$ ($\lambda = 0.363$ Å), $R = 10$ cm.
Lower curve: $\mu = 0.3$ ($\lambda = 0.363$ Å), $R = 2.5$ cm.
Dotted curve: relative values $\frac{D_x}{D_0}$ of the intensity of direct radiation.

the total X-ray intensity J_x has a maximum at some distance under the water surface. A tendency to the formation of such a maximum is also found in the lower full curve, but here is not actually present. The latter curve refers to

Case I*b*. $\mu = 0.30$, $R = 2.5$ cm.

Here we have $k_2 = 0.085$, $\gamma_2 = 2.12$ (see fig. 2), and the resulting formula for J_x is

$$J_x = D_0 (1.917\, e^{-0.3x} - 0.614\, e^{-0.528x} - 0.094\, e^{-2.80x}). \quad . \quad (14)$$

The dotted curve in fig. 3 gives the contribution to J_x which is due to direct radiation, and so the figure clearly shows the importance of the process of scattering.

A comparison of the full curves shows us the influence of the size of the incident beam. The influence of the wavelength becomes apparent when we compare these curves with the full curve of fig. 4 which corresponds to either of the following two cases:

Case II*a*. $\mu = 0.60$, $R = 10$ cm.

$$\lambda = 0.552 \text{ Å} \qquad k_2 = 0.108 \qquad \gamma_2 = 1.53$$
$$J_x = D_0 (1.916\, e^{-0.6x} - 0.750\, e^{-0.794x} - 0.042\, e^{-5.80x}). \quad . \quad (15)$$

Case II*b*. $\mu = 0.60$, $R = 2.5$ cm.

$$\lambda = 0.552 \text{ Å} \qquad k_2 = 0.101 \qquad \gamma_2 = 1.68$$
$$J_x = D_0 (1.607\, e^{-0.6x} - 0.453\, e^{-0.894x} - 0.042\, e^{-5.80x}). \quad . \quad (16)$$

In these cases the influence of scattering is not so important as in the cases Ia and I*b*.

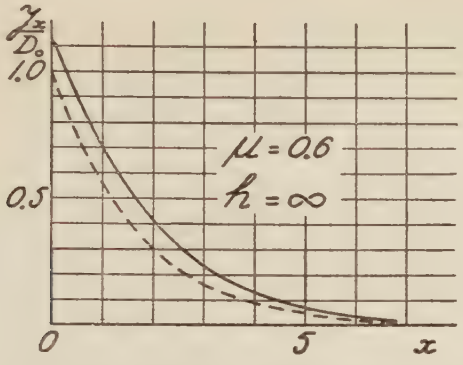


Fig. 4. Full curve: calculated relation between $\frac{J_x}{D_0}$ and x for $\mu = 0.6$ ($\lambda = 0.552$ Å), $R = 2.5$ cm.
This curve is not sensibly different from that for $\mu = 0.6$ ($\lambda = 0.552$ Å), $R = 10$ cm.
Dotted curve: relation between $\frac{D_x}{D_0}$ and x.

5. *Watersheets of moderate thickness.*

The formulae (11) and (12) obtained in the preceding section for the total X-ray intensity J_x refer to the limit case of infinitely thick watersheets. In the present section we will derive formulae applying to the case of watersheets of moderate thickness.

It is obvious that we can obtain general formulae for the coefficients g figuring in (10) by solving the set of linear equations (8). We have for instance:

$$g_{11} = \frac{\begin{vmatrix} \frac{1}{a_3} & \frac{1}{b_1} \\ \frac{1}{a_4} & \frac{1}{b_2} \end{vmatrix} \begin{vmatrix} \frac{1}{e_3} & \frac{1}{b_1} \\ \frac{1}{e_4} & \frac{1}{b_2} \end{vmatrix} e^{(-\mu+\eta)h} - \begin{vmatrix} \frac{1}{e_1} & \frac{1}{b_1} \\ \frac{1}{e_2} & \frac{1}{b_2} \end{vmatrix} \begin{vmatrix} \frac{1}{a_1} & \frac{1}{b_1} \\ \frac{1}{a_2} & \frac{1}{b_2} \end{vmatrix} e^{(\xi+\eta)h} + 4 \text{ other terms}}{(1-m)\left\{ 2 \begin{vmatrix} \frac{1}{a_1} & \frac{1}{a_3} \\ \frac{1}{a_2} & \frac{1}{a_4} \end{vmatrix} \begin{vmatrix} \frac{1}{b_3} & \frac{1}{b_1} \\ \frac{1}{b_4} & \frac{1}{b_2} \end{vmatrix} + \begin{vmatrix} \frac{1}{a_3} & \frac{1}{b_1} \\ \frac{1}{a_4} & \frac{1}{b_2} \end{vmatrix}^2 e^{(-\xi+\eta)h} + 3 \text{ other terms} \right\}}$$

We will now confine ourselves to the consideration of cases where the watersheet in question is more than 4 cm thick. Numerical calculations show that several of the terms figuring in the expression for g_{11} may then be neglected and that the expression can be reduced to

$$g_{11} = \frac{\frac{a_2}{a_4}\frac{a_2}{e_4} e^{(-\mu-\xi)h} - \frac{a_2}{e_2}\left\{1 - \frac{b_1}{b_2}\left(\frac{e_2}{e_1} - \frac{a_2}{a_1}\right)\right\}}{(1-m)\left\{1 - \left(\frac{a_2}{a_4}\right)^2 e^{-2\xi h}\right\}}$$

In a similar way can be obtained the following expressions for the other quantities g:

$$g_{12}=\frac{\left(-\frac{a_2}{e_4}+\frac{a_2}{b_2}\frac{b_1}{e_3}\right)\left(1+\frac{a_2}{a_1}\frac{b_1}{b_2}\right)e^{-\mu h}+\frac{a_2}{e_2}\frac{a_2}{a_4}\left\{1+\frac{b_1}{b_2}\left(2\frac{a_2}{a_1}-\frac{e_2}{e_1}-\frac{a_4}{a_3}\right)\right\}e^{-\xi h}}{(1-m)\left\{1-\left(\frac{a_2}{a_4}\right)^2 e^{-2\xi h}\right\}}e^{-\xi h}$$

$$g_{21}=\frac{-\frac{b_1}{e_1}+\frac{a_2}{a_1}\frac{b_1}{e_2}-\frac{a_2}{a_4}\frac{b_1}{a_3}\frac{a_2}{e_2}\left(1-\frac{a_3}{a_4}\frac{e_2}{e_1}\right)e^{-2\xi h}}{(1-m)\left\{1-\left(\frac{a_2}{a_4}\right)^2 e^{-2\xi h}\right\}}$$

$$g_{22}=\frac{\left(-\frac{b_1}{e_3}+\frac{a_2}{a_1}\frac{b_1}{e_4}\right)\left(1+\frac{a_2}{a_1}\frac{b_1}{b_2}\right)e^{-\mu h}+\frac{a_2}{e_2}\left(\frac{b_1}{a_3}-\frac{b_1}{a_4}\frac{a_2}{a_1}\right)\left\{1-\frac{b_1}{b_2}\left(\frac{e_2}{e_1}-2\frac{a_2}{a_1}\right)\right\}e^{-\xi h}}{(1-m)\left\{1-\left(\frac{a_2}{a_4}\right)^2 e^{-2\xi h}\right\}}e^{-\eta h}$$

We will now apply these formulae to some cases where $h = 5$ cm, 10 cm and 20 cm, and compare the results with those obtained in section 4 for watersheets of infinite thickness.

Case Ia. $\mu = 0.30$, $R = 10$ cm.

The coefficients figuring in the right hand member of (10) are given by the formulae

$$\xi = 0.315 \qquad \eta = 2.80 \qquad \frac{1}{1-m} = 15.533$$

$$g_{11}=\frac{0.567\,e^{-0.615h}-14.102}{1-0.036\,e^{-0.63h}}$$

$$g_{12}=\frac{-3.042\,e^{-0.3h}+2.710\,e^{-0.315h}}{1-0.036\,e^{-0.63h}}e^{-0.315h}$$

$$g_{21}=\frac{-0.103-0.122\,e^{-0.63h}}{1-0.036\,e^{-0.63h}}$$

$$g_{22}=\frac{-0.730\,e^{-0.3h}+0.663\,e^{-0.315h}}{1-0.036\,e^{-0.63h}}e^{-2.80h}$$

From the latter four of these formulae can be derived the values of the quantities g corresponding to different water-thicknesses h. The values for $h = \infty$, $h = 20$ cm, $h = 10$ cm and $h = 5$ cm are given in Table Ia.

With the help of these values were obtained the J_x, x-curves corresponding to $h = \infty$, $h = 10$ cm and $h = 5$ cm which are shown in fig. 5;

TABLE Ia ($\mu = 0.30$, $R = 10$ cm).

h	∞	20 cm	10 cm	5 cm
g_{11}	-14.102	-14.102	-14.101	-14.076
$g_{12}e^{0.315h}$	0	-0.003	-0.035	-0.118
g_{21}	-0.103	-0.103	-0.103	-0.108
$g_{22}e^{2.976h}$	0	-0.001	-0.008	-0.026

a curve for $h = 20$ cm has been omitted since it would not have differed much from that for $h = \infty$. Just as in figs. 3 and 4 the dotted curve gives the contribution to J_x which is due to direct radiation.

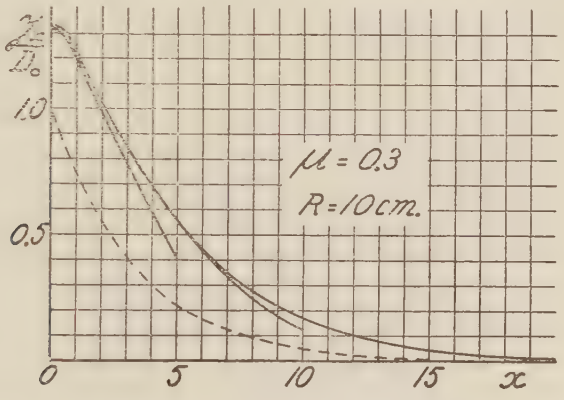


Fig. 5. Full curves: calculated relations between $\frac{J_x}{D_0}$ and x corresponding to the following cases:

Upper curve: $\mu = 0.3$, $R = 10$ cm, $h = \infty$
Middle curve: $\mu = 0.3$, $R = 10$ cm, $h = 10$ cm
Lower curve: $\mu = 0.3$, $R = 10$ cm, $h = 5$ cm

Dotted curve, relation between $\frac{D_x}{D_0}$ and x.

Case I*b*. $\mu = 0.30$, $R = 2.5$ cm.
Here we have

$$\xi = 0.528 \qquad \eta = 2.80 \qquad \frac{1}{1-m} = 1.917.$$

$$g_{11} = \frac{0.021\, e^{-0.828h} - 0.619}{1 - 0.009\, e^{-1.056h}}$$

$$g_{12} = \frac{-0.225\, e^{-0.3h} + 0.059\, e^{-0.528h}}{1 - 0.009\, e^{-1.056h}}\, e^{-0.528h}$$

$$g_{21} = \frac{-0.094 - 0.002\, e^{-1.056h}}{1 - 0.009\, e^{-1.056h}}$$

$$g_{22} = \frac{-0.100\, e^{-0.3h} + 0.032\, e^{-0.528h}}{1 - 0.009\, e^{-1.056h}}\, e^{-2.80h}$$

If in the latter four of these formulae we substitute $h = \infty$, $h = 20$ cm, $h = 10$ cm and $h = 5$ cm we find the values of the quantities g which are given in Table I*b*.

TABLE I*b* ($\mu = 0.30$, $R = 2.5$ cm).

h	∞	20 cm	10 cm	5 cm
g_{11}	-0.614	-0.614	-0.614	-0.614
$g_{12}e^{0.528h}$	0	-0.001	-0.011	-0.046
g_{21}	-0.094	-0.094	-0.094	-0.094
$g_{22}e^{2.80h}$	0	0	-0.005	-0.020

The J_x, x-curves corresponding to $h = \infty$ and $h = 5$ cm are shown in fig. 6.

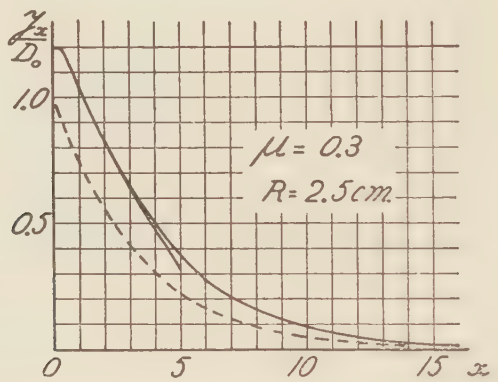


Fig. 6. Full curves: calculated relations between $\frac{J_x}{D_0}$ and x corresponding to the following cases:

Upper curve: $\mu = 0.3$, $R = 2.5$ cm, $h = \infty$
Lower curve: $\mu = 0.3$, $R = 2.5$ cm, $h = 5$ cm

Dotted curve: relation between $\frac{D_x}{D_0}$ and x.

As to the cases IIa ($\mu = 0.60$, $R = 10$ cm) and II*b* ($\mu = 0.6$, $R = 2.5$ cm) it may be observed that the differences between the J_x, x-curves for $h = 20$ cm, $h = 10$ cm and $h = 5$ cm and those for $h = \infty$ given in fig. 4 are negligible.

6. *General discussion. Comparison with experimental data.*

The formulae for J_x obtained in this paper were derived from the integral equation (3) by a method of approximation which is perfectly justified. The integral equation itself, however, needs some discussion. One of the assumptions on which it is based is the validity of J. J. THOMSON's

classical theory of X-ray scattering. We know that this theory gives a reasonable approximation of facts when the wavelengths concerned are large with respect to the Compton-wavelength $\lambda_c = 0.024$ Å, but that it does not hold for smaller wavelengths (e.g. < 0.24 Å). When we have to deal with such smaller wavelengths we must account for a reduction of sideward and especially of backward scattering and, moreover, for the increase of wavelength known as the Compton-effect. Whilst these phenomena will only slightly reduce the influence of scattering in forward directions they may cause a considerable decrease of the influence of scattering in backward directions, and so we must expect that their neglection will, in the main, result in an overvaluation of the influence of backward scattering in the theory. To this divergence must be added an error due to another source. It was assumed in the deduction of the integral equation that scattered radiation of any order originating in some part of the watersheet has the same distribution over various directions as secondary radiation, and this results in the influence of both forward and backward scattering being exaggerated, particularly when the incident beam is large. In cases of large wavelengths and incident beams of moderate size, however, the final formulae for J_x may be fairly reliable, and the question arises whether we have at our disposal experimental data with which these formulae can be compared numerically. Now this appears to be actually the case. The curves of figs. 3 and 4 refer to incident beams of wavelengths $\lambda = 0.363$ Å and $\lambda = 0.552$ Å, and the formula

$$\mu = 37\,\lambda^3 + 0.38$$

for the coefficient of enfeeblement in aluminium gives the corresponding half value layers for this metal to 3.0 mm and 1.0 mm respectively. Now depth dose measurements on radiations with this sort of half value layers were carried out during the war by C. B. BRAESTRUP, and his results can be found in tables $A - F$ at the end of the book "Physical Foundations of Radiology" by O. GLASSER, E. H. QUIMBY, L. S. TAYLOR and J. L. WEATHERWAX. However, the conditions of the experiments were rather different from those to which the calculations of this paper apply. The main differences are:

1. The calculations apply to cylindriform beams of parallel X-rays whereas the experiments were carried out with divergent beams, and

2. In the calculations the incident X-radiation is supposed to have one definite wavelength whereas in the experiments it formed a rather extensive continuous spectrum.

Now it is easy to see what effect these differences will have when we compare cases of equal half value layer and equal size of the exposed area of the water surface. Let us first consider the influence of the divergence of the incident beams which BRAESTRUP applied in his measurements. It is obvious that this divergence will result in a more rapid decrease of X-ray intensity on the way from the surface to the lower parts of the watersheet.

For the direct radiation the extra decrease is given by the inverse square law; in scattered radiation, however, a still stronger decrease of intensity must be expected on account of the fact that the mean distance of any part of the incident beam to the other parts increases with increasing cross-section.

As to the spectral composition of the incident radiation applied in the experiments it is clear that the more penetrating components will have greater importance according as we consider a lower level in the water-sheet. From this fact will result an increase of X-ray intensity in the lower levels and this increase may overcompensate the decrease due to the divergence of the incident beam.

There is, however, one point which is not affected by the discussed differences between theoretical and experimental conditions. This point is the X-ray intensity at the surface of the watersheet which is due to downward incident radiation and to upward backscatter. If, therefore, we express backscatter or total X-radiation at the water surface in terms of the incident radiation we should find close agreement between calculated and measured values. Now, in fact, the agreement between calculated and measured backscatter values is very satisfactory indeed. This is clearly shown by the curves of figs. 7 and 8 which refer to cases with half value

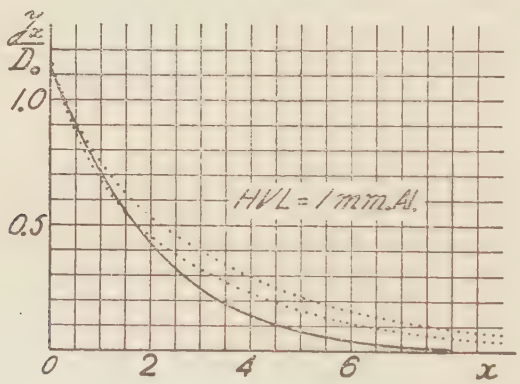


Fig. 7. Curves giving calculated and measured relations between $\frac{J_x}{D_0}$ and x for radiations with half-value layer of 1 mm Al.
Full curve (calculated): parallel X-rays of wavelength $\lambda = 0.552$ Å ($\mu = 0.6$), field of incidence 20 cm² or greater ($R = 2.5$ cm).
Dotted curves (measurements by C. B. BRAESTRUP): high tension 100 kV, inherent filter only, focus-surface distance 30 cm.
Upper dotted curve: field of incidence 100 cm² ($R = 11.3$ cm).
Lower dotted curve: field of incidence 25 cm² ($R = 5.6$ cm).

layers of 1 mm Al ($\mu = 0.6$ and $\lambda = 0.552$ Å in the calculations) and 4 mm Al ($\mu = 0.257$ and $\lambda = 0.313$ Å in the calculations). Corresponding to the lower levels of the watersheet the curves show discrepancies of the sort predicted in the above qualitative considerations, and so the formulae

obtained in this paper look like giving a good approximation of facts when the wavelength of the incident radiation is between 0.3 and 0.6 Å. It is therefore of interest that they give quantitative information on certain

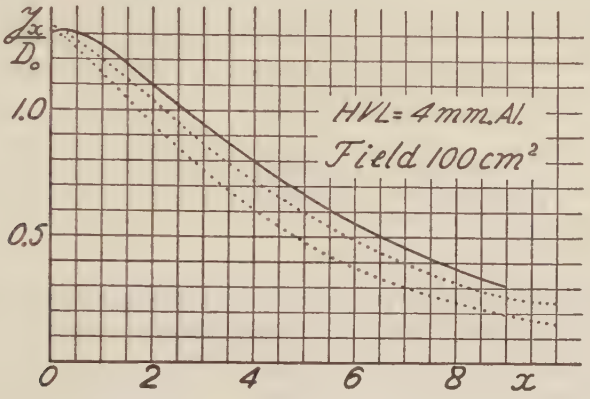


Fig. 8. Curves giving calculated and measured relations between $\frac{J_x}{D_0}$ and x for radiations with half-value-layer of 4 mm Al. and field of incidence of 100 cm² ($R = 5.64$ cm). Full curve (calculated): parallel X-rays of wavelength $\lambda = 0.363$ Å ($\mu = 0.3$). Dotted curves (measurements by C. B. BRAESTRUP): high tension 120 kV, filter 3 mm Al.

Upper dotted curve: focus surface distance 40 cm.
Lower dotted curve: focus surface distance 20 cm.

features which are not taken into account in current medical applications. In conclusion some of these features may be mentioned here:

a. The part of total X-radiation which is due to scattering is more important according as we have to do with thicker watersheets,

b. When compared with its general course in the central parts of the watersheet the intensity of total X-radiation shows a marked decrease not only towards the surface, but also towards the bottom of the sheet.

Summary.

A method is given by which can be approximately calculated the total intensity of X-radiation in various levels of a horizontal watersheet which is exposed to a cylindriform beam of soft vertical X-rays. The resulting formulae are in satisfactory agreement with the results of a series of measurements carried out during the war by C. B. BRAESTRUP. They give some quantitative information on certain features which are not taken into account in current medical applications.

Zoology. — *The Oestrus Cycle of Rhodeus Amarus Bloch* ♀. By J. MELTZER. (Communication No. 20 of the Workcommunity for Endocrinology from the Laboratory of Comparative Physiology and the Laboratory of General Zoology, University Utrecht.) (Communicated by Prof. CHR. P. RAVEN.)

(Communicated at the meeting of May 31, 1947.)

1. Introduction.

Rhodeus Amarus is often kept in aquaria. It is a gratifying object as it is easily brought to spawn. Since the investigations of NOLL (1869, 1877) the female is known to lay her eggs in mussels of the genera *Anodonta* and *Unio*. When ovulating the female inserts the ovipositor into the gills of the mussel. Out of the spawning season the ovipositor is hardly existing or not at all; only a slight elevation immediately behind the anus, the urogenital papilla, indicates the place where the excretory ducts emerge. In springtime this papilla starts developing into an ovipositor which may reach beyond the tailfin. Figure 1 shows the position of the ovipositor and the internal organs.

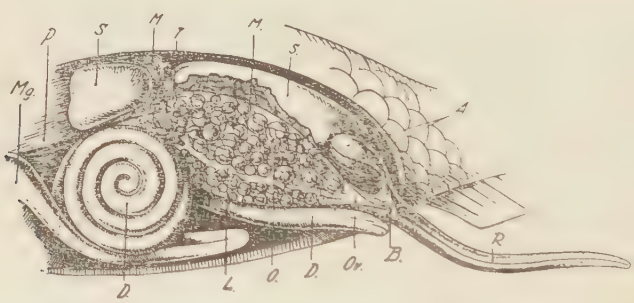


Fig. 1. Position of the ovary and the internal organs during the spawning season. The liver has been removed. O = ovary; OV = internal oviduct; A = bladders; R = ovipositor; S = swimming bladder; P = ductus pneumaticus; D = intestine. After *OLT* (1893).

It is remarkable that the ovipositor only grows if both mussel and male are present, though WUNDER (1933, 1934) reports that the mere presence of the mussel is sufficient. Reports of experiments as to that will be issued elsewhere. Here a short summary of the mating-biology in so far as necessary for a good understanding of the oestrus cycle, may suffice.

When in early spring a couple of *Rhodeus* is brought together with an *Unio*, and then observed, one can trace in what manner the ovipositor develops. On a given day both fishes begin to show an interest in the mus-

sel. And it is not long before the mussel is being guarded by the male only. All congenera, male as well as female are chased away from the territory of the mussel. In the mean time we see how the ovipositor of the female is gradually beginning to grow. When the ovipositor has reached its full length, we can observe that all of a sudden the behaviour of the male towards the female entirely changes. Instead of chasing her he lures the female with shuddering movements of his entire body towards the mussel. Once near the mussel the male sheds his milt-liquid above the siphon. Under the influence of the trembling male the female is thrown into extasy; with her head directed towards the discharge aperture of the mussel and her tail pointing upward she awaits the right moment for shooting downward. Quick as lightning the base of the ovipositor is now brought above the siphon of the mussel and at the same moment an egg slips through the ovipositor, driven forward by a liquid (urine?). Thus a wateraxis is formed, by which the ovipositor becomes rigid and disappears into the mussel (c.f. DUYVENÉ DE WIT and BRETSCHNEIDER 1940). In a fraction of a second the egg is laid and the female swims away.

Without going further into the behaviour of the roach, it must be remarked that the cause of the change in the behaviour of the male towards the female must be sought in the typical movement of the female when about to ovulate. When ripe for ovulating the female assumes an oblique pose whereby the head is directed downward and the tail upward. I should like to call this position the inclination-pose. It goes without saying that this inclination-pose is extremely important for the external recognition of the condition of oestrus of the female. It is interesting that JASKI (1939) discovered a similar pose with *Lebistes reticulatus*. With *Lebistes* the females that are ripe for copulation assume the so-called elevation-pose. There are important differences, however, between the elevation-pose of *Lebistes* and the inclination-pose of *Rhodeus*. With the first the head is directed upward, whereas with the last it is directed downward. From this different behaviour it is evident, however, that the position cannot be merely the result of the physical conditions caused by the heavy ovary. Rather a hormonal and psychical regulation should be considered here. Another difference is to be found in the way in which the elevation or inclination-pose is brought about. With *Lebistes* the elevation takes place gradually during several days, i.e. there is a gradual increase of the gradient. With *Rhodeus*, on the other hand, the inclination can be seen to take place suddenly and frequently during about 24 hours, whenever an egg is to be laid. For the eggs are laid one by one, with intervals of two minutes to even one or more hours. A non-interrupted inclination-pose would moreover result in the female being continually importuned.

2. The periodicity of the growth of the ovipositor.

DUYVENÉ DE WIT (1939) already supposes that the phenomenon of the

growth of the ovipositor under normal conditions probably would appear periodically. Herewith corresponds the observation of NOLL (1877) that with some couples of *Rhodeus* ovulations took place several times in one season.

From the 15th of April till the 10th of June 1942 in the aggregate some 40 females were being watched daily. The length of the ovipositor was measured daily in anal units (A.U.) and set out in a diagram.

One anal unit is ⅛th part of the anal. The relation between the length of the fish and the length of the anal appears to be almost constant. An exact measurement in mm would of course be no criterion, considering the great variation in size of the fishes. Moreover the fishes would be too much upset when being fetched out of the water for each measurement. After some excercise the estimation of the length of the ovipositor is done with fair accuracy. The average fault amounts to about 0,5 A.U.

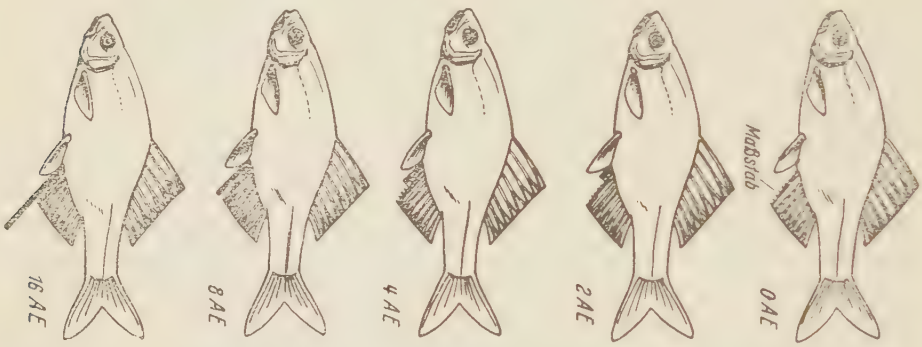


Fig. 2. The foremost radius of the anal has been divided into 8 parts and serves as a measure for the length of the ovipositor. ⅛th part is called 1 anal unit (A.U.). With the above fishes the ovipositors are resp. 0, 2, 4, 8, and 16 A.U.

Of those fourty females one or two did not ovulate at all. A few others ovulated once or twice, but most of them three or four times. One female even came to ovulating seven times in this one season. It appears from the diagrams (Fig. 3—9) that the ovipositor develops in periods that slightly vary with the several individuals. Mostly a period was 6 to 8 days, but with some also periods of 11 till 13 days occurred. It is quite possible, however, that these animals simply passed over an ovulation-phase.

According to WUNDER there are two stages of growth of the ovipositor to be distinguished: In the first stage the ovipositor reaches a length of a few mm, whereas in the second stage the ovipositor develops into the complete length of some cm within a week. The diagrams however give a totally different aspect of the process. Fig. 3 e.g. shows that the ovipositor is oscillating round 3 A.U. for a long time, until on the 16th of May a continuous growth sets in, culminating in a top on the 23th of May.

As a matter of fact the growth of the ovipositor takes place in two stages, but different from the way WUNDER reports. The increase from 0 to 5 A.U. may take weeks. The increase from 5 A.U. to the complete

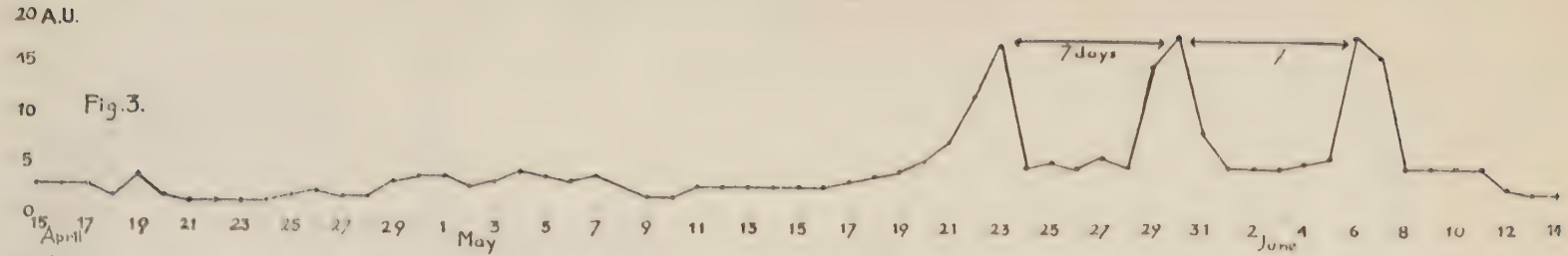


Fig. 3.

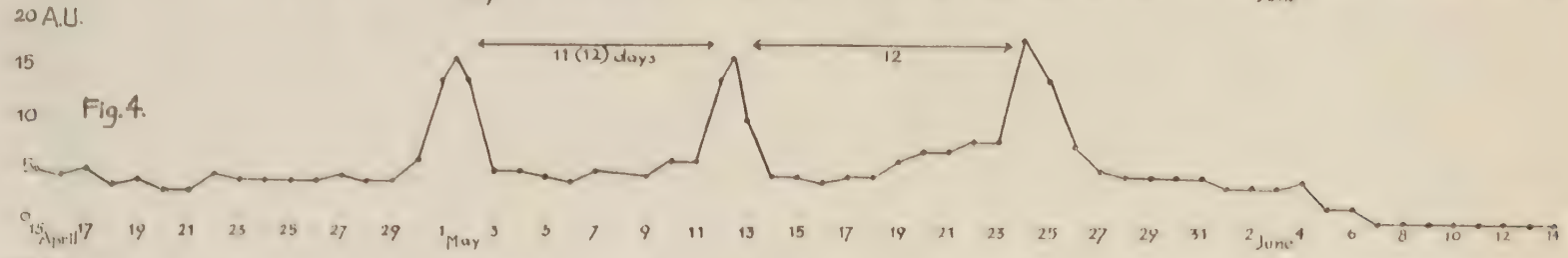


Fig. 4.

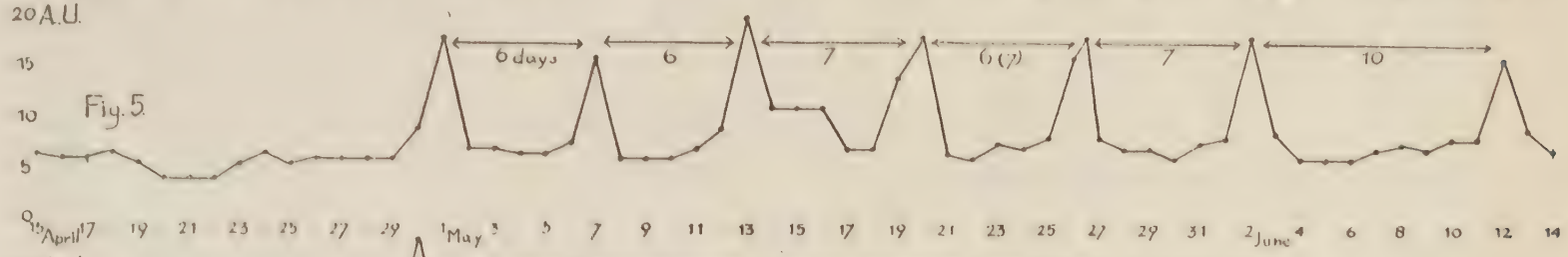


Fig. 5

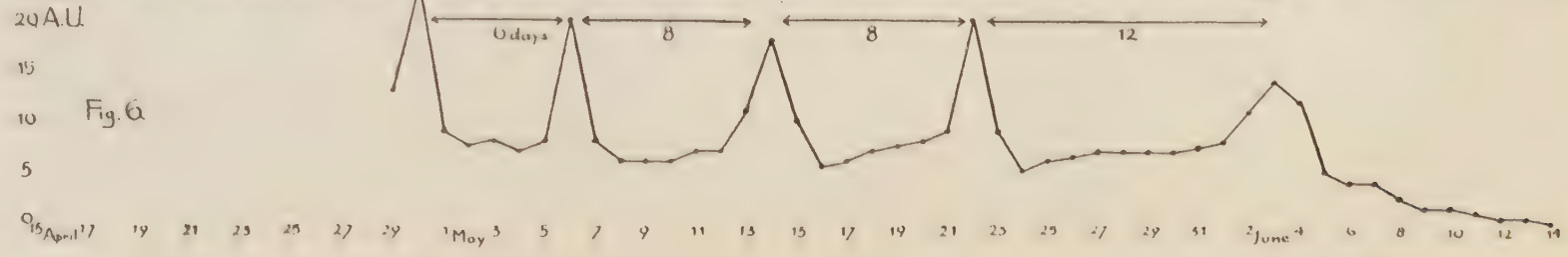
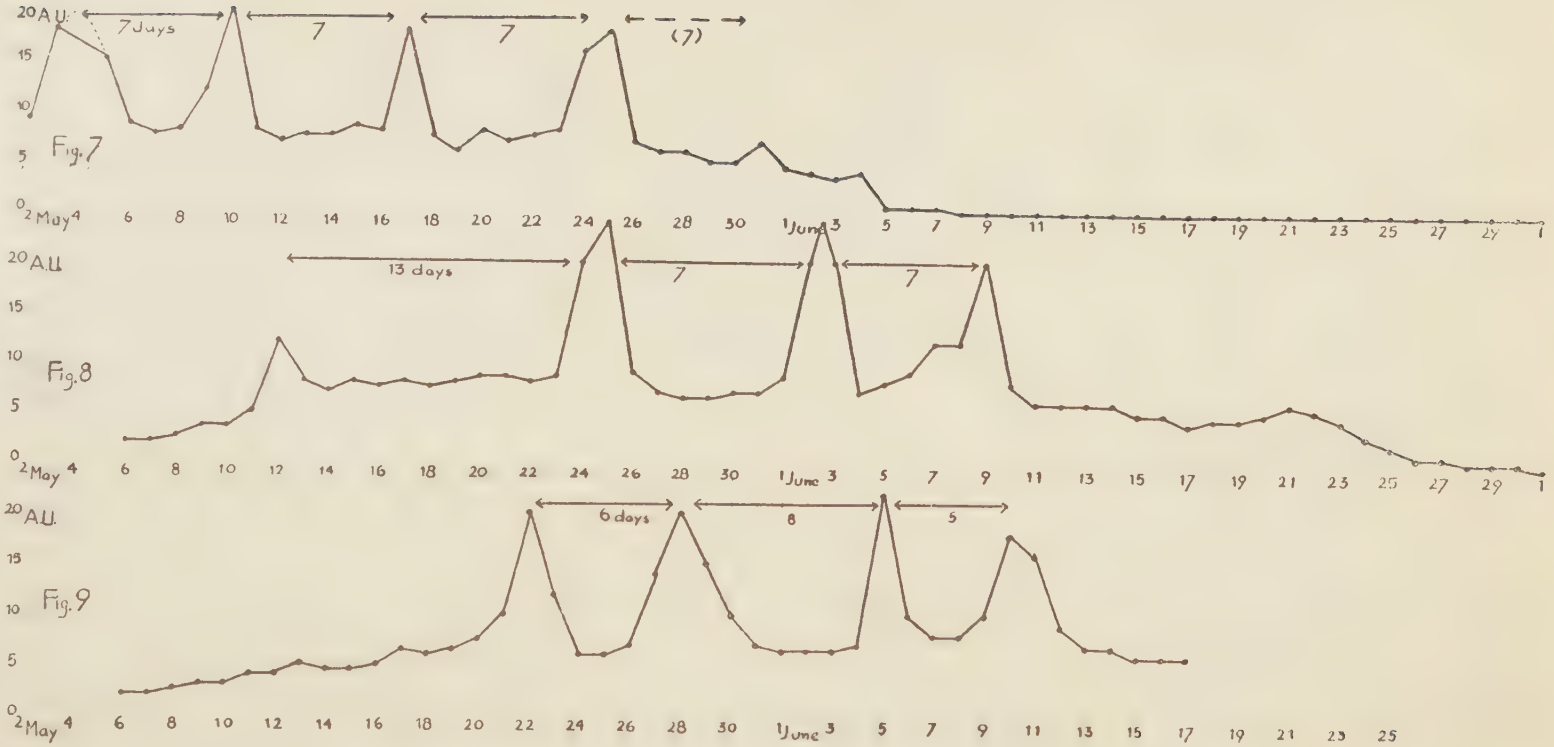


Fig. 6.


20 A.U.
15
10
5
0
7 days
7
7
(7)
Fig. 7
2 May 4 6 8 10 12 14 16 18 20 22 24 26 28 30 1 June 3 5 7 9 11 13 15 17 19 21 23 25 27 29 1
20 A.U.
15
10
5
0
13 days
7
7
Fig. 8
2 May 4 6 8 10 12 14 16 18 20 22 24 26 28 30 1 June 3 5 7 9 11 13 15 17 19 21 23 25 27 29 1
20 A.U.
15
10
5
0
6 days
8
5
Fig. 9
2 May 4 6 8 10 12 14 16 18 20 22 24 26 28 30 1 June 3 5 7 9 11 13 15 17 19 21 23 25

length on the other hand takes place very rapidly, generally within 24 hours. The decrease following the top has a quick course too; its original length is mostly regained within twenty-four hours. The ovipositor, however, is never reduced to zero during the spawning season. During this period the ovipositor is oscillating between 5 and 8 A.U., reaching a top with intervals of about a week. In the diagrams three stages of the ovipositor are to be discerned, i.e.:

1st stage: Length 0—5 A.U. The ovipositor is in the period of growth. Histologically this period is characterised by the appearance of several mitoses (c.f. BRETSCHNEIDER and DUYVENÉ DE WIT 1941).

2nd stage: Length 5—8 A.U. The ovipositor is in the period of disposition, i.e. it is periodically able to quickly attain a toplength, so that ovulation is possible. The length of 5—8 A.U. is characteristic only for the phase of rest. According to the processes inacting themselves in and round the ovipositor, the following phases may be distinguished:

a. Phase of rest, length 5—8 A.U.

b. Phase of ovulation, i.e. the period during which the ovipositor after a quick increase as quickly decreases. In this phase we may discern again:

α. *Phase of infiltration.* The ovipositor is lengthened by infiltration with lymphe and by swelling of the tissue until a length of 16—20 A.U. (individual differences).

β. *Phase of inclination.* During this phase the female ovulates. The ovipositor may lengthen here, but the extension amounts to 3 A.U. at most. With several individuals the length varies from 16—24 A.U. Characteristic for this phase is the inclination-pose of the female.

γ. *Phase of defiltration.* The ovipositor is reduced to the length of the phase of rest.

3rd stage: Period of reduction. After the spawning season the ovipositor is reduced again to zero.

In the first stage the growth of the ovipositor is very slow. The artificial lengthening of the ovipositor as described by DUYVENÉ DE WIT (1939) belongs to this part of the growth curve of the ovipositor. With the artificial growth of the ovipositor the rate of growth however is accelerated. Under natural conditions the mitoses will not appear as quickly and numerously, and probably the actual growth of the cells will take place directly after the division.

The varying length of the ovipositor also during this period proves, however, that the infiltration with lymphe is not exclusively restricted to the

infiltration-phase of the second stage. Moreover BRETSCHNEIDER and DUYVENÉ DE WIT (1941) pointed out that also in the period of mitoses the infiltration with lymphe plays an important part. Under natural conditions the part of the lymphe-infiltration during the growing stage is certain to be less important, whereas the cells originating by division have sufficient time to grow. Hyperaemia and lymphe-infiltration are much important quantitatively in the second stage of the development of the ovipositor.

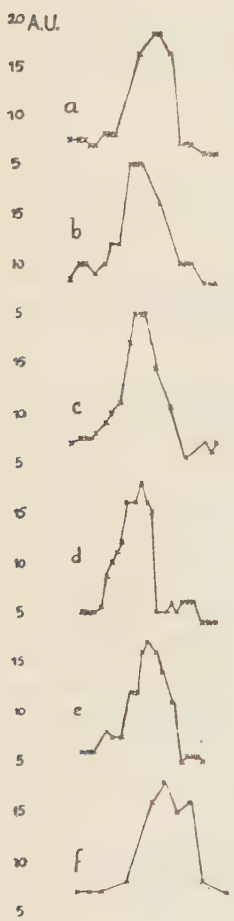


Fig. 10. Different types of tops occurring. The phase of infiltration varies from 10—24 hours. The inclination phase lasts about 24 hours. Only at *f* another top occurs. The defiltration-phase varies from 12—24 hours.

The disosition period is the actual spawning season of the female concerned; only during this time ovulation can take place. According to the diagrams the spawning season of the test animals fell between the end of April and the middle of June. Without artificial interference the females could not be brought to ovulate before the 29th of April. After the 15th of June only those females ovulated who were kept in water of 13° C.

The phase of infiltration often lasts twelve hours, but may extend to twenty-four hours (Fig. 10). So within twenty-four hours the greatest lengthening of the ovipositor takes place. The phase of inclination as a rule lasts about twenty-four hours. Between 16 and 24 A.U. ovulation takes place, i.e. some females already begin at a length of 16 A.U., whereas others ovulate not until 22 A.U. Only once an ovulation at a length of 12 A.U. occurred. During the phase of ovulation the ovipositor mostly gets another slight increase. Once the curve even shows two tops (Fig. 10). This is another illustration of the fact that infiltration in itself is not a characteristic of the infiltration-phase.

During the inclination-phase the inclination-pose is assumed by the female, with interruptions as remarked before. The defiltration-phase

finally lasts again 12—24 hours. A schematical curve of the oestrus cycle, only two ovulation-phases being drawn, may elucidate facts mentioned (Fig. 11).

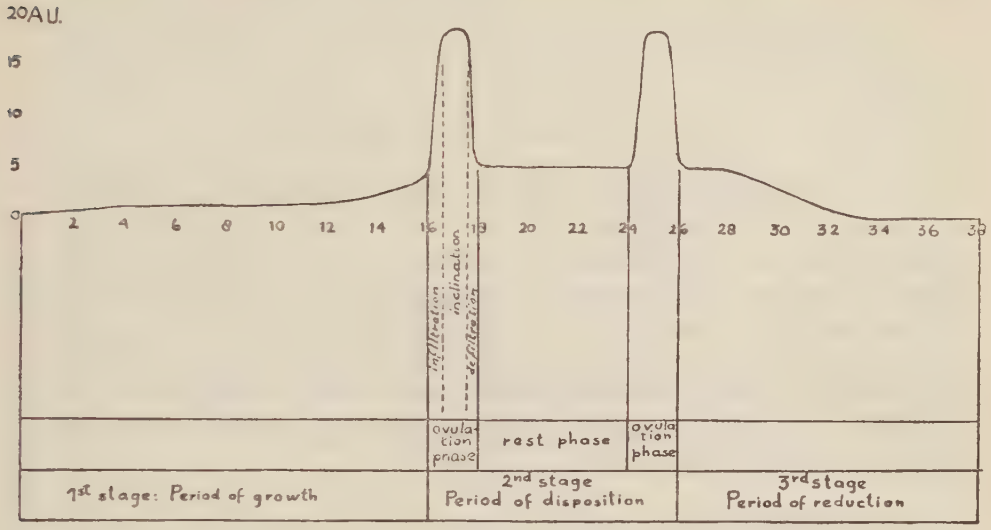


Fig. 11. Diagram of the development of the ovipositor during the spawning season. The first stage requires as a rule a long time. In the second stage the duration depends on the number of ovulation phases of the individual concerned. The third stage is finished as a rule within a week.

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Mathematics. — *On the minimum determinant and the circumscribed hexagons of a convex domain.* By K. MAHLER. (Communicated by Prof. J. G. VAN DER CORPUT.)

(Communicated at the meeting of May 31, 1947.)

In his "Diophantische Approximationen", MINKOWSKI gave a simple rule for obtaining the critical lattices of a convex domain by means of the *inscribed* hexagons (see Lemma 2). I study here an analogous method based instead on the *circumscribed* hexagons. In the special case of a convex polygon, a simple rule for finding all critical lattices and the minimum determinant is obtained. I also show the surprising result that the boundary of an irreducible convex domain not a parallelogram has in all points a continuous tangent. Finally the lower bound of $Q(K)$ is evaluated for all convex octagons.

§ 1. Notation.

The same notation as in earlier papers of mine is used [1]). In particular, the determinant of a lattice Λ is called $d(\Lambda)$; $V(K)$ and $\triangle(K)$ are the area and the minimum determinant of a domain K, and $Q(K)$ is the absolute affine invariant

$$Q(K) = \frac{V(K)}{\triangle(K)}.$$

The letter L is used for straight lines not passing through the origin $O = (0, 0)$, and $-L$ is then the line symmetrical to L in O.

All domains K considered in this paper are assumed to be symmetrical in O; the boundary of K is called C. A convex polygon of $2n$ sides and symmetrical in O will be denoted by Π_n, its boundary by Γ_n. The indices of its vertices P_k and its sides L_k are always chosen in such a way that if Γ_n is described in positive direction, then the successive vertices are

$$Q_1, Q_2, \ldots, Q_n, Q_{n+1} = -Q_1, Q_{n+2} = -Q_2, \ldots, Q_{2n} = -Q_n,$$

and the successive sides are

$$L_1 = Q_1 Q_2, \quad L_2 = Q_2 Q_3, \ldots, L_n = Q_n Q_{n+1},$$

$$L_{n+1} = Q_{n+1} Q_{n+2} = -L_1, \quad L_{n+2} = Q_{n+2} Q_{n+3} = -L_2, \ldots,$$

$$L_{2n} = Q_{2n} Q_1 = -L_n.$$

§ 2. Basic lemmas.

The following lemmas are essential for our investigations.

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[1]) See, e.g. Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, **50**, 98—107 and 108—118 (1947). These two papers will be quoted as *A* and *B*, respectively.

Lemma 1: *Let K be a convex domain; let $\mp P_1$, $\mp P_2$, $\mp P_3$ be six points on C such that $P_1 + P_3 = P_2$, and let Λ be the lattice generated by P_1 and P_2. Then Λ is K-admissible.*

Proof: Obvious from the convexity.

Lemma 2: *Let Λ be any critical lattice of the convex domain K. Then Λ contains three points P_1, P_2, P_3 on C such that,* (i) *P_1, P_2 is a basis of Λ, and* (ii) *$OP_1P_2P_3$ is a parallelogram of area $d(\Lambda) = \triangle(K)$. Conversely, if P_1, P_2, P_3 are three points on C such that $OP_1P_2P_3$ is a parallelogram, then the area of this parallelogram is not less than $\triangle(K)$, and it is equal to $\triangle(K)$ if and only if the lattice of basis P_1, P_2 is critical* [2]).

Lemma 3: *The convex domain K is irreducible if and only if every boundary point of K belongs to a critical lattice of K* [3]).

Lemma 4: *For every parallelogram Π_2,*

$$\triangle(\Pi_2) = \tfrac{1}{4} V(\Pi_2), \quad Q(\Pi_2) = 4.$$

Moreover, every such parallelogram is an irreducible domain [4]).

Lemma 5: *For every convex hexagon Π_3,*

$$\triangle(\Pi_3) = \tfrac{1}{4} V(\Pi_3), \quad Q(\Pi_3) = 4.$$

Moreover, every such hexagon has only one critical lattice, and this lattice has just six points on Γ_3, viz. the midpoints of the six sides of Π_3 [5]).

§ 3. Two formulae for $\triangle(K)$.

Let K be a convex domain symmetrical in O. From Lemma 2, we immediately obtain the formula

(I): $$\triangle(K) = \tfrac{1}{3} \operatorname{fin} \inf_{h \in I_K} V(h)$$

for $\triangle(K)$; here I_K denotes the set of all hexagons h which have their six vertices $\mp P_1$, $\mp P_2$, $\mp P_3$ on the boundary C of K and for which

$$P_1 + P_3 = P_2.$$

For this relation implies evidently that

$$V(h) = 3V(p)$$

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[2]) This is Lemma 3 of paper *A*.

[3]) See Lemmas 8 and 12 of paper *A*.

[4]) The first part of the assertion is equivalent to MINKOWSKI's theorem on linear forms; for the second part see Lemma 1 of paper *A*.

[5]) The assertion follows from the fact that the whole plane can be covered in just one way without overlapping by means of hexagons congruent to Π_3; see paper *B*, § 7.

An entirely different result holds for non-convex hexagonal star domains Π_3 symmetrical in O, viz.

$$\triangle(\Pi_3) = \tfrac{1}{4} V(\Pi_2), \quad Q(\Pi_3) > 4;$$

here Π_2 is the inscribed parallelogram of maximum area. There are an infinity of critical lattices, and every critical lattice has points only on four of the sides of Π_3.

where p is the parallelogram $OP_1P_2P_3$. Since in (I) the lower bound is attained, it is allowed to replace "*fin inf*" by the sign "*min*".

The following theorem gives a formula analogous to (I) but involving the circumscribed hexagons.

Theorem 1: *Let K be an convex domain symmetrical in O, and let U_K be the set of all hexagons H bounded by any three pairs of tac-lines $\mp L_1$, $\mp L_2$, $\mp L_3$ of K* [6]*. Then*

(II): $$\Delta(K) = \tfrac{1}{4} \operatorname*{fin\,inf}_{H \in U_K} V(H).$$

Proof: By the Lemmas 4 and 5, since K is a subset of every hexagon H,

$$\Delta(K) \leq \Delta(H) = \tfrac{1}{4} V(H),$$

hence

$$\Delta(K) \geqq \tfrac{1}{4} \operatorname*{fin\,inf}_{H \in U_K} V(H). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Next choose any critical lattice Λ of K, and denote by $\mp P_1$, $\mp P_2$, $\mp P_3$, where $P_1 + P_3 = P_2$, its points on C (Lemma 2), and by $\mp L_1$, $\mp L_2$, $\mp L_3$ three pairs of symmetrical tac-lines of K at these points. The hexagon H bounded by these tac-lines is convex; hence, by Lemma 1, Λ is H-admissible, and so by Lemmas 4 and 5,

$$\Delta(K) = d(\Lambda) \geqq \Delta(H) = \tfrac{1}{4} V(H). \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Since H belongs to U_K, the assertion follows from (1) and (2).

By this proof, the lower bound is attained also in (II); hence the sign "*fin inf*" may also in this formula be replaced by the sign "*min*".

§ 4. Properties of critical lattices.

The two formulae (1) and (2) of the last paragraph imply that

$$V(H) = 4\,\Delta(K) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

for every hexagon H belonging to a critical lattice. Hence we find:

Theorem 2: *Let K be a convex domain symmetrical in O which is not a parallelogram; let Λ be any critical lattice of K; and let $\mp P_1$, $\mp P_2$, $\mp P_3$, where $P_1 + P_3 = P_2$, be the points of Λ on C. Then,* (i) *there are unique tac-lines $\mp L_1$, $\mp L_2$, $\mp L_3$ of K at these points* [7]*;* (ii) *no two of these tac-lines coincide;* (iii) *the hexagon H bounded by the tac-lines is of area $V(H) = 4\,\Delta(K)$;* (iv) *each side $\mp L_k$ of H is bisected at the lattice point $\mp P_k$ where it meets and touches C.*

Proof. The notation can be chosen such that when C is described in positive direction, then the six lattice points follow one another in the sequence

$$P_1, P_2, P_3, P_4 = -P_1, P_5 = -P_2, P_6 = -P_3.$$

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[6]) Parallelograms are considered as limiting cases of hexagons and must be included in U_K.

[7]) These tac-lines are therefore tangents of C.

Since K is not a parallelogram, none of the six arcs

$$\overset{\frown}{P_1P_2},\ \overset{\frown}{P_2P_3},\ \overset{\frown}{P_3P_4},\ \overset{\frown}{P_4P_5},\ \overset{\frown}{P_5P_6},\ \overset{\frown}{P_6P_1}$$

of C is a line segment [8]), and so (ii) is true. Hence H is a proper hexagon, and the tac-lines L_1 at P_1 and L_3 at P_3 are not parallel or coincident. Assume there is more than one tac-line L_2 at P_2; then this tac-line can vary over a whole angle, and so $V(H)$ is also variable and not constant, contrary to (3). Therefore the assumption is false and (i) is true. The assertion (iii) is identical to (3); from it, Λ must be a critical lattice of H, and so (iv) follows at once from Lemma 5.

One consequence of Theorem 2 is of particular interest:

Theorem 3: *Let K be an irreducible convex domain symmetrical in O which is not a parallelogram. Then the boundary C of K has everywhere a continuous tangent.*

Proof: Obvious from Lemma 3 and the last theorem.

This theorem is rather surprising, since the boundary of non-convex irreducible star domains may have angular points.

§ 5. **An inequality property of convex domains.**

Theorem 4: *To every convex domain K symmetrical in O, there exist an inscribed hexagon h and a circumscribed hexagon H both symmetrical in O such that*

$$4V(h) = 3V(H).$$

Proof: Obvious from (I) and (II), since the bounds are attained.

We deduce that if h runs over all inscribed symmetrical hexagons and H over all circumscribed symmetrical hexagons, then

$$4 \text{ fin sup } V(h) \geq 3 \text{ fin inf } V(H);$$

and here the ratio $4/3$ of the constants can not be replaced by a smaller one, as the example of the ellipse shows [8a]).

§ 6. **The case of a polygon.**

Let Π_n be a convex polygon of $2n$ sides $\mp L_1, \mp L_2, \ldots, \mp L_n$ where $n \geq 3$, and let $H_{\alpha\beta\gamma}$ be the proper hexagon bounded by $\mp L_\alpha, \mp L_\beta, \mp L_\gamma$ where α, β, γ run over all systems of three different indices $1, 2, \ldots, n$. The number of such hexagons is thus

$$\binom{n}{3} = \frac{n(n-1)(n-2)}{6}.$$

Theorem 5: *If Π_n is a polygon of $2n \geq 6$ sides symmetrical in O, then*

(III):
$$\Delta(\Pi_n) = \tfrac{1}{4} \min_{\alpha,\beta,\gamma} V(H_{\alpha\beta\gamma}).$$

Every critical lattice of Π_n is also a critical lattice of at least one hexagon $H_{\alpha\beta\gamma}$; hence Π_n has at most $\binom{n}{3}$ different critical lattices.

[8]) See paper *A*, Lemma 5.

[8a]) Theorem 4 is a special case of a more general result of L. FEJES, Compositio Mathematica **6**, 456—467 (1939), § 3.

Proof: Analogous to that of Theorem 1, except that U_K is replaced by the set of all hexagons $H_{\alpha\beta\gamma}$.

The upper bound $\binom{n}{3}$ for the number of critical lattices of Π_n is attained for $n = 3$ and $n = 4$, but not for larger n: it would therefore be of interest to find then the exact upper bound for this number.

§ 7. **The constants Q and Q_n.**

The lower bound

$$Q = \text{fin inf } Q(K)$$

extended over all convex domains symmetrical in O exists and satisfies the inequalities [9])

$$\sqrt{12} < Q < \frac{2\pi}{\sqrt{3}}. \quad \ldots\ldots\ldots \quad (4)$$

Moreover, there exist convex domains for which this bound is attained: they are called extreme domains.

Let, similarly, Q_n denote the lower bound

$$Q_n = \text{fin inf } Q(\Pi_n)$$

extended over all convex polygons Π_n of $2n \geq 4$ sides. It is evident that this limit exists and that $Q_n \geq Q$. From Lemmas 4 and 5.

$$Q_2 = Q_3 = 4.$$

We call Π_n extreme if

$$Q(\Pi_n) = Q_n.$$

§ 8. **The existence of extreme polygons Π_n.**

Theorem 6: *If $n \geq 3$, then there exists to every given polygon Π_n of $2n$ sides a polygon Π_{n+1} of $2(n+1)$ sides such that*

$$Q(\Pi_{n+1}) < Q(\Pi_n).$$

Proof: From Lemma 3 and any one of the Theorems 1, 3, or 5, every polygon not a parallelogram is reducible. Hence Π_n contains a convex domain K symmetrical in O and satisfying

$$V(K) < V(\Pi_n), \quad \Delta(K) = \Delta(\Pi_n).$$

At least one pair of vertices of Π_n, say the vertices $\mp Q_1$, lie outside K. Therefore there exist a pair of symmetrical tac-lines $\mp L$ of K such that L separates Q_1 and $-L$ separates $-Q_1$ from O, while all the other vertices of Π_n lie between these two lines. Denote by Π_{n-1} the set of all points of Π_n lying between L and $-L$. Then Π_{n-1} is a proper polygon of $2(n-1)$ sides, and from the construction

$$V(\Pi_{n+1}) < V(\Pi_n), \qquad \Delta(\Pi_{n+1}) \geq \Delta(\Pi_n),$$

hence

$$Q(\Pi_{n+1}) = \frac{V(\Pi_{n+1})}{\Delta(\Pi_{n+1})} < \frac{V(\Pi_n)}{\Delta(\Pi_n)} = Q(\Pi_n),$$

as asserted.

[9]) See paper *B*, §§ 1 and 5.

Theorem 7: *For every $n \geq 2$, there exists a polygon Π_n such that*

$$Q(\Pi_n) = \mathbf{Q}_n,$$

and this polygon is a proper $2n$-side.

Proof: There exists an infinite sequence of polygons

$$\Pi_n^{(1)} \quad , \quad \Pi_n^{(2)} \quad , \quad \Pi_n^{(3)} , \quad \ldots \ldots \ldots \quad (5)$$

satisfying

$$\lim_{r \to \infty} Q(\Pi_n^{(r)}) = \mathbf{Q}_n.$$

By affine invariance, these polygons may be assumed to satisfy the two conditions,

(*a*): $$Q(\Pi_n^{(r)}) = \frac{\sqrt{3}}{2} \qquad (r = 1, 2, 3, \ldots).$$

(*b*): The six fixed points

$$p_1 = (1, 0), \qquad p_2 = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right), \qquad p_3 = \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right),$$

$$p_4 = -p_1, \qquad p_5 = -p_2, \qquad p_6 = -p_3$$

lie on the boundary of each polygon $\Pi_n^{(r)}$.

Denote by H the regular hexagon of vertices $p_1, \ldots, p_6$, and by S the figure consisting of six equilateral triangles of unit side, where each such triangle has its base on one of the sides of H, while its opposite vertex lies outside H. From (*b*) and from the assumed convexity, all $2n$ vertices of each polygon $\Pi_n^{(r)}$ belong to the finite set S. It is therefore possible to select an infinite subsequence

$$\Pi_{n,1} = \Pi_n^{(r_1)} , \ \Pi_{n,2} = \Pi_n^{(r_2)} , \ \Pi_{n,3} = \Pi_n^{(r_3)}, \ldots \qquad (r_1 < r_2 < r_3 < \ldots)$$

of (5) such that the vertices of these polygons tend to $2n$ limiting points,

$$\mp \mathbf{Q}_1, \pm \mathbf{Q}_2, \ldots, \mp \mathbf{Q}_n, \quad \text{say.}$$

Let Π_n be the polygon which has these points as its vertices. Then by the continuity of V and $\triangle$,

$$\triangle(\Pi_n) = \lim_{r \to \infty} \triangle(\Pi_{n,r}) = \lim_{r \to \infty} \triangle(\Pi_n^{(r)}) = \frac{\sqrt{3}}{2},$$

hence

$$V(\Pi_n) = \lim_{r \to \infty} V(\Pi_{n,r}) = \lim_{r \to \infty} V(\Pi_n^{(r)}) = \frac{\sqrt{3}}{2} \lim_{r \to \infty} Q(\Pi_n^{(r)}) = \frac{\sqrt{3}}{2} \mathbf{Q}_n,$$

whence

$$Q(\Pi_n) = \mathbf{Q}_n,$$

so that Π_n is an extreme polygon. This implies that Π_n is a proper $2n$-side, since it would otherwise be possible, by Theorem 6, to inscribe a polygon Π_n^* of at most $2n$-sides for which

$$Q(\Pi_n^*) < Q(\Pi_n) = \mathbf{Q}_n,$$

contrary to the definition of $\mathbf{Q}_n$.

§ 9. **Properties of the constants $\mathbf{Q}$ and $\mathbf{Q}_n$.**

Theorem 8: *The constants $\mathbf{Q}_n$ and $\mathbf{Q}$ satisfy the relations,*

$$4 = \mathbf{Q}_2 = \mathbf{Q}_3 > \mathbf{Q}_4 > \mathbf{Q}_5 > \ldots > \mathbf{Q},$$

$$\lim_{n \to \infty} \mathbf{Q}_n = \mathbf{Q}.$$

Proof: The inequalities $\mathbf{Q}_n > \mathbf{Q}_{n+1}$ for $n \geq 3$ follow at once from the last two theorems. The further inequality $\mathbf{Q}_n > \mathbf{Q}$ holds since every polygon which is not a parallelogram is reducible. Finally, for the proof of the limit formula, denote by K any extreme convex domain, so that

$$Q(K) = \mathbf{Q}.$$

Given $\varepsilon > 0$, it is possible to approximate to K by a polygon Π_n of sufficiently large n such that

$$V(\Pi_n) < (1 + \varepsilon)\, V(K), \quad \triangle(\Pi_n) \geq \triangle(K),$$

hence

$$Q(\Pi_n) < (1 + \varepsilon)\, Q(K) = (1 + \varepsilon)\, \mathbf{Q}.$$

On allowing ε to end to zero, the assertion becomes obvious.

§ 10. **The triangles T_k belonging to an extreme octagon.**

The preceding results enable us to determine the extreme octagons Π_4 and to evaluate the constant $\mathbf{Q}_4$, as follows.

Let Π_4 be a fixed extreme octagon; for its vertices and sides, we use the notation of § 1, and we denote by k one of the four indices 1, 2, 3, 4.

On omitting the pair of sides $\mp L_k$ of Π_4, the remaining sides

$$\mp L_h, \text{ where } h \neq k, \ 1 \leqq h \leqq 4,$$

form the boundary of a hexagon, H_k say. This hexagon contains Π_4 as a subset and is, in fact, the sumset of Π_4 and two triangles T_k and $-T_k$ symmetrical to one another in O. Let T_k be that triangle with its base on L_k, and $-T_k$ the triangle with its base on $-L_k$. Then

$$V(H_k) = V(\Pi_4) + 2V(T_k),$$

whence by Theorem 5,

$$\triangle(\Pi_4) = \tfrac{1}{4}\, V(\Pi_k) + \tfrac{1}{2} \min_{1 \leq k \leq 4} V(T_k).$$

Therefore,

$$Q(\Pi_4)^{-1} = \tfrac{1}{4} + \tfrac{1}{2}\, M(\Pi_4), \text{ where } M(\Pi_4) = \min_{1 \leq k \leq 4} \frac{V(T_k)}{V(\Pi_4)}. \quad . \quad (6)$$

For an extreme octagon, $M(\Pi_4)$ evidently assumes its largest value.

Theorem 9: *If Π_4 is an extreme octagon, then*

$$V(T_1) = V(T_2) = V(T_3) = V(T_4).$$

Proof: It suffices to show that if these equations are not all satisfied, then there exists an octagon Π_4^* satisfying

$$M(\Pi_4^*) > M(\Pi_4). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

We may assume, without loss of generality, that T_2 is the triangle of smallest area and that, say,

$$V(T_1) \geqq V(T_2), \qquad V(T_3) > V(T_2). \quad . \quad . \quad . \quad . \quad . \quad (8)$$

The line L_2 intersects L_1 at the vertex Q_2 of Π_4, and it intersects $-L_4$ at a point R_1 which is a vertex of T_1. Denote by Q_2^* an *inner* point of the line segment Q_1Q_2, and by R_1^* the point on $-L_4$ near to R_1 for which the triangle $T_1^* = Q_1R_1^*Q_2^*$ is of equal area to T_1:

$$V(T_1^*) = V(T_1). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

Let further L_2^* be the line through Q_2^* and R_1^*, and let Π_4^* be the octagon bounded by the sides $\mp L_1$, $\mp L_2^*$, $\mp L_3$, $\mp L_4$. Then, firstly,

$$V(\Pi_4^*) < V(\Pi_4), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

since Π_4^* is contained in Π_4. Next let T_1^*, T_2^*, T_3^*, T_4^* be the triangles analogous to T_1, T_2, T_3, T_4 which belong to Π_4^*, and assume that Q_2^* is chosen sufficiently near to Q_2. Then $V(T_3^*)$ differs arbitrarily little from $V(T_3)$; further, from the construction,

$$V(T_2^*) > (T_2), \qquad V(T_3^*) < V(T_3), \qquad V(T_4^*) = V(T_4), \quad . \quad (11)$$

the last formulae holding since T_4^* and T_4 are the same triangle. On combining (8), (9), and (11), secondly,

$$\min_{1 \leq k \leq 4} V(T_k^*) \geqq \min_{1 \leq k \leq 4} V(T_k). \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

The assertion (7) follows now immediately from (6), (10), and (12).

§ 11. **Determination of the extreme octagons.**

We determine now the octagons Π_4 for which

$$V(T_1) = V(T_2) = V(T_3) = V(T_4), \quad . \quad . \quad . \quad . \quad . \quad (13)$$

and select from among these the extreme ones. Since $M(\Pi_4)$ is an affine invariant, it suffices to consider octagons which are normed in the following way:

Denote by R_1, R_2, R_3, R_4 the points of intersection of $-L_4$ and L_2, L_1 and L_3, L_2 and L_4, and L_3 and $-L_1$, respectively, and by $\Pi_2^{(1)}$ the parallelogram of vertices $\mp R_1$, $\mp R_3$, and by $\Pi_2^{(2)}$ the parallelogram of vertices $\mp R_2$, $\mp R_4$. Hence $\Pi_2^{(1)}$ has the sides $\mp L_2$, $\mp L_4$, and $\Pi_2^{(2)}$ has the sides $\mp L_1$, $\mp L_3$, and Π_4 is the intersection of $\Pi_2^{(1)}$ and $\Pi_2^{(2)}$. Apply an affine transformation such that $\Pi_2^{(1)}$ becomes the square of vertices

$$R_1 = (1, -1), \; R_3 = (1, 1), -R_1, -R_3.$$

The second parallelogram $\Pi_2^{(2)}$ is then subject only to the conditions that its sides intersect those of $\Pi_2^{(1)}$ so as to form together a convex octagon Π_4. Let the sides of $\Pi_2^{(2)}$ be, say,

$$L_1 : \; x_2 = tx_1 - \tau; \quad L_3 : \; x_2 = -sx_1 + \sigma;$$
$$-L_1 : \; x_2 = tx_1 + \tau; \quad -L_3 : \; x_2 = -sx_1 - \sigma;$$

its vertices are therefore

$$R_2 = \left(\frac{\sigma+\tau}{s+t}, \frac{\sigma t - s\tau}{s+t}\right), \quad R_4 = \left(\frac{\sigma-\tau}{s+t}, \frac{\sigma t + s\tau}{s+t}\right), -R_2, -R_4.$$

On intersecting the sides of $\Pi_2^{(1)}$ and $\Pi_2^{(2)}$, the vertices of Π_4 become,

$$Q_1 = \left(\frac{\tau-1}{t}, -1\right); \quad Q_2 = (1, t-\tau); \quad Q_3 = (1, -s+\sigma); \quad Q_4 = \left(\frac{\sigma-1}{s}, 1\right);$$

$$-Q_1, -Q_2, -Q_3, -Q_4.$$

From the construction, L_1 is of positive and L_3 of negative gradient, and these lines meet the coordinate axes outside $\Pi_2^{(1)}$; hence

$$s > 0, \ t > 0, \ \sigma > 1, \quad \tau > 1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

The conditions that the four points R_1, Q_2, Q_3, R_3 on L_2, and the four points R_3, Q_4, $-Q_1$, $-R_1$ on L_4, follow one another in this order, give the further inequalities,

$$\xi > 0, \quad \eta > 0, \quad \xi + \eta < 2, \quad 2st - t\xi - s\eta > 0, \quad . \quad . \quad (15)$$

where ξ and η are defined by

$$\xi = s - \sigma + 1, \quad \eta = t - \tau + 1.$$

The areas of the triangles T_k are easily obtained; on substituting in (13), these equations take the form,

$$2V(T_k) = \frac{\xi^2}{s} = \frac{\eta^2}{t} = \frac{(2-\xi-\eta)^2}{s+t} = \frac{(2st - t\xi - s\eta)^2}{st(s+t)}, \qquad = \frac{1}{\lambda} \text{ say},$$

where, from (14) and (15), λ is positive; hence

$$s = \lambda\xi^2, \ t = \lambda\eta^2, \ s + t = \lambda(2-\xi-\eta)^2, \ st(s+t) = \lambda(2st - t\xi - s\eta)^2.$$

From these equations, firstly

$$\xi^2 + \eta^2 = (2-\xi-\eta)^2, \quad \text{hence} \quad 2-\xi-\eta = \xi + \eta - \xi\eta, \quad . \quad . \quad (16)$$

and secondly,

$$\lambda^3\xi^2\eta^2(2-\xi-\eta)^2 = \lambda(2st - t\xi - s\eta)^2 = \lambda^3\xi^2\eta^2(2\lambda\xi\eta - \xi - \eta)^2,$$

whence, from (15),

$$2 - \xi - \eta = \mp(2\lambda\xi\eta - \xi - \eta),$$

and so, either

(A): $$2-\xi-\eta = +(2\lambda\xi\eta - \xi - \eta), \qquad \lambda = \frac{1}{\xi\eta},$$

or

(B): $$2-\xi-\eta = \xi + \eta - \xi\eta = -(2\lambda\xi\eta - \xi - \eta), \qquad \lambda = \tfrac{1}{2}.$$

In case (A),

$$s = \frac{\xi}{\eta}, \quad t = \frac{\eta}{\xi}, \quad st = 1,$$

so that adjacent sides of $\Pi_2^{(2)}$ are perpendicular; hence $\Pi_2^{(2)}$ is a rectangle. It is even a square congruent to $\Pi_2^{(1)}$, since the distances

$$\delta_1 = +\tau(1+t^2)^{-\frac{1}{2}} = \left|\left(\frac{\eta}{\xi} - \eta + 1\right)\left(1+\frac{\eta^2}{\xi^2}\right)^{-\frac{1}{2}}\right| = \left|\frac{\xi\eta - \xi - \eta}{\sqrt{\xi^2+\eta^2}}\right|,$$

$$\delta_3 = +\sigma(1+s^2)^{-\frac{1}{2}} = \left|\left(\frac{\xi}{\eta} - \xi + 1\right)\left(1+\frac{\xi^2}{\eta^2}\right)^{-\frac{1}{2}}\right| = \left|\frac{\xi\eta - \xi - \eta}{\sqrt{\xi^2+\eta^2}}\right|,$$

of L_1 and L_3 from O are both equal to unity, as follows from (16). The four triangles T_k are therefore congruent and of area

$$V(T_k) = \frac{\xi^2}{2s} = \frac{\xi\eta}{2}.$$

Further

$$V(\Pi_4) = V(\Pi_2^{(1)}) - 4\,V(T_k) = 4 - 2\xi\eta;$$

hence

$$M(\Pi_4) = \frac{\xi\eta}{4(2-\xi\eta)}$$

is an increasing function of $\xi\eta$. By (15) and (16),

$$\xi > 0, \quad \eta > 0, \quad \xi + \eta < 2, \quad (2-\xi)(2-\eta) = 2,$$

and so $M(\Pi_4)$ attains its maximum when

$$\xi = \eta = 2 - \sqrt{2}, \quad \xi\eta = 6 - 4\sqrt{2}, \quad s = t = 1, \quad \sigma = \tau = \sqrt{2},$$

that is, when Π_4 is a regular octagon. For such an octagon,

$$M(\Pi_4) = \frac{6-4\sqrt{2}}{4(4\sqrt{2}-4)} = \frac{\sqrt{2}-1}{8}, \quad Q(\Pi_4) = \left\{\frac{1+2M(\Pi_4)}{4}\right\}^{-1} = \frac{16}{7}(3-\sqrt{2}).$$

Next, in case (B),

$$s = \frac{\xi^2}{2}, \quad t = \frac{\eta^2}{2},$$

whence from (15) and (16),

$$\xi\eta > 0, \ \xi + \eta < 2, \ 2st - t\xi - s\eta = \frac{\xi\eta}{2}(\xi\eta - \xi - \eta) = \frac{\xi\eta}{2}(\xi + \eta - 2) > 0,$$

which is impossible; this case therefore cannot arise.

We have thus proved [10])

Theorem 10: *For every convex octagon Π_4 symmetrical in O,*

$$Q(\Pi_4) \geqq \frac{16}{7}(3-\sqrt{2}),$$

with equality if and only if Π_4 is affine-equivalent to the regular octagon.

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[10]) Dr. LEDERMANN, to whom I showed this paper, has since found a much simpler proof of Theorem 10.

§ 12. **An upper bound for Q.**

The last theorem implies that

$$Q_4 = \frac{16}{7}(3-\sqrt{2}) = 3.624654715\ldots \quad [11].$$

This result is rather surprising, since in the case of an ellipse E [12]

$$Q(E) = \frac{2\pi}{\sqrt{3}} = 3.627598727\ldots > Q_4.$$

As we show now, one can construct an irreducible convex domain K for which $Q(K)$ is even smaller.

Let again Π_4 be the regular octagon which is the intersection of the square $\Pi_2^{(1)}$ of vertices

$$R_1 = (1,-1),\ R_3 = (1,1),\ -R_1,\ -R_3,$$

and the square $\Pi_2^{(2)}$ of vertices

$$R_2 = (\sqrt{2},0),\quad R_4 = (0,\sqrt{2}),\ -R_2,\ -R_4.$$

The vertices of Π_4 itself are

$$Q_1 = (\sqrt{2}-1,-1),\quad Q_2 = (1,1-\sqrt{2}),\quad Q_3 = (1,\sqrt{2}-1),\quad Q_4 = (\sqrt{2}-1,1),$$
$$-Q_1,\ -Q_2,\ -Q_3,\ -Q_4,$$

and further

$$V(\Pi_4) = 8(\sqrt{2}-1),\quad \triangle(\Pi_4) = \sqrt{2}-\frac{1}{2},\quad Q(\Pi_4) = \frac{16}{7}(3-\sqrt{2}). \tag{17}$$

There are four hexagons H_k circumscribed to Π_4, namely,

the hexagon H_1 of vertices $R_1,\ Q_3,\ Q_4,\ -R_1,\ -Q_3,\ -Q_4$;
the hexagon H_2 of vertices $R_2,\ Q_4,\ -Q_1,\ -R_2,\ -Q_4,\ Q_1$;
the hexagon H_3 of vertices $R_3,\ -Q_1,\ -Q_2,\ -R_3,\ Q_1,\ Q_2$;
the hexagon H_4 of vertices $R_4,\ -Q_2,\ -Q_3,\ -R_4,\ Q_2,\ Q_3$.

Each hexagon H_k possesses just one critical lattice Λ_k, and this is also a critical lattice of Π_4. On the boundary of Π_4, Λ_k has exactly six points, say the points

$$\mp U_k,\quad \mp V_k,\quad \mp W_k,$$

namely the midpoints of the sides of H_k. The coordinates of these points are given in the following table:

$U_1 = (\sqrt{\tfrac{1}{2}}-1,-1)$,	$V_1 = (\sqrt{\tfrac{1}{2}},-\sqrt{\tfrac{1}{2}})$,	$W_1 = (1,1-\sqrt{\tfrac{1}{2}})$,
$U_2 = (\tfrac{1}{2},\tfrac{1}{2}-\sqrt{2})$,	$V_2 = (1,0)$,	$W_2 = (\tfrac{1}{2},\sqrt{2}-\tfrac{1}{2})$,
$U_3 = (1,\sqrt{\tfrac{1}{2}}-1)$,	$V_3 = (\sqrt{\tfrac{1}{2}},\sqrt{\tfrac{1}{2}})$,	$W_3 = (\sqrt{\tfrac{1}{2}}-1,1)$,
$U_4 = (\sqrt{2}-\tfrac{1}{2},\tfrac{1}{2})$,	$V_4 = (0,1)$,	$W_4 = (\tfrac{1}{2}-\sqrt{2},\tfrac{1}{2})$.

Evidently,

$$U_k + W_k = V_k,\qquad \{U_k, W_k\} = \triangle(\Pi_4)\qquad (k = 1,2,3,4). \tag{18}$$

[11] I am in great debt to Mr. D. F. FERGUSON, M. A., for the evaluation of this constant and the two other ones.

[12] See paper *B*, § 1.

Consider now two variable points

$$P_1 = (1, \alpha), \qquad P_3 = (\beta, \beta + \sqrt{2})$$

on the line segments joining V_2 to W_1 and $-U_2$ to $-V_1$, respectively, and assume that the determinant of these two points has the value,

$$\{P_1, P_3\} = \triangle(\Pi_4). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

Then the point

$$P_2 = (x_1, x_2) = P_1 + P_3 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

describes a hyperbola arc A_4 connecting W_2 with $-U_1$. Since by (19),

$$\beta - \alpha\beta + \sqrt{2} = \sqrt{2} - \tfrac{1}{2},$$

and by (20),

$$x_1 = 1 + \beta, \qquad x_2 = \alpha + \beta + \sqrt{2},$$

this hyperbola has the equation,

$$x_2 = x_1 + \frac{1}{2(x_1 - 1)} + \sqrt{2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

The arc A_4 touches the boundary Γ_4 of Π_4 at the two points $-U_1$ and W_2, and together with this boundary encloses a curvilinear triangle, τ_4 say, which is of area,

$$\left.\begin{aligned} V(\tau_4) = \{[\tfrac{1}{2} - (1 - \tfrac{1}{2}\sqrt{\tfrac{1}{2}})] \cdot 1 - \tfrac{1}{2}[\tfrac{1}{2} - (\tfrac{1}{2}\sqrt{2} - 1)]^2\} - \\ - \int_{1-\sqrt{\frac{1}{2}}}^{\frac{1}{2}} \left(x_1 + \frac{1}{2(x_1 - 1)} + \sqrt{2}\right) dx_1 \\ = (-\tfrac{21}{8} + 2\sqrt{2}) - (\tfrac{3}{8} - \tfrac{1}{4}\log 2) = 2\sqrt{2} - 3 + \tfrac{1}{4}\log 2. \end{aligned}\right\} \quad (22)$$

In just the same way, each vertex $\mp Q_k$ of Π_4 can be separated from O by means of a hyperbola arc $\mp A_k$; this arc is congruent to A_4 and touches Γ_4, and it encloses, together with Γ_4, a triangle τ_k congruent to τ_4.

Let now K be the convex domain obtained from Π_4 by cutting off all eight triangles $\mp \tau_k$. Then every point on the boundary C of K belongs to a lattice of determinant $\triangle(\Pi_4)$ which has on C just six points $\mp P_1, \mp P_2, \mp P_3$ satisfying $P_1 + P_3 = P_2$, and is therefore K-admissible (Lemma 1). Hence K is irreducible and of determinant

$$\triangle(K) = \triangle(\Pi_4) = \sqrt{2} - \tfrac{1}{2}.$$

(Lemmas 2 and 3). On the other hand, from (17) and (22),

$$V(K) = V(\Pi_4) - 8\,V(\tau_4) = 16 - 8\sqrt{2} - \log 4.$$

By combining these two equations, we find that

$$Q(K) = \frac{32 - 16\sqrt{2} - 4\log 2}{2\sqrt{2} - 1} = 3.609656737\ldots$$

This is an upper bound for $\mathbf{Q}$, and possibly even its exact value.

Mathematics Department, Manchester University.

March 20, 1947.

Biochemistry. — *Distribution of the complexcomponent which is present in excess, between Complexcoacervate and Equilibrium liquid.* By H. G. BUNGENBERG DE JONG.

(Communicated at the meeting of April 26, 1947.)

1. *Introduction.*

At a definite mixing proportion of the two equally concentrated (2 %) isohydric Gelatin and Gum Arabic Sols, complexcoacervation is a maximum [1]). At this mixing proportion the Gelatin-cations and Arabinate-anions unite in aequivalent proportions. As the (apparent) aequivalent weights of both colloids are functions of pH, this "aequivalent" mixing proportion shifts to other values if the pH is changed.

From previous publications it was already known in which direction the compositions of the coacervate and of the equilibrium liquid are altered, if at constant pH the mixing proportion is changed [1]), respectively if at constant mixing proportion the pH is changed [2]).

Below it will be shown, that both changes can be summarized from one simple point of view.

2. *Variation of the mixing ratio of the sols at constant pH.*

The following table contains the analytical results for pH 3.51 [3]). The mixing proportion of the isohydric 2 % sols (column 1) is here varied. The colloid contents (G = Gelatin, A = gum arabic) of the coacervate layer and of the equilibrium-liquid are represented in the columns 2 and 3, respectively in 4 and 5.

The "colloidcompositions", i.e. 100 A/A + G, to be found in the last two columns, were calculated from the data of columns 2, 3, 4 and 5.

These colloid compositions as function of the mixing ratio are represented

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[1]) H. G. BUNGENBERG DE JONG and W. A. L. DEKKER, Kolloid Beih. **43**, 213 (1936). The analytical results, which will serve as a base for what follows, are to be found on p. 222—223. For a discussion of the changes in composition of coacervate and equilibrium liquid, see p. 233 and fig. 11.

[2]) H. G. BUNGENBERG DE JONG and B. KOK, Proc. Ned. Akad. v. Wetensch., Amsterdam, **45**, 51 (1942).

[3]) It has been recognized since a few years, that in the calculating the pH from E.M.F. measurements with the H electrode, a systematic error was introduced. All pH values in the publication mentioned in note [1]) must be increased with 0.33. The isohydric mixing series thus actually was not pH 3.51 but 3.84. In the same way all pH values in table 2 must be increased with 0.33. As also the I.E.P. of the gelatin used must be increased with the same amount and as in the discussion of the experimental result only pH differences and not the absolute values of the pH play a rôle, general conclusions rest the same. In the aim not to cause confusion the originally given pH values have been maintained in this publication.

in fig. 1 by curve C (coacervate, column 6) and curve E (equilibrium liquid, column 7).

Mixing ratio of sols $100 \frac{A}{A+G}$	Composition coacervate		Composition equilibrium liquid		$100 \frac{A}{A+G}$	
	% G	% A	% G	% A	in coacervate	in equilibrium liquid
33.6	7.69	6.11	0.686	0.234	44.3	25.4
40.4	7.86	6.60	0.419	0.151	45.6	26 5
45.4	7.55	6.77	0.256	0.154	47.3	37.6
50.4	7.01	6.93	0.170	0.200	49.7	54.1
55.4	6.15	7.30	0.162	0.318	54.3	66.3
60.4	5.88	7.38	0.195	0.535	55.7	73.3
65.3	5.41	7.15	0.261	0.839	56.9	76.3

These curves intersect each other at the aequivalent mixing proportion (colloid composition of coacervate = colloid composition of the equilibrium liquid). At this or very near to this mixing ratio lays also the electro-

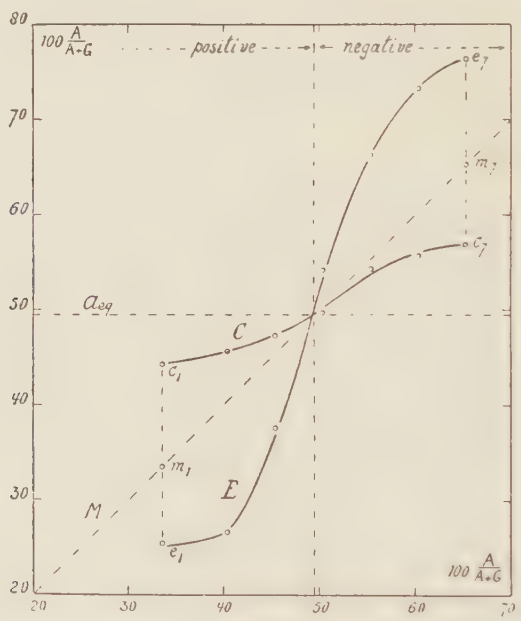


Fig. 1.

phoretic reversal of charge of the boundary coacervate drops/equilibrium liquid.

The figure further contains three dotted lines, all three going through the intersecting point of the curves C and E. The vertical one divides the plane of the figure into a left half, in which the coacervate is positively charged, and a right half, in which the coacervate is negatively charged.

The horizontal dotted line ("Aeq") gives the colloidcomposition of the coacervate (and equilibrium liquid) at the aequivalent mixing ratio.

The third dotted line ("M"), drawn at an angle of 45° represents the colloid compositions of the total mixtures (as they were prepared by mixing equally concentrated Gelatin and Gum Arabic Sols).

As at the aequivalent mixing proportion, the colloid composition of the coacervate and of the equilibrium liquid are equal, these colloid compositions are of cause here also equal to that of the total solmixture. Therefore the intersecting point of the curves C and E must lay on the dotted line "M".

We must now draw our attention to the relative positions of the curves C, E and of the dotted lines "Aeq" and M.

From the relative position of C, E and "Aeq" follows, that if one of the colloid components is present in excess, this component distributes itself in such a way, that the colloid composition of the equilibrium liquid is much more changed than that of the coacervate.

A total mixture m_7 thus divides in an equilibrium liquid e_7, which is still richer in A than m_7, and in a coacervate c_7, which comparatively is but little richer in A than the aequivalent coacervate (c_7 lies not far from the horizontal dotted line "Aeq").

The same can be observed with positively charged coacervates: the equilibrium liquid e_1 is still richer in G than m_1, but the coacervate c_1 is only little richer in G than the aequivalent coacervate.

From this much smaller shift in composition of the coacervate compared with the shift in composition of the equilibrium liquid appears that the processes at work in complexcoacervation tend to the separation of the aequivalent coacervate.

The colloid component in excess mainly accumulates in the equilibrium liquid. A smaller part of it nevertheless enters the coacervate changing it in two ways:

1) the coacervate boundary assumes the electrical sign of the component in excess,
2) the reciprocal solubility of coacervate and equilibrium liquid is increased.

Fig. 1 is valid for one chosen pH; at other pH values for the isohydric mixing-series the system of intersecting curves C and E shifts in such a way that its intersecting point always lies on the dotted line "M". At higher pH this intersecting point moves to the left, at lower pH to the right. For the rest, the relative positions of the curves C, E and the lines (through the intersecting point) "Aeq" and "M" are not altered in principle, and therefore the conclusions drawn are the same as above.

3. *Variation of the pH at constant mixing ratio of the sols.*

Combining the analytical result for five isohydric mixing series, it is possible to survey the changes in G and A composition of the coacervate and equilibrium liquid at one constant mixing ratio and variation of the pH.

For this aim we choose the mixing ratio of 50 %, for alone this ratio occurs in all five mixing series.

We then find the colloid percentages as given in the columns 2, 3, 4 and 5 of the following table. From these figures are then calculated the "Colloidcompositions" of the coacervate and the equilibrium liquid as given in columns 6 and 7.

pH	Coacervate		Equilibrium liquid		$100\frac{A}{A+G}$		Mixing ratio of aequivalent Coacervation
	% G	% A	% G	% A	Coacervate	Equilibrium liquid	
3.00	5.44	6.92	0.59	0.495	56.0	45.6	62.2
3.29	6.68	7.39	0.277	0.183	52.5	39.8	55.2
3.51	7.06	6.92	0.175	0.195	49.5	52.7	49.5
3.80	6.25	5.27	0.324	0.446	45.7	57.8	40.4
4.00	3.9	3.0	0.80	0.90	43.5	52.9	32.7

These colloid compositions as function of pH are represented in fig. 2 by the curves C (coacervate) and E (equilibrium liquid). The colloid

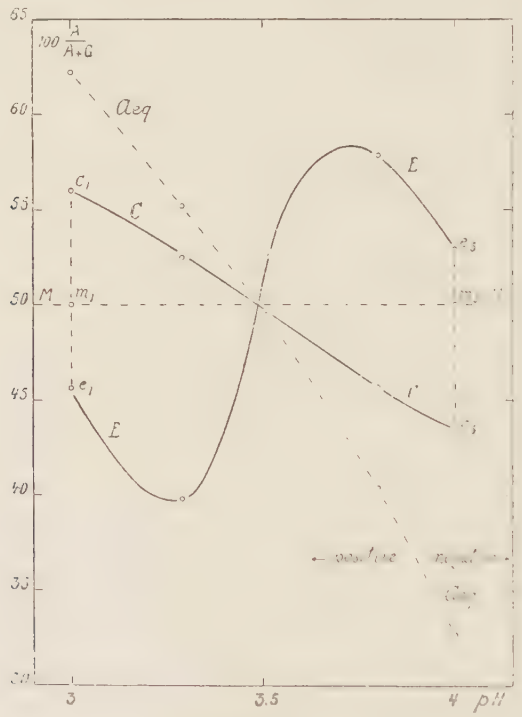


Fig. 2.

compositions of the total mixtures lie, as they all have now the same mixing ration (50 %), on the dotted horizontal line "M". Also in this fig. 2 the curves C and E intersect and the intersecting point necessarily must lie on the dotted line M.

This intersecting point lies at that pH, whereby the chosen mixing ratio of the sols, is at the same time the aequivalent mixing ratio.

Fig. 2 further contains a curve "Aeq", indicating the colloid compositions of the aequivalent coacervates at different pH values (from column 8).

As at or very near to the aequivalent mixing ratio reversal of charge is obtained, curve "Aeq" divides the plane of the figure into a positive half (to the left) and a negative half (to the right).

From the relative positions of the curves C, E, Aeq and the line M one may conclude that here once more the same tendency of the processes at work in complexcoacervation may be perceived as in fig. 1: these processes tend to the separation of the aequivalent coacervate at the prevailing pH.

As the mixing ratio is now held constant, this tendency can only be satisfied completely at one pH value. At the remaining pH values this tendency reveals itself in the fact that a total mixture (e.g. m_1 or m_5) separates into a coacervate (c_1 or c_5) the colloid composition of which is much nearer to that of the aequivalent coacervate belonging to that pH, and an equilibrium liquid (e_1 or e_5), the colloid composition of which is still farther different from that of the aequivalent coacervate as the colloid composition of the total mixture.

Summary.

1) The distribution of the complexcomponent which is in excess, between complexcoacervate and equilibrium liquid, is discussed using formerly published analytical data.

2) In both cases considered:

a) variation of mixing ratio of sols at constant pH,

b) variation of pH at constant mixing ratio,

the above distribution takes place in such a way, that therein a tendency can be perceived to maintain the separation of a complexcoacervate of aequivalent colloid composition, belonging to the prevailing pH.

Anatomy. — *Examen Clinico-Anatomique de l'encéphalite chronique de l'hypothalamus et du territoire avoisinant.* By B. BROUWER. (Travail de la Clinique et du Laboratoire de Neurologie de l'Hôpital „Wilhelmina" et de l'Institut Central Néerlandais pour les recherches du Cerveau à Amsterdam.)

(Communicated at the meeting of June 28, 1947.)

Il a été prouvé par des recherches expérimentale-anatomiques et clinico-anatomiques, que l'hypothalamus forme, dans l'organisation des agissements du système nerveux végétatif et de sa collaboration avec le système nerveux de la vie animale, un centre d'embranchement important. Le traité, récemment paru, de ROUSSY et MOSINGER [10]), et l'ouvrage sur l'hypothalamus édité en 1940 par l'„Association for research in nervous and mental disease" [9]) résument nos connaissances actuelles sur ce sujet. J'ai, dans les dernières années, avec mes collaborateurs, réuni un important matériel dont les données cliniques ont été vérifiées par l'examen du cerveau sur des coupes sériées de la région hypothalamique et de son entourage. L'une de ces observations sera décrite dans le présent article.

Le syndrome suivant, qui s'étendit lentement et progressivement, concerne un homme de 33 ans:

a. Obésité et troubles sexuels, mais sans atrophie des organes génitaux. Diminution de la croissance de la barbe et du système pileux du mons Veneris. Abaissement du métabolisme basal (—10.9 %).
b. Troubles de la thermorégulation centrale.
c. Accès d'hypéridrose dans la moitié gauche de la face et, plus tard, dans toute la moitié gauche du corps.
d. Tic facial à gauche.
e. Parole monotone et murmurante.
f. Troubles cérébraux de la vessie et, plus tard, de la défécation.
g. Inversion du mécanisme du sommeil (veillée nocturne, sommeil diurne).
h. Psychose de Korsakoff.
i. Grisonnement précoce de la chevelure.
j. Amblyopie à gauche, avec restriction concentrique du champ visuel.
k. Anomalie des mouvements du regard.
l. Anomalie de la fonction vestibulaire.

Nous n'avons point observé, pendant l'année que le malade passa à la clinique, divers autres symptômes du métabolisme que l'on a vus fréquemment lors d'affections hypothalamiques: le diabète insipide, par exemple, le diabète sucré, des anomalies de l'appareil vaso-moteur, etc.

L'autopsie (Prof. Dr. T. DEELMAN, numéro de l'autopsie 42299) ne révéla

aucune anomalie en dehors du cerveau (à l'exception de quelques foyers frais de bronchopneumonie). L'examen microscopique de l'hypophyse, fait au laboratoire d'anatomie pathologique sur des coupes en série, montra, dans le lobe postérieur, du pigment brun, granuleux, donnant en partie une ferro-réaction positive. Il y avait croissance de cellules basophiles dans le lobe postérieur. La pars intermédia contenait des kystes recouverts parfois de cellules basophiles et, parfois, de cellules indifférentes. Il y avait, dans la pars antérior, quelques petits follicules avec du colloïde. Les cellules basophiles présentaient une grande quantité de vacuoles. Les rapports cellulaires étaient normaux. Le Prof. DEELMAN me céda le cerveau pour me permettre de plus amples recherches; je l'en remercie ici tout cordialement.

L'examen histologique ne décèle rien d'important à l'écorce du pallium et du cervelet. L'examen des coupes sériées (coloration: en partie selon Nissl et à l'hématoxyline-éosine, en partie selon Weigert–Pal et van Gieson), ne laisse voir aucune anomalie dans la moelle oblongue. Le pont de Varole, le tegmentum et l'entourage de l'aqueduc de Sylvius ne contiennent, ici et là, que quelques infiltrations perivasculaires qui sont évidemment les jets de l'encéphalite hypothalamique qui sera décrite ci-dessous. Il en est de même quant au mésencéphale. Il y a quelque augmentation des vaisseaux sanguins et quelques infiltrats inflammatoires dans les corpora quadrigemina antica et postica, mais les cellules dans l'entourage immédiat de ces derniers ne sont pas endommagées. Il en est de même quant au pulvinar. La commissure postérieure, le noyau de Darkewitch, le noyau interstitiel, le noyau rouge et la substance noire sont pratiquement sans anomalie. Par contre, le procès inflammatoire est plus intense dans le gyrus hippocampi gauche; il y a aussi ici perte parenchymateuse avec prolifération secondaire de la névroglie. On ne remarque, dans le gyrus hippocampi droit, que des réactions vasculaires.

Dans le néothalamus, le noyau antérieur est occupé par des infiltrations inflammatoires. Il n'y a, dans les autres noyaux thalamiques que peu d'altérations. Le corps sous-thalamique est normal. Le globus pallidus et le putamen sont à peu près intacts, le noyau caudé montre de la réaction mésodermale, de même que le gyrus cinguli et quelque lobules de l'écorce de l'insula gauche. Mais l'hypothalamus est intensivement altéré. Il y a ici une encéphalite de caractère lymphocytaire, laquelle, à certains endroits, est accompagnée d'une méningite circonscrite. Nous n'avons pas trouvé de cellules géantes.

Nous suivons, dans la description des altérations de l'hypothalamus, la nomenclature selon LE GROS CLARK [7]).

Dans la partie caudale de l'hypothalamus, les corps mamillaires ont perdu, bilatéralement, beaucoup de cellules et ces ganglions contiennent un amassement de cellules de microglie et de lymphocytes. Il y a bien encore quelques cellules à la place des nuclei intercalati, mais on ne peut distinguer, comme tels, des groupes cellulaires nucléaires. La paroi de la moitié ventrale de l'infundibulum contient des colliers d'inflammation typiques, tandis que le noyau hypothalamique postérieur a perdu de nombreuses cellules. On ne voit dans les coupes de Nissl et d'hématoxyline-éosine du faisceau de Vicq d'Azyr, aucunes cellules de névroglie de quelque importance. Les fornix sont pour ainsi dire incrustés dans du tissu enflammé. Le procès est évidemment avant tout une polioencéphalite et

non une leucoencéphalite. La partie des fornix descendants qui est attachée au corps calleux est pour ainsi dire intacte, dans toute la série, et contient seulement, à quelques endroits, un infiltrat périvasculaire. Quand les corps mamillaires ont disparu de la série, on constate que le tuber cinereum est complètement occupé par des cellules névrogliques, des lymphocytes et de nombreux vaisseaux sanguins (Photo I). Il n'y a pas de cellules ganglionnaires bien nettes. L'épendyme de l'infundibulum n'est pas altéré et le plexus choroïde ne présente aucune altération vasculaire, mais la substance grise qui se trouve près du ventricule est, à ce niveau, fortement altérée elle-aussi. Dans la moitié orale du tuber cinereum, étudiée sur des coupes de Weigert–Pal et de van Gieson, on remarque une démyélinisation des fibrilles plus fines du tuber. On ne peut plus reconnaître comme tels les noyaux tubériens latéral, médial et intermédiaire. On ne retrouve non plus aucun vestige de noyaux hypothalamique ventro-médial ou ventro-dorsal dans la partie médiale de la substance grise. On ne voit pas de cellules appartenant au noyau infundibulo-mamillaire. Les noyaux paraventriculaires sont perdus. Le procès inflammatoire est plus étendu vers la droite que vers la gauche (voir foto 2). Les coupes du niveau des noyaux supra-optiques révèlent que ces noyaux sont intacts, et le procès pathologique a de même épargné les parois latérales du tuber cinereum.

En direction orale dans la série, le procès inflammatoire diminue rapidement d'intensité et d'extension, de sorte qu'au niveau de la région préoptique, on ne voit plus, ici et là, que quelques parties de tissu altéré par une inflammation et ayant subi de la perte nerveuse. Les tractus optiques, le chiasma et les nerfs optiques sont intacts.

La figure No. 3 montre un dessin de l'extension maximale du procès pathologique en direction vertico-frontale.

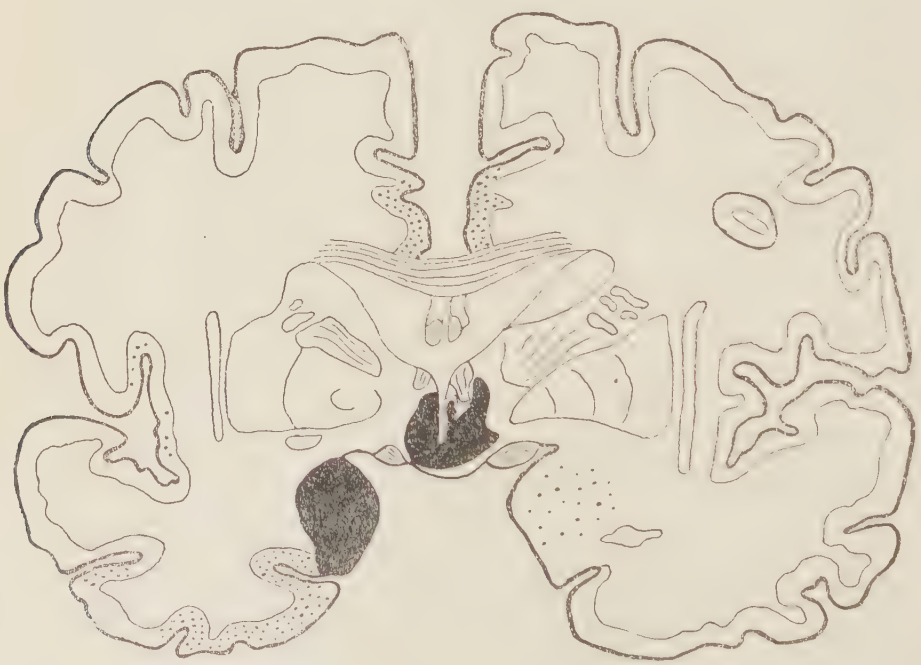


Fig. 3.
Extension maximale du procès pathologique en direction vertico-frontale.
(Les spaces noires indiquent l'extension de la perte nerveuse, les spaces pointillés de la réaction mésodermale.)



Fig. 1. Encéphalite chronique du tuber cinereum et du gyrus hyppocampi gauche.
(Méthode de NISSL.)

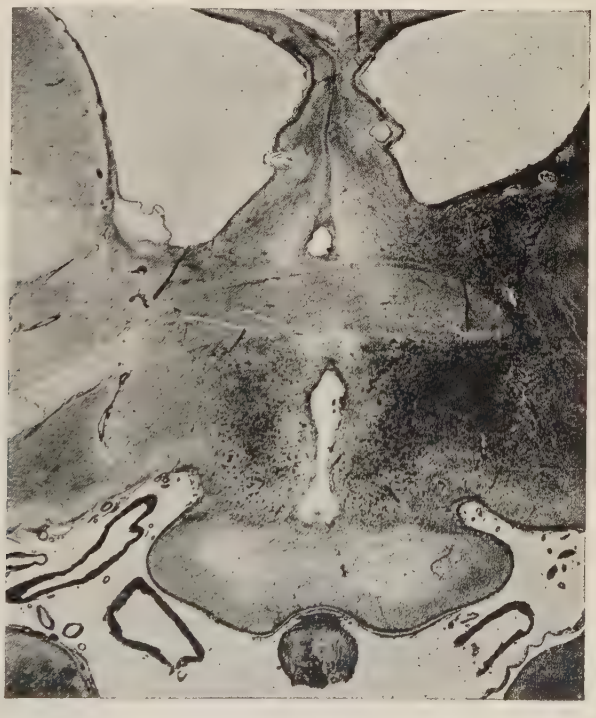


Fig. 2. Région supra-chiasmatique. Encéphalite chronique de la substance grise latérale du troisième ventricule.
(Méthode de VAN GIESON.)

Par conséquent, en résumé, cette encéphalite a provoqué, exclusivement, une perte nerveuse intense des corps mamillaires, du tuber cinereum, de la substance grise des parois du troisième ventricule et du pôle temporal gauche.

Si nous vérifions maintenant les symptômes cliniques au substrat anatomique, le groupe des symptômes mentionnés sub *a*. trouve une explication satisfaisante dans le champ du procès hypothalamique. Le fait qu'il n'y a pas encore eu atrophie des organes génitaux provient de la rapidité assez grande du cours de l'affection. Dans un second de mes cas d'encéphalite de l'hypothalamus, la dystrophie adiposo-génitale était accompagnée de diabète insipide. Toutefois, en ce cas, le procès s'étendait davantage vers l'avant et avait causé de même de la dégénération des noyaux supra-optiques.

b. J'ai déjà décrit ailleurs les troubles de la thermorégulation centrale qui furent observées dans ce cas [1]). Ces troubles sont rares chez l'homme; nos connaissances principales sous ce rapport reposent sur des bases expérimentalo-anatomiques et les données concernant la précision de la localisation dans l'hypothalamus présentent une grande variété. On peut avancer des arguments sérieux en faveur du concept que c'est surtout à la substance grise latérale du troisième ventricule que revient un rôle important dans la thermorégulation. Notre cas a cela de remarquable qu'il y avait eu non-seulement une fièvre de longue durée non provoquée par des anomalies organiques cliniques ou pathologo-anatomiques, mais, de plus, hypéridrose à gauche (*c*). Il est intéressant de noter, à ce sujet, que le procès de la substance grise était plus étendu, près de l'infundibulum, au côté droit que sur le côté gauche. Cette hypéridrose doit être interprétée, de même que le tic facial de gauche (*d*), comme un symptôme d'irritation. Le tic avait absolument de même caractère que celui observé chez les psychasthéniques. Dans une phase ultérieure de la vie, ces symptômes de tic s'étendent aussi aux muscles du cou et à ceux des deux bras. Le travail expérimental de HESS [4, 5]) et ses collaborateurs, principalement, nous a clairement appris que l'on peut éveiller des réactions, par de faibles courants électriques à partir d'endroits nettement circonscrits de l'hypothalamus du chat, non-seulement dans le tissu des muscles lisses, mais aussi dans celui des muscles striés. HESS considère par conséquent l'hypothalamus comme une zone importante de coordination entre le système nerveux de la vie animale et le système nerveux végétatif. L'altération caractéristique de la parole présentée par le malade dans la phase ultérieure de sa maladie, altération rappelant celle de la parole de certains Parkonsonniens, doit être mise, elle-aussi, sur le compte d'un procès hypothalamique, d'autant plus que les corps striés, le néo-thalamus et la substance noire ne montraient aucune altération importante.

f. Des recherches expérimentales ont prouvé que l'hypothalamus exerce de l'influence sur les mouvements de la vessie et du rectum. On voit ordinairement, chez l'homme, des troubles cérébraux de la vessie lors des cas

de procès bilatéraux des hémisphères de l'encéphale, des noyaux caudés et de leur entourage et du néothalamus. Notre observation démontre que ces troubles peuvent, en effet, avoir l'hypothalamus pour point de départ. Dans un second cas de notre matériel d'affections hypothalamiques, examinées sur des coupes en série, les troubles cérébraux de la vessie formaient un symptôme précoce et constant.

g. L'inversion du mécanisme du sommeil repose sur un trouble du rythme du rapport constamment varié établi entre l'activité parasympathique et l'activité sympathique. Il est tout naturel d'attribuer le trouble impliqué aux rapports pathologiques de la substance grise dans l'hypothalamus. On sait qu'il est possible de provoquer le sommeil expérimental, tout particulièrement à partir de la région de transition entre l'infundibulum et l'aquéduc de Sylvius (HESS), mais les données relatives à l'inversion du mécanisme du sommeil, qui pourraient nous guider dans une plus ample analyse de notre observation clinico-anatomique, sont insuffisantes. Le symptôme de l'inversion du sommeil n'est présent dans aucun de nos autres cas hypothalamiques et, par ailleurs, la bibliographie clinico-anatomique ne nous donne point, elle non plus, de réponse définitive.

h. Psychose de Korsakoff. Il a été constaté qu'il peut survenir des troubles psychiques lors d'une affection de l'hypothalamus. Comparer ici, entre autres, ALPERS, dans la monographie de l'hypothalamus [9]) (1940), et LHERMITTE [8]) et ses collaborateurs (1942). Notre observation apporte ici une nouvelle contribution. Notre série de 10 cas, examinés clinico-anatomiquement, en contient 4 ayant présenté le syndrome pur de Korsakoff, deux autres cas montrèrent des anomalies psychiques nettement accusées. Notre matériel n'est pas défavorable au concept que le syndrome de Korsakoff a affaire avec les corps mamillaires et leur entourage (GAMPER [3]) et autres), mais on ne saurait plus amplement conclure. Le syndrome de Korsakoff, en effet, peut survenir lors de diverses autres localisations de procès cérébraux. Au point de vue physiologico-anatomique, on peut dire seulement, à l'heure actuelle, que le corps mamillaire doit avoir de l'importance dans l'élaboration de stimulations de l'odorat et, vraisemblablement, de quelques formes de la sensibilité. Quoi qu'il en soit, le corps mamillaire, en effet, se trouve en corrélation avec de nombreuses parties autres du système nerveux central par des systèmes de voies, de sorte qu'il doit y avoir un centre d'embranchement important dans le système des réflexes qui créent la possibilité de fonctions plus compliquées et, entre autres, de celles qui se rattachent à la vie psychique.

i. Le grisonnement précoce de la chevelure est à considérer comme un trouble de l'influence trophique que l'hypothalamus exerce normalement sur les tissus.

j. L'amblyopie gauche (visus 4/60) a été accompagnée par un rétrécissement concentrique générale du champ visuel. Il y avait des altérations myopiques dans le fond de l'oeil, avec un staphylome nasal du nerf optique. Il n'y a jamais eu d'enflure de la papille. La réaction des pupilles à

la lumière n'a jamais été troublée. La vue comportait, à droite, $\frac{3}{4}$; elle baissa peu à peu jusqu'à $\frac{1}{4}$, tandis que, sur ce même côté, les limites du champ visuel rétrécissaient.

Bien que les altérations de la rétine, qui ont été mentionnées, aient dû exister depuis fort longtemps et même avoir été partiellement congénitales, le malade ne se rendit compte de leur présence que quelques années avant son entrée à la clinique. Il se plaignait alors d'avoir mauvaise vue à gauche et disait qu'il lui semblait souvent un voile devant l'oeil. L'examen du cerveau révéla que le procès inflammatoire avait épargné les nerfs optiques, le chiasma et les tractus optiques. Ceci est important par rapport aux recherches pratiquées en Suisse dans les dernières années. FRANCESCHETTI [2]) a exposé que certains troubles visuels, et particulièrement le rétrécissement concentrique du champ visuel, peuvent se produire lors d'une lésion de la région diencéphalo-mésencéphalaire, en dehors des voies optiques. Il relève, à ce sujet, les recherches de HESS [6]) auquel ses expériences chez des chats permirent de déterminer une diminution de l'acuité visuelle à partir du thalamus optique et de l'hypothalamus, sans qu'il y eut lésion des voies optiques centrales. HESS est d'avis qu'une influence inductrice indirecte s'exerce sur la capacité de réaction de l'appareil optique central, à partir du diencéphale, via le système nerveux végétatif.

k. Les anomalies des mouvements du regard sont les suivantes.

Dans les derniers temps qui précédèrent son entrée à la clinique, le malade avait remarqué qu'il ne pouvait plus diriger rapidement ses yeux d'un côté vers l'autre. A la clinique, il avait constamment le regard fixe. Les mouvements du regard se faisaient difficilement en directions horizontale et verticale, mais le degré de difficulté variait beaucoup, de sorte que l'on pensa tout d'abord à une origine psychogène. Toutefois, ces anomalies augmentèrent peu à peu et il devint évident que le malade ne pouvait en effet diriger volontairement ses globes oculaires vers le haut, le bas et les côtés, tandis que, par contre, les mouvements oculaires réflectifs (lors de flexions passives de la tête en avant ou en arrière) étaient intacts. Quand l'examen vestibulaire eut de même révélé des anomalies, le Prof. A. DE KLEYN, fit un examen plus ample et constata, lui-aussi, parésie du regard dans toutes les directions. L'examen vestibulaire révéla, il est vrai, une phase rapide, mais la différence qu'elle présentait avec la phase lente était bien moindre qu'en des circonstances normales. Il en était de même quant au nystagmus optocinétique souscortical, tandis que la phase rapide faisait complètement défaut dans la forme corticale. Le Prof. DE KLEYN en vint ainsi à supposer que les voies supranucléaires du regard étaient affectées, mais que la région nucléo-vestibulaire était intacte, ce qui fut confirmé par notre examen anatomique. De plus, le Prof. DE KLEYN constata la présence du symptôme *d'Eagleton*, c'est à dire possibilité de provoquer un nystagmus horizontal par excitation vestibulaire, mais non point de nystagmus rotatoire. Pareil contraste se présente le plus souvent lors de procès de la fossa cranienne postérieure, mais notre examen anatomique

a prouvé que ceci n'était point le cas dans notre observation. Bien que l'examen expérimental ait montré parfois qu'une influence hypothalamique s'exerce sur les mouvements des globes oculaires, je crois pourtant qu'il n'y a pas, dans ce cas, de raisons sufficantes à l'admission d'un rapport causal entre le procès hypothalamique et les anomalies citées, d'autant plus que l'on n'a pu encore établir avec certitude l'existence d'une connexion entre les centres vestibulaires et cette partie de l'encéphale. Vu l'état actuel de nos connaissances relatives à cette connexion, il me paraît plus juste de rapporter les anomalies des mouvements du regard et l'appareil vestibulaire aux altérations mésodermales de l'entourage de l'aquéduc de Sylvius et les corpora quadrigemina. Bien qu'il n'y eut point ici de perte nerveuse, il reste possible que des troubles circulatoires aient provoqué une diminution de la fonction.

Résumé: Il y avait ici une encéphalite chronique de l'hypothalamus et des territoires avoisinants, laquelle a causé le vaste syndrome clinique décrit au début de cet article. Les différents symptômes ont été vérifiés par l'examen pathologo-anatomique. L'importance de cette observation sera plus amplement exposée ailleurs, comparativement à neuf autre cas qui furent examinés sur des coupes en série.

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Physics. — *Recovery and recrystallization viewed as processes of dissolution and movement of dislocations.* III. By W. G. BURGERS. (Laboratorium voor Physische Scheikunde der Technische Hoogeschool, Delft.) (Communicated by Prof. J. M. BURGERS.)

(Communicated at the meeting of April 26, 1947.)

III. *Dissolution and movement of dislocations during heat treatment.*

III. 1. *Displacements of atoms in the boundary layers.*

We have now to consider what changes will take place in a deformed metal with a structure of the general type discussed in II, when subjected to heat treatment. Here again a precise treatment is wholly impossible. We may best start from BRAGG's conception (21), mentioned in II, 5, according to which the deformed block structure is in a state of dynamic equilibrium, in which the system of boundary layers may be considered as a foam with a definite energy. As the temperature is raised, the mobility of the atoms increases and atomic displacements will take place so that the free energy of the structure diminishes. This, presumably, can best be realized by displacements of the atoms in the dislocated transition layers. Two processes can be discerned, (*a*) such displacements which diminish the "tension" in the layers without displacing them as a whole and (*b*) displacements of the layers themselves. This latter process would be most effective if the layers could be "pushed" and "pulled" up to the boundaries of the test-piece, so that they would finally disappear, leaving in their "wake" an "ideal" single crystal.

We shall consider these two types of displacement somewhat more in detail for the "simplified" block-structure, considered also when discussing the stability in deformed metals in II, 5, to know a "two-dimensional" [18]) block-aggregate with only positive and negative dislocations of "edge-type" in the transition layers separating the blocks.

a. In transition layers containing dislocations of opposite sign, we have discerned between two types of pattern, represented in fig. 10 *a* and *b*: in *a* opposite dislocations have coinciding directions of "easy mobility" (directions of slip), in *b* they lie on alternating lines. In the first case a mutual neutralization of pairs of adjoining dislocations seems possible. A similar process cannot be realized in *b*: here, however, a change in the stress acting parallel to the direction of "easy mobility" over the whole pattern can displace the whole set of positive dislocations with regard to the set of negative dislocations, as discussed in II, 5. Both

[18]) "Two-dimensional' in the sense indicated in footnote [11]).

processes entail no growth of one lattice block at the expense of one of its neighbours. On the contrary, one may expect the involved structural changes to be in general so small, that they are "invisible" by ordinary means of observation (cf. I, 1).

On the other hand the mutual neutralization of opposite dislocations causes an elimination of the stress regions of which they were the centres. The disappearance of these stresses will likewise be noticeable at larger distances. Due to it, for example by a displacement of whole sets of dislocations as occurring in the pattern of fig. 10*b*, a release of stress extending over a whole lattice block may be brought about [19]). This relief of stress is directly visible in transparent crystals, like rocksalt, by means of observation in polarized light (LASCHKAREW and ALICHANIAN (47); BRILLIANTOW and OBREIMOW (48)). Also the decrease of the broadening of DEBYE–SCHERRER lines during annealing is for the greater part due to this process [(37), (38), (49)]. Finally all physical and mechanical properties, which are influenced by the stressed state of the transition layers and the blocks, will be affected. As most probably different properties depend on stresses of different period, it can be foreseen and is experimentally observed that different properties recover in different temperature regions and with different rates (BURGERS (49); (1), § 72). On all these grounds *it seems reasonable to indicate the process of dissolution of pairs of* TAYLOR*-dislocations of opposite sign* ("DEHLINGER–VERHAKUNGEN") *as the fundamental action of what "macroscopically" is called recovery.* This conception shows a close analogy with the ideas developed by KORNFELD (2), DEHLINGER and KOCHENDÖRFER (3), (39) [20]).

b. We now consider transition layers containing only dislocations of one sign, or at least a surplus of them. The most simple type is that shown in fig. 4, where the boundary is built up of a series of equally spaced dislocations of the same sign only. As already remarked while discussing fig. 4 and 5 in II, such a boundary necessarily separates lattice blocks differing in orientation. This type of boundary can never be wholly removed by mutual dissolution of opposite dislocations. The transition region can, however, be displaced as a whole by means of relatively small atomic displacements, as indicated by BRAGG (23) and already mentioned at the end of II, 5. This occurrence entails the enlargement of one lattice block at the expense of a neighbouring block and is *essentially a process of crystal growth.* It sets in when the resultant stress acting on the series of dislocations parallel to the direction of "easy mobility" attains a definite value, this value presumably depending, in the simple case of fig. 4, on the density of succession of dislocations, that is on the angular separation of the adjoining blocks.

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[19]) As remarked in footnote [13]), it is also possible to assume sets of dislocations to be present *inside* the blocks.

[20]) As KORNFELD puts it, recovery "mollifies the glue" between the blocks.

In what follows we intend to discuss to what extent some of the characteristics of both recovery and recrystallization can be understood with the help of these types of atomic displacement. Again we must limit our discussion to the "two-dimensional" structures considered so far.

III, 2. *Recovery considered as due to mutual neutralization of dislocations of opposite sign.*

From the foregoing it is at once clear that recovery can be realized at its best in cold-worked test-pieces, where the deformation has produced equal numbers of positive and negative dislocations, as arranged in patterns like those shown in fig. 10. Such arrangements may be produced most exclusively in sheared single crystals, where the presence of adjacent lattice blocks with different orientations is limited to at its most very small regions ["local curvatures" of glide-lamellae: see LEBBINK and BURGERS (34); BARRETT (51), p. 14)] [21]).

It may be expected that in such deformed specimina the mayor part of the stresses can be eliminated by neutralization of pairs of dislocations ("Verhakungen") and by a relative displacement of sets of dislocations, as discussed under *a*) in the preceding section. It is therefore not surprising that, as mentioned in section I, 1, precisely in deformed *single* crystals a far advanced recovery without "visible" recrystallization has been realized.

With increasing inhomogeneity of the deformation process the percentage of blocks in non-parallel positions will increase and the same holds for deformed polycrystalline test-pieces, as a consequence of the hampering effect of the crystal boundaries on the glide-process. Here, therefore, "pure" recovery can never do away with the whole stress system, produced by cold-working. Dissolution of pairs and displacements of whole sets of dislocations must finally be followed or accompanied by a boundary-displacement, as discussed in III, 1 under *b*). That, therefore, pure recovery in such cases is only possible to a limited extent, is not surprising.

Far less understandable is the experimental fact, shown in fig. 1, that annealing at different temperatures both in single- and polycrystals gives rise to different "rest-values". It might be understood to some extent by making the plausible assumption, that the cold-worked test-piece contains groups of dislocations with different "neutralization-energies" Q for individual pairs. Then, even if these energies within each group remain constant during the annealing treatment, the whole course of the recovery might be conceived as composed of a superposition of e-curves of formula [4] in I, 1 with different exponents: dislocation-pairs for which the neutralization-energy is smallest, are eliminated at the lowest temperature and after the shortest time of heating. With suitable values for these

21) Their presence follows from "asterism" in Laue-photographs (BURGERS and LEBBINK), and from x-ray microscopic observations by a new method, developed by BARRETT.

energies a course as found experimentally might be realized. It seems, moreover, reasonable to suppose that this effect will be more pronounced if the considerations, developed by DEHLINGER regarding the stability of linear groups of "Verhakungen" (see II, 5), might be applied also to the corresponding two-dimensional arrangements. According to his quantitative estimation, the energy required to dissolve a dislocation-pair decreases with decreasing "density" (that is number of pairs per cm, resp. cm^2). This energy therefore becomes less and less in the course of the dissolution of the group, until finally, when the density has reached a critical value, the remaining group collapses suddenly and completely. It follows that the rate of removal of a given number of dislocation-pairs in the course of the annealing process depends on its *distribution* over the deformed test-piece. The more numerous in membership the groups of which it is built up, the more difficult will be its dissolution: not only the initial value of the dissolution-energy in each group increases with its "size", but also a larger fraction of the total number of dislocations must disappear before collapse of the rest will set in. It seems possible that such circumstances might favour a course of the recovery-isotherms as found experimentally [22]).

Leaving this question as it stands, and assuming the occurrence of "rest-values" as an experimental fact, it is perhaps easier to understand the occurrence, discussed in I, 2, of recovery-isotherms approaching rest-values which lie "higher" for larger degrees of deformation. Reasoning along the lines applied above, this behaviour seems to fit in with the assumption that, taken as a whole, the density of dislocation-pairs, and thus the threshold-values of the energy required for their dissolution, increases with the degree of deformation.

We are well aware that the foregoing considerations are highly speculative. They have only been given with the intention to bring forward some characteristic features of the recovery process, which, as far as known to us, have not been given much attention up to the present [23]).

III, 3. *Recrystallization viewed as a process of displacement of transition-layers.*

We now turn to recrystallization proper and consider as its fundamental process the displacement of a boundary layer between non-parallel lattice

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[22]) The picture implies that after *very prolonged* heat-treatment for all temperatures, where broadly speaking recovery is not yet accompanied by recrystallization, the *same* end-value must be reached, only the time to attain this state being different for different temperatures. According to KOCHENDÖRFER (39) (p. 240), this is actually the case. The occurrence of different *approximately constant* "rest-values" for different temperatures remains, however, very remarkable.

[23]) Part of these speculations were given in (1), § 76. Here it was moreover suggested that the observation of KORNFELD, also mentioned in I, 2, of a larger rate of anneal in deformed single crystals as compared with polycrystals, might be "understood" on similar grounds, but the speculations given are too vague to have value.

blocks. Limiting our considerations as before to the simple transition layer shown in fig. 4, such a displacement can only set in when this layer is subjected to a stress of sufficient size, acting in the direction of "easy mobility" of the layer (that is, in the case of fig. 4, in a horizontal direction). It must be assumed that the required stress arises by a "redistribution" of the stress-system, existing in the cold-worked specimen, caused by dissolution of a number of dislocation-pairs (recovery). This leads us to a conception of the nature of "nuclei of recrystallization" and of their "period of incubation", which is essentially that given as long ago as 1929 by DEHLINGER (3), (50). The nuclei in this picture are lattice-blocks already present in the cold-worked state. The time of incubation is merely the time, elapsing from the beginning of the annealing process, until, by the preceding recovery (dissolution of a sufficient number of dislocation-pairs), the critical stress, necessary for movement of the transition layer, has been attained [24]).

It seems natural to suppose that displacement of a transition-layer only occurs if the "structural" conditions prevalent at both sides of the boundary are different. Only under such circumstances the development of a resultant stress acting in one of the two possible directions of mobility can be expected. Otherwise stresses will balance each other and no uni-directional displacement is possible. This conclusion is confirmed by the behaviour of recrystallized copper or nickel-iron sheet with so-called cube-texture: in such material, as is well known, by far the major part of all crystallites lie with a scattering of 5—10° with a cubic-plane and -axis parallel to the plane and direction of rolling. On this and other grounds (for example etch-figures) one feels justified to assume that the foil consists of blocks in nearly parallel and "equivalent" positions with regard to their neighbours, and may be considered as a "pseudo single crystal in 100-position" with an "extreme" mosaic structure. If such material is subjected to a prolonged heating at for example 1000° or 1100°, one never observes (at least as far as we know), that the structure is transformed into one "real" single crystal *of the same position*: elimination of the boundaries between the nearly equivalent blocks appears to be impossible [25]).

If, as often occurs (see (1), § 121), the annealing process gives rise to the formation of large crystals (by what is called "secundary recrystallization" or "exaggerated graingrowth"), these new crystals possess always orientations which *differ considerably* from the 100-orientation. This means, in our conception, that the finely grained material, besides the 100-crystallites, contains a small percentage of blocks with a different

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[24]) See, however, p. 727.

[25]) This case thus represents an example of "capillary equilibrium" by the boundaries in the sense as meant by BENEDICKS (44); cf. also BRAGG (21), when discussing the stability of the domain structure, formed when order sets in in the alloy Cu_3Au. There, due to the existence of four equivalent super-lattice arrangements, a relatively stable state of dynamic equilibrium can be realized.

orientation, which, in the course of the annealing process, obtained the faculty to grow at the expense of the 100-crystallites [26]).

From the foregoing it follows that a lattice-block, which can serve as nucleus for a new crystal, must be in a state of stress different from its neighbours. This condition would be satisfied by a block in a position like *b* in fig. II, which one can consider to lie at the "point of inflexion" of the

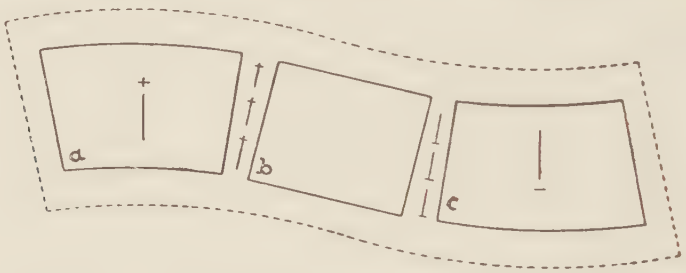


Fig. 11. Three adjoining lattice blocks, which, taken together, can be considered to form a "*S*-curved" lattice region. Block *b* in the "inflexionpoint" can presumably function as a nucleus for recrystallization.

lattice region *a-b-c*. In this block, which is the least curved and thus contains (or is surrounded by: compare footnote [13])) an equal number of dislocations of opposite sign, "true" recovery by dissolution of dislocation-pairs can eliminate a large part of the stresses it contains, so that displacement of the boundaries between this block and its neighbours *a* and *c* can set in. As *b* is least "deformed", we should expect *b* to grow at the expense of *a* and *c*: here again we are in agreement with DEHLINGER's conception of the most probable nuclear spot [27]).

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[26]) It is not yet fully understood what factors govern the occurrence of secundary recrystallization in these and other metals and neither what factors determine the orientations of the new crystals formed in this way. From the work of several investigators (for example COOK, MACQUARIE and RICHARDS (52), CUSTERS and RATHENAU (53) and others (see (1), § 121)) definite orientation-relations, partly consisting of twin-positions, have been found. We are inclined to think that the phenomenon is related to that of "stimulated" crystalgrowth mentioned at the end of this paper.

[27]) The problem how actually nuclei for recrystallization are formed, either in a way analogous to nucleus formation in vapour or solution by a favorable thermal fluctuation (in such case the time of nucleation represents something like the time elapsing from the beginning of the annealing before this fluctuation occurs) or by the growth of lattice blocks in favorable state of stress as represented in the foregoing, has not yet been solved and has given rise to much discussion and controversy. DEHLINGER (3), KORNFELD (58) and the author are inclined to consider the second alternative more probable, ANDERSON and MEHL (10) the first one. To decide this question, the experimental data available, especially those relating to the dependence of rate of nucleation and rate of growth on degree of deformation and temperature, must be critically analyzed. Also the influence of a recovery treatment of the deformed test-piece, preceding the recrystallization process, must be taken into account. Most of the data regarding these factors available up to 1940 have been critically considered by the author in (1) § 109; some later considerations are given in ANDERSON and MEHL's paper (10). The question cannot be

III, 4. *Rate of growth of crystals.*

In the foregoing section the growth of one block at the cost of an adjoining block was conceived as a displacement of a transition layer as a whole under the influence of a crtical stress, rather than as a process of individual jumps of atoms. Its rate of propagation probably depends on the size of potential barriers in a similar way as the propagation of a single dislocation along the glide plane in a crystal. As set forth by OROWAN (57), in this latter case the rate is given by a BOLTZMANN-formula if the average stress acting upon the dislocation is less than the stress necessary to set it into movement without the help of the thermal agitation; if, however, the average stress surpasses this critical value, it will attain a constant rate determined by the energy dissipated to the surrounding lattice parts.

As to the displacement of a whole transition layer, it seems natural to assume that the rate so obtained will remain unchanged only, if the "structural characteristics" at both sides of the boundary do not change. These, however, generally change continuously in the course of the devoring of a lattice block, due to a continuous change in its orientation-relations with regard to adjoining lattice blocks. This will occur certainly when a lattice block has been wholly devored by a growing block, as now by a "melting together" of both the moving and the new transition layers, completely different boundary structures (patterns of dislocations) are formed. These may be either such that they require a smaller critical stress for displacement and thus facilitate the displacement, or, what seems perhaps more plausible, the new boundary requires a much larger stress and thus brings the displacement practically to a stop. In that case only dissolution of more dislocation pairs can start the process anew. In this picture the displacement of boundaries is thus conceived to proceed more or less in a jumpy way from boundary to boundary, or from lattice block to lattice block. The final rate is determined by two component processes,

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regarded as resolved with any degree of certainty. It is no use to put the suggestions given in this paper to the test with regard to the available data, as these suggestions are vague and moreover the actual structure of the transition layers is far more complicated than the simple type considered here. [Consider for example the influence of a recovery treatment of the deformed test-piece before recrystallization. On the basis of our considerations one might expect this to *shorten* the incubation period, as part of the necessary dissolution of dislocations can take place during this preliminary process. On the other hand, on the basis of the "fluctuation-theory", one might sooner anticipate an *extension* of the incubation period, as recovery will cause a release of stress especially in the points of highest stress concentration, where fluctuations will preferably occur. In reality, in most cases experiments point to an extension (KORNFELD and PAWLOW (54), MATHEWSON and COLLINS (55), although the opposite has apparently also been found (KORNFELD and SCHAMARIN (56); see (1), § 103). — On the other hand, the fact that in rolled sheet the crystal orientations after recrystallization can be found or are in some way connected with those in the cold-worked state of the specimen, is most easily understood on the assumption that the nuclei of the new crystallites formed part of the cold-worked structure.]

one consisting of dissolution of dislocations in order to create after every step the stress required for further displacement, the other that of the displacement proper under influence of these stresses. That process which is the slower determines the final value of the rate of displacement [28]).

A jumpy displacement of the boundary of a growing crystal was actually observed in MÜLLER's experiments (11) on the recrystallization of rocksalt. It may be doubted, however, whether this fact may be regarded as confirming the conceptions given above, as the displacement during *one* jump could attain 20—30 μ at 770° C., a displacement much larger than the average size assumed for a mosaic block. (According to MÜLLER, the jumps are due to the accumulation of foreign atoms in the transition layers, and the occasional "piercing" of these layers).

Leaving this question as it stands, it would follow from these considerations, that a constant rate of growth of new crystals can only be attained when the growing nucleus has grown beyound the region of inhomogeneous deformation, where it is supposed to have started (see above fig. 11; this region may be of (sub-) microscopic size) and has entered that part of the test-piece, where the deformation is (quasi)-homogeneous. If, during this first period, the total rate is slower than the final "constant" rate, the curve which gives the rate of growth as a function of time of heating, which is linear in a macroscopically homogeneously deformed specimen, does not

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[28]) In this connection we may insert the following remark, to which I am indebted to Prof. MOTT. In ANDERSON and MEHL's paper the rate of growth of a new crystal is given by an expression

$$G = B \cdot e^{-Q_G/RT}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\alpha)$$

where B is a constant and Q_G an activation energy. The numerical value of B (which depends on the degree of deformation of the matrix) is of the order 10^{14}—10^{16} cm per sec. Now we consider a transition boundary of 1 cm^2. If a is the atomic distance, than the number of atoms in this surface is $\sim 1/a^2$. The number of these atoms, which undergo displacement by virtue of the activation energy Q_G, will be of the order

$$\nu \cdot 1/a^2 \cdot e^{-Q_G/RT}$$

per second, where ν represents the atomic frequency (which we may put to about 10^{12}). Suppose further that the displacement of every one of these atoms involves, by a kind of "chain reaction" (or gliding), a displacement of N atoms, that is growth over a volume of $N \cdot a^3$ cm^3, thus a displacement for a surface of 1 cm^2 over $N \cdot a^3$ cm, then the displacement of the boundary per second, that is the rate of growth of the recrystallization process, is given by

$$G = N a^3 \cdot \nu \, 1/a^2 \cdot e^{-Q_G/RT} = N \nu a \cdot e^{-Q_G/RT}$$

so that from (α):

$$B = N \nu a. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\beta)$$

With $B = 10^{15}$ cm/sec, $a = 2 \times 10^{-8}$ cm and $\nu = 5 \times 10^{12}$ sec^{-1}, we find for N an order of magnitude of 10^{10}. This corresponds to a volume $N \cdot a^3 = (0.4\,\mu)^3$, that is about the size of a mosaic block, and the above calculation might be considered in favour of the conception that recrystallization proceeds in "steps" of one mosaic block at a time. (See, however, Addendum at the end of this paper.)

pass through the origin (see fig. 12): the "incubation period" thus found may be the *sum* of the "true" incubation period as defined in section III, 3 on p. 723 and the "period of invisible growth" during the consumption of the inhomogeneously deformed region around the nucleus. This conception approaches those of MÜLLER (11) and of KORNFELD (58) (see (1), § 109). The constant rate itself presumably depends in the first place on the mutual orientation of growing and disappearing block. In fact, it seems natural to

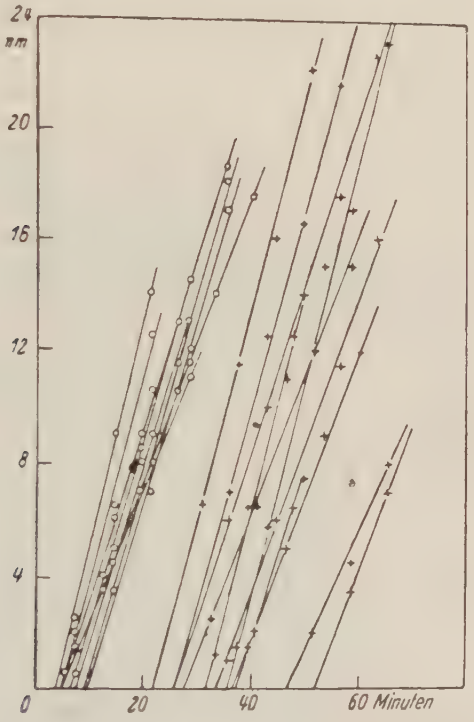


Fig. 12. Constant rate of growth of crystals, formed by recrystallization, in polycristalline aluminium wire (after KORNFELD and PAWLOW (54)). Degree of deformation 3%. Temperature of anneal 450° C.

assume that the critical stress, required for displacement of the boundary layer, depends on its "structure", for example in the simple case of fig. 4, on the density of the dislocation-series, that is on the orientation between the two lattice blocks [29]).

The influence of a difference of orientation of neighbouring blocks upon the rate of displacement of the common boundary can be clearly demonstrated in recrystallization experiments with slightly deformed single

[29]) In this simple case one might perhaps expect this stress to be larger for larger deviation [at least as long as the difference in orientation increases and does not, as a consequence of symmetry relations, becomes less pronounced again (see footnote [12])], as displacement of a "boundary" in an ideal lattice (with density of dislocation zera) may be said to require no force at all.

crystals: here new grains often show more or less straight boundaries, parallel to certain directions. Fig. 13 gives a striking example, it shows the progressive growth of a new crystal in the unicrystalline matrix parallel to a definite direction. Other examples are given in experiments by KORNFELD and RYBALKO (59), also with aluminium, where the new crystals are of lozengeshape [30]).

In a different way the same result is confirmed by recrystallization of polycrystalline material. If such material has no preferential orientation, the new grains have approximately a circular appearance (fig. 14*a*), showing that the average rate of growth is the same in all directions, so that individual differences, which may be expected due to the different orientation between growing crystal and disappearing crystallites (see fig. 15) are effaced. That, however, such individual differences are actually

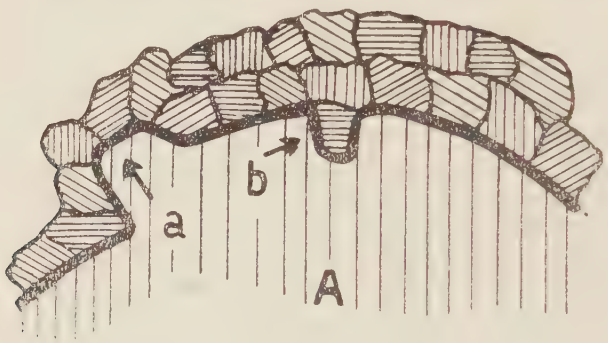


Fig. 15. Schematic representation of the growth of a large crystal at the cost of small surrounding crystals. Growth may proceed "from below" (at *a*) or "sideways" (at *b*). Due to differences in mutual orientation, differences in rate of consumption of individual grains are to be expected.

present is clear when the mother-material is rather coarsely-grained: the corrugated character of the boundaries of the new grains, as shown in fig. 14*b*, is due to this effect. On the basis of this "effacing-effect", one might expect that new crystals, formed in homogeneously deformed finely grained material (without any preferential orientation) *all* show the same rate of growth, this rate being the mean value of the rates between the growing crystal and the surrounding individual crystallites, which are being devored [31]).

Direct determinations of the growth of crystals under such circumstances in various metals by KARNOP and SACHS (60) and by KORNFELD and

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[30]) The exact nature of the directions of preferential growth have still to be found as determinations of various authors gave no corresponding results (see fig. 13). According to KORNFELD and RYBALKO they are related to octahedral planes, which would not seem surprising. Further work on this question is necessary.

[31]) Moreover, the rate can be averaged still more due to the possibility (see fig. 15) that the growing crystal approaches the small crystallites not "directly" but via an adjoining crystallite.

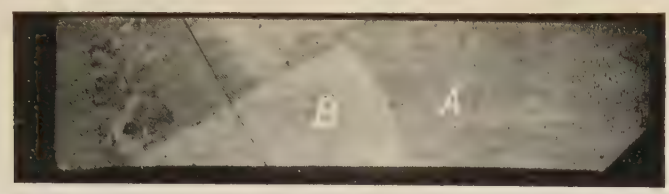


Fig. 13. Directional growth of new crystal (*B*) in a stretched single crystal (*A*) of aluminium. The boundary is indicated after 15 and 25 minutes of heating at 600° C. The rate of displacement in a direction perpendicular to the oblique boundary was zero. This boundary was parallel to the trace of a 111-plane of the growing crystal (*not* of the mothercrystal *). Nat. size.

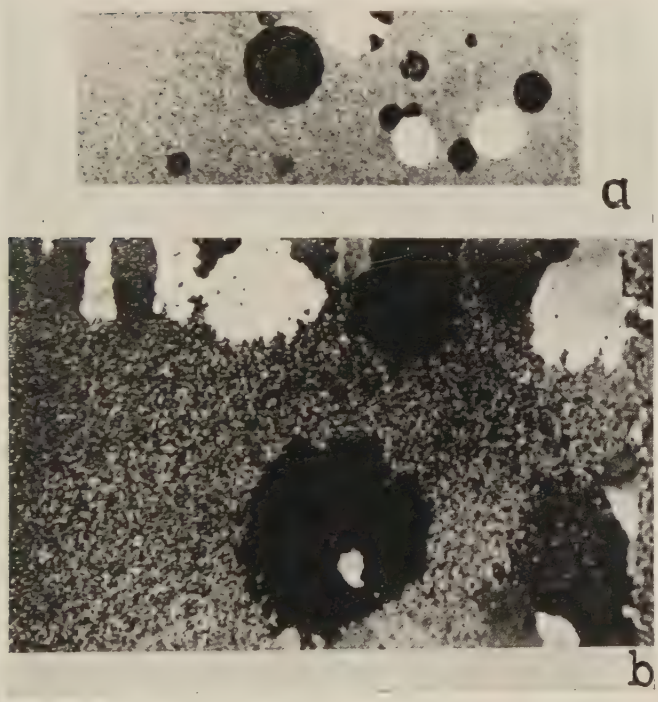


Fig. 14. Formation of circular crystals in quasi-isotropic, finegrained aluminium sheet, which has been subjected to a (macroscopically) homogeneous stretching of a few percent. *The mean rate of growth is equal in all directions.*

a. original material finegrained: *smooth* boundaries.

b. original material somewhat coarser than *a*): *corrugated* boundaries, due to differences in the rate of consumption of individual crystallites (compare fig. 15). Nat. size.

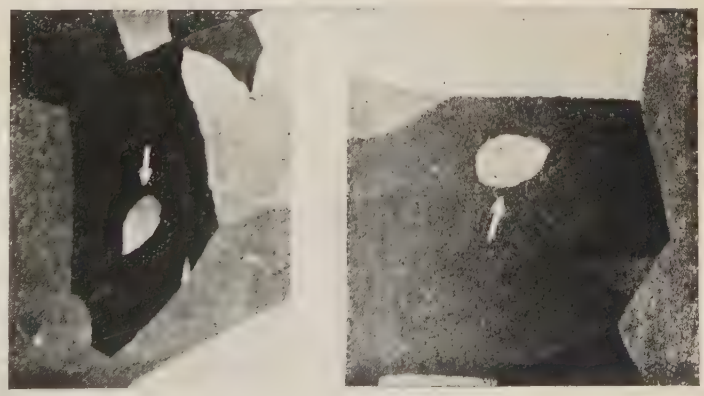


Fig. 16. Aluminium plates at approximately natural size, showing large crystals, formed by recrystallization of finegrained material. The presence of pear-shaped crystals, inside larger surrounding crystals, is due to the smaller rate of growth of the former with regard to the latter: The ratio is given by the cosine of half the angle at the pointed and of the enclosed crystal.

*) The contrary was found in KORNFELD and RYBALKO's experiments, mentioned in the text.

co-workers (54) (61) carried out by measuring the diameter of the crystals as a function of time of heating (see example in fig. 12), are not sufficiently accurate for this purpose: the difference in inclination of the lines of growth can easily be due to the impossibility to measure the size of the crystals with sufficient precision.

III, 5. *Occurrence of crystals with different rates of growth.*

Indirectly, however, the presence of considerable differences follows from an analysis of the *form* of the boundaries of the new crystals in recrystallized aluminium plates, as done first by SANDEE (62), and afterwards by BURGERS and MAY (63). Considering the problem as "two-dimensional" and assuming a constant rate of growth in all directions, SANDEE showed that special types of boundaries and especially the presence of pear-shaped crystals *within* larger surrounding crystals (examples are given in fig. 16) can only be explained by assuming a constant ratio between the rates of growth of both crystals during the whole course of their growth. From the analysis it follows that this ratio is given by the cosine of half the angle at the pointed end of the enclosed crystal. As angles of up to 80° occur, the ratio can assume values up to about $\frac{3}{4} : 1$.

The occurrence of these differences in rate of growth between various crystals growing at the expense of the same matrix may mean that the "effacing-effect", discussed in the foregoing section, is not so completely realized as we expected there, so that crystals with different orientations have still different rates of growth. So far, however, we have not succeeded in finding a definite relation between crystal-orientation and rate of growth. This suggests the possibility that the observed differences find their origin in the internal state of the growing crystals themselves. In view of the mechanism of growth discussed above, this would lead to the hypothesis that besides the block structure of the disappearing crystallites *also that of the growing crystal* has an influence on the growing process. It is not easy to understand in what way this might be possible. It might imply that some imperfection, created in the very beginning of the growth, would enforce itself upon the whole future crystal. A schematic picture as given in fig. 17 (which is given by TAYLOR (46) in another connection) might be suggestive here: suppose two blocks are growing while including a small angle, then perhaps each time the gap between them attains one atomic distance, a break in the growth process may occur, which influences its final rate. To explain the difference in rate of growth encountered with different crystals, it would moreover be necessary to assume that different crystals are characterized by different "imperfections". Considering the findings of DEHLINGER and GISEN (19), mentioned in II, 1, as to the presence of differences between aluminium crystals grown by recrystallization and from the melt, this possibility need not be rejected a priori.

For recrystallization-crystals, it must be kept in mind that they start from nuclei which certainly are in different states of stress with regard to their immediate surroundings.

For the moment, in absence of further experimental work, this question must be left as it stands. Perhaps it will be possible by applying special methods of x-ray research (for example Mrs. LONSDALE's "divergent beam"



Fig. 17. Boundary of two crystals at slightly different orientations, showing boundaries of separate crystalblocks and a regular repetition of "surface of misfit" (after TAYLOR (46)). The figure is given here as an endeavour to understand the repetition, in a growing crystal, of a definite imperfection.

method (64) or BARRETT's "x-ray microscope" (51)) to get some further experimental evidence regarding the presence or absence of differences in mosaic character of different crystals [32]).

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[32]) Perhaps recrystallization of silverhalide crystals can also be of help while investigating this problem. Preliminary experiments by BURGERS and TAN KOEN NIOK (65) have shown that such crystals, after irradiation with ultraviolet light, show an optical "etching effect" parallel to cube-planes, which seem to form an indication of the presence of a mosaic structure with blocks parallel to these planes. Differences might be found in this "optical" way between various crystals. [In (65) it was suggested that the diffuse bands on Laue-photographs of these crystals was partly due to this supposed mosaic character. Further work has shown, however, that the bands are of thermal nature, similar to those found for other crystals by PRESTON (66), LONSDALE and SMITH (67, GUINIER (68), KRONING and ARLMAN (69) and others.]

(To be concluded.)

Addendum (June 1947).

With regard to the remark in footnote [28]), Mr. F. R. N. NABARRO drew my attention to the fact that measurements of rates of growth by other investigators [KORNFELD and PAWLOW (9) with aluminium; MÜLLER (11) with rocksalt] lead with the same reasoning to much smaller values of the number of atoms (N), involved in a "chain of jumps". In this connection it might be remarked that the mosaic character of a crystal, even of the same substance, may be largely dependant on its mode of formation or on its degree of purity: cf. section II, 1. The question, however, may be raised whether the introduction of a quantity N, as done in footnote [28]), can be maintained.

Mathematics. — *Einige Anwendungen des Dualitätsprinzips in topologischen Strukturen.* By J. RIDDER. (Communicated by Prof. W. VAN DER WOUDE.)

(Communicated at the meeting of May 31, 1947.)

I. Charakterisierung des offenen Kernes durch ein einziges Axiom.

§ 1 [1]). Wir betrachten eine BOOLEsche Algebra S, definiert durch die nachfolgenden Axiome 1°—7°; ihre Elemente wollen wir *Somen* nennen.

Axiom 1°. $\alpha)$ $a \subset a$; $\beta)$ aus $a \subset b$ und $b \subset c$ folgt $a \subset c$.

Definition. $a = b$, falls $a \subset b$ und $b \subset a$.

Definition. Ein Soma $\prod_{a_\iota \in K} a_\iota$ wird *Produkt der zu der Klasse K gehörenden Somen* (a_ι) genannt, falls: $\alpha)$ $\prod_{a_\iota \in K} a_\iota$ jedes a_ι; $\beta)$ aus $b \subset a_\iota$ für jedes $a_\iota \in K$ folgt $b \subset \prod_{a_\iota \in K} a_\iota$.

Axiom 2°. Für jedes Somenpaar a, b gibt es ein Produkt ab.

Definition. Ein Soma $\sum_{a_\iota \in K} a_\iota$ wird *Summe der zu einer Klasse K gehörenden Somen* (a_ι) genannt, falls: $\alpha)$ jedes $a_\iota \subset \sum_{a_\iota \in K} a_\iota$; $\beta)$ aus $a_\iota \subset b$ für jedes $a_\iota \in K$ folgt $\sum_{a_\iota \in K} a_\iota \subset b$.

Axiom 3°. Für jedes Somenpaar a, b gibt es eine Summe $a + b$.

Axiom 4°. Es gibt ein kleinstes (leeres) Soma 0; das soll heissen: $0 \subset a$ für jedes $a \in S$.

Axiom 5°. Es gibt ein grösstes Soma 1; das soll heissen: $a \subset 1$ für jedes $a \in S$.

Axiom 6°. $ac + bc = (a + b)c$.

Axiom 7°. Zu jedem Paar von Somen a, b, mit $a \subset b$, gibt es ein Soma $b - a$ derart, dass $a + (b - a) = b$ und $a \cdot (b - a) = 0$ ist.

Definition. *Das Komplement von a, a'*, sei das Soma $1 - a$.

§ 2 [1]). Jedem Soma $a \in S$ seien in eindeutiger Weise zugeordnet das Soma $\bar{a}$, *die abgeschlossene Hülle von a,* und $\underline{a}$, *der offene Kern von a;* zwischen diesen soll folgender Zusammenhang bestehen:

$$\underline{a} = (\overline{a'})', \text{ oder } \bar{a} = (\underline{a'})' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

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[1]) Siehe J. RIDDER, Verhand. Ned. Akad. v. Wetensch., Amsterdam, Sekt. 1, **18**, Nr 4 (1944), 43 Seiten.

Die topologischen Axiome, die im folgenden mit lateinischen Ziffern angedeutet werden, sind paarweise aequivalent; das eine Axiom eines solchen Paares wird durch ein Ziffer ohne-, das andere durch dasselbe Ziffer mit einem Asterisk angedeutet.

$T.A.I$: $a \subset \bar{a}$.	$T.A.I^*$: $\underline{a} \subset a$.
$T.A.II$: $\bar{0} = 0$.	$T.A.II^*$: $\underline{1} = 1$.
$T.A.III$: $a \subset b \to \bar{a} \subset \bar{b}$.	$T.A.III^*$: $b \subset a \to \underline{b} \subset \underline{a}$.
$T.A.IV$: $\bar{\bar{a}} = \bar{a}$.	$T.A.IV^*$: $\underline{\underline{a}} = \underline{a}$.

Dualitätsprinzip. *Zu jedem in einer* BOOLEschen *Algebra aus einem oder mehreren der vorangehenden topologischen Axiome ableitbaren Satz gibt es einen dualen, welchen man erhält, wenn im ursprünglichen Theorem überall:* $\alpha) x \subset y$ *durch* $y \subset x$ *(und somit* xy *durch* $x + y$*),* $\beta)$ 0 *durch* 1, $\gamma) \bar{a}$ *durch* $\underline{a}$*, und umgekehrt, ersetzt wird.*

§ 3. **Theorem** (von MONTEIRO) [2]. In einer BOOLEschen Algebra ist das von den Axiomen I, III und IV gebildete Axiomensystem aequivalent mit dem einzigen Axiom:

$$y + \bar{y} + \bar{\bar{x}} \subset \overline{\overline{x + y}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Wir wollen zeigen, dass Anwendung des Dualitätsprinzips aus diesem Satz ableiten lässt das

Theorem. *In einer* BOOLEschen *Algebra ist das von den Axiomen I*, III* und IV* gebildete Axiomensystem aequivalent mit dem einzigen Axiom:*

$$\underline{xy} \subset y \cdot \underline{y} \cdot \underline{x} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Beweis. Die Axiome I*, III* und IV* seien erfüllt: dann gilt dasselbe von den Axiomen I, III und IV, also nach dem Theorem von MONTEIRO auch von (2). Nach dem Dualitätsprinzip folgt daraus dann die Gültigkeit von (3).

Nehmen wir, umgekehrt, an, dass als einziges Axiom (3) erfüllt sei. Komplementbildung liefert dann

$$(xy)' \supset (y \cdot y \cdot x)'$$

oder, unter Anwendung von (1),

$$\overline{(xy)'} \supset y' + \underline{(y)'} + \underline{(x)'}$$

oder

$$\overline{x' + y'} \supset y' + \overline{y'} + \overline{(x)'}$$

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[2] Siehe A. MONTEIRO, Portugaliae Math. 4 (1943/45), S. 158—160.

oder

$$\overline{x' + \overline{y'}} \supset y' + \overline{y'} + \overline{\overline{x'}},$$

somit auch (2).

Aus (2) folgt wieder nach dem Theorem von MONTEIRO, dass die Axiome I, III und IV erfüllt sind, und dadurch, nach dem Dualitätsprinzip, ebenfalls die Axiome I*, III* und IV*.

Bemerkung. Das Theorem von MONTEIRO gilt schon in einer Struktur, welche nur die Axiome 1°, 3° und 4° erfüllt. Ebenso *gilt das letzte Theorem schon in einer Struktur, welche nur die Axiome* 1°, 2° *und* 5° *erfüllt*; um dies zu zeigen hat man nur den MONTEIROschen Beweis [2]) seines Satzes dual umzusetzen (gemäss α), β), γ) von § 2).

Analoge Charakterisierungen anderer Grundbegriffe.

§ 4 [1]). Jedem Soma a einer BOOLEschen Algebra S sei in eindeutiger Weise das Soma a^e zugeordnet. Zwischen a^e und $\underline{a}$ soll folgender Zusammenhang bestehen:

$$a^e = \underline{a}', \text{ oder } \underline{a} = (a')^e. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Dann sind die folgenden topologischen Axiome paarweise aequivalent; das eine Axiom eines solchen Paares wird durch ein Ziffer mit einem Asterisk, das andere durch dasselbe Ziffer mit einem „e" angedeutet.

$T.A.I^*$: $\underline{a} \subset a$.	$T.A.I^e$: $a^e \subset a'$.
$T.A.II^*$: $\underline{1} = 1$.	$T.A.II^e$: $0^e = 0'$.
$T.A.III^*$: $b \subset a \rightarrow \underline{b} \subset \underline{a}$.	$T.A.III^e$: $a \subset b \rightarrow b^e \subset a^e$.
$T.A.IV^*$: $\underline{\underline{a}} = \underline{a}$.	$T.A.IV^e$: $\{(a^e)'\}^e = a^e$.

Mittels (4) lässt sich aus (3) ableiten

$$(x + y)^e \subset y' \cdot y^e \cdot \{(x^e)'\}^e, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

und umgekehrt.

Da jedes der Axiomensysteme: I^e, III^e, IV^e; I^*, III^*, IV^*; einziges Axiom (3); einziges Axiom (5) aequivalent ist mit vorangehendem und (oder) nachfolgendem System folgt sofort das

Theorem. *In einer* BOOLEschen *Algebra ist das von den Axiomen* I^e, III^e *und* IV^e *gebildete Axiomensystem aequivalent mit dem einzigen Axiom:*

$$(x + y)^e \subset y' \cdot y^e \cdot \{(x^e)'\}^e, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

§ 4bis [1]). Durch duale Umsetzung von § 4 erhält man:

Jedem Soma a einer BOOLEschen Algebra S sei in eindeutiger Weise

das Soma a_e zugeordnet. Zwischen a_e und $\bar{a}$ soll folgender Zusammenhang bestehen:

$$a_e = \overline{a'}, \text{ oder } \bar{a} = (a')_e. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Dann sind die folgenden topologischen Axiome paarweise aequivalent; das eine Axiom eines solchen Paares wird durch ein Ziffer, das andere durch dasselbe Ziffer mit einem „e" angedeutet.

$T.A.I$: $a \subset \bar{a}$.	$T.A.I_e$: $a_e \supset a'$.
$T.A.II$: $0 = 0$.	$T.A.II_e$: $1_e = 1'$.
$T.A.III$: $a \subset b \rightarrow \bar{a} \subset \bar{b}$.	$T.A.III_e$: $b \subset a \rightarrow a_e \subset b_e$.
$T.A.IV$: $\overline{\bar{a}} = \bar{a}$.	$T.A.IV_e$: $\{(a_e)'\}_e = a_e$.

Mittels (6) lässt sich aus (2) ableiten

$$y' + y_e + \{(x_e)'\}_e \subset (xy)_e, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

und umgekehrt.

Da jedes der Axiomensysteme: I_e, III_e, IV_e; I, III, IV; einziges Axiom (2); einziges Axiom (6) aequivalent ist mit vorangehendem und (oder) nachfolgendem System folgt das

Theorem. *In einer* BOOLEschen *Algebra ist das von den Axiomen I_e, III_e und IV_e gebildete Axiomensystem aequivalent mit dem einzigen Axiom:*

$$y' + y_e + \{(x_e)'\}_e \subset (xy)_e \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

§ 5[1]). In einer BOOLEschen Algebra folgt aus

$$a^g = \bar{a} \cdot \overline{a'} \text{ und } a \subset \bar{a},$$

dass

$$\bar{a} = a + a^g \text{ und } a^g = (a')^g$$

ist. Umgekehrt, aus

$$\bar{a} = a + a^g \text{ und } a^g = (a')^g$$

folgt:

$$a^g = \bar{a} \cdot \overline{a'} \text{ und } a \subset \bar{a}.$$

In S sind somit

$$T.A.I: a \subset \bar{a} \text{ und } T.A.I^g: (a')^g = a^g$$

aequivalent. Zwischen ihren Grundbegriffen $\bar{a}$ und a^g gelten in beiden Systemen die Relationen:

$$a^g = \bar{a} \cdot \overline{a'} \text{ und } \bar{a} = a + a^g.$$

Die Aequivalenz bleibt bestehen nach Hinzufügung von

$$T.A.\,III\colon\ a \subset b \to \bar{a} \subset \bar{b} \quad \text{bzw.} \quad T.A.\,III^g\colon\ a \subset b \to a\,(a^g)' \subset b\,(b^g)'$$

[oder $a \subset b \to a + a^g \subset b + b^g$],

und nach abermaliger Hinzufügung von

$$T.A.\,IV\colon\ \bar{\bar{a}} = \bar{a} \quad \text{bzw.} \quad T.A.\,IV^g\colon\ (a + a^g)^g \subset a + a^g.$$

Das System I, III und IV ist aequivalent mit dem einzigen Axiom (2); (2) und

$$x^g + y^g + (x + x^g)^g \subset (x + y) + (x + y)^g \quad . \quad . \quad . \quad . \quad (8)$$

lassen sich unter Annahme einer der Relationen $\bar{a} = a + a^g$, $a^g = \bar{a} \cdot \overline{a'}$ (wobei dann auch die Gültigkeit der anderen sofort folgt) ineinander überführen. Wir haben somit das

Theorem. *In einer* Boole*schen Algebra ist das von den Axiomen I^g, III^g und IV^g gebildete Axiomensystem aequivalent mit dem einzigen Axiom:*

$$x^g + y^g + (x + x^g)^g \subset (x + y) + (x + y)^g \quad . \quad . \quad . \quad . \quad (8)$$

§ 5bis. Duale Umsetzung von § 5 liefert das

Theorem. *In einer* Boole*schen Algebra ist das von den Axiomen:* I_g: $(a')_g = a_g$; III_g: $b \subset a \to b + (b_g)' \subset a + (a_g)'$ [oder $b \subset a \to b \cdot b_g \subset a \cdot a_g$]; IV_g: $a \cdot a_g \subset (a \cdot a_g)_g$ *gebildete Axiomensystem aequivalent mit dem einzigen Axiom:*

$$x_g \cdot y_g \cdot (x \cdot x_g)_g \supset x \cdot y \cdot (x \cdot y)_g \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

Der Zusammenhang mit $\underline{a}$ wird hier geliefert durch

$$a_g = \underline{a} + \underline{a}' \text{ und } \underline{a} = a \cdot a_g.$$

II. Eindeutige Bestimmtheit des offenen Kernes durch die offenen Somen.

§ 6. **Definition.** a heisst *abgeschlossen,* falls $a = \bar{a}$; a heisst *offen,* falls $a = \underline{a}$.

Theorem (von Monteiro und Ribeiro) [3]. In einer Booleschen Algebra S sei die abgeschlossene Hülle $\bar{x}$ eines jeden Somas x eindeutig festgelegt, und genüge den topologischen Axiomen I, III und IV (§ 2). Dann ist

$$\bar{x} = \prod_{y \in P(x)} y, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

wobei $P(x)$ alle und nur alle diejenigen abgeschlossenen Somen y umfasst, für die $x \subset y$ gilt.

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[3]) Siehe A. Monteiro und H. Ribeiro, Portugaliae Math. **3** (1942), S. 171—183.

Umgekehrt, gibt es eine Klasse K von Somen (y) derartig, dass für jedes Soma $x \in S$ die Somen $y \in K$ mit $x \subset y$ ein Produkt haben, das ebenfalls zu K gehört, und dass wir durch $\bar{x}$ andeuten wollen, so genügt die Operation $\bar{x}$ den topologischen Axiomen I, III und IV; für jedes $y \in K$ ist dann $\bar{y} = y$, und jedes abgeschlossene Soma z (d.h. mit $\bar{z} = z$) gehört zu K.

Das Dualitätsprinzip lässt nun ableiten das

Theorem. *In einer* BOOLEschen *Algebra S sei der offene Kern $\underline{x}$ jedes Somas x eindeutig festgelegt, und genüge den topologischen Axiomen I*, III* und IV** (§ 2). *Dann ist*

$$\underline{x} = \sum_{y \in Q(x)} y, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

wobei $Q(x)$ alle und nur alle diejenigen offenen Somen y umfasst, für die $y \subset x$ gilt.

Umgekehrt, gibt es eine Klasse L von Somen (y) derartig, dass für jedes Soma $x \in S$ die Somen $y \in L$ mit $y \subset x$ eine Summe haben, die ebenfalls zu L gehört, und die wir durch $\underline{x}$ andeuten, so genügt die Operation $\underline{x}$ den topologischen Axiomen I, III* und IV*; die Klasse der offenen Somen fällt mit L zusammen.*

Beweis. Die Axiome I*, III* und IV* seien erfüllt; dann gilt, unter Annahme der Relationen (1), dasselbe von den Axiomen I, III und IV. Nach dem Theorem von MONTEIRO und RIBEIRO gilt dann (10), und dadurch nach dem Dualitätsprinzip auch (11).

Definieren wir, umgekehrt, für die Somen einer Klasse L, welche den Bedingungen der zweiten Hälfte unseres Theorems genügt, die Operation $\underline{x}$ mittels (11). Komplementbildung liefert dann

$$(\underline{x})' = [\sum_{y \in Q(x)} y]' = \prod_{y \in Q(x)} y'.$$

Setzt man $(\underline{x})' = x$, und nennt man L' die Klasse von Somen, deren jedes Komplement eines Somas von L ist, so folgt aus der zweiten Hälfte des Theorems von M. und R., dass x den topologischen Axiomen I, III und IV, und somit (siehe auch (1)) $\underline{x}$ den Axiomen I*, III* und IV* genügt; L' fällt mit der Klasse der abgeschlossenen Somen, somit L mit der Klasse der offenen Somen zusammen.

Bemerkung. Das Theorem von M. und R. gilt schon in einer Struktur, welche nur das Axiom 1° erfüllt. *Ebenso gilt das letzte Theorem schon in einer Struktur, welche nur das Axiom* 1° *erfüllt*; man hat nur wieder den Beweis von M. und R., loc. cit. 3), S. 176, 177, dual umzusetzen (gemäss α), β), γ) von § 2).

Diese duale Umsetzung von Beweisen liefert aus in der zitierten Arbeit vorkommenden Sätzen u.a.:

Theorem. *Damit in einer dem Axiom* 1° *genügenden Struktur die Bildung des offenen Kernes $\underline{x}$ für den immer $\underline{x} \subset x$ gelten soll* (T.A. I*),

eindeutig festgelegt sei durch seine Invarianten (d.h. durch die Klasse der offenen Somen), *ist notwendig und hinreichend, dass* $\underline{x}$ *den Axiomen III* und IV* genügt.*

§ 7. **Theorem.** *Eine Struktur S genüge dem Axiom* 1°. *In S seien zwei Operationen definiert, die zu jedem* $x \in S$ *ein offener Kern,* $\underline{x}^{(1)}$ *bzw.* $\underline{x}^{(2)}$, *hinzufügen, deren jeder die Axiome I*, III* und IV* erfüllt.*

Damit für jedes $x \in S$ $\underline{x}^{(1)} \subset \underline{x}^{(2)}$ *sei, ist notwendig und hinreichend, dass die für die Operation* $\underline{x}^{(1)}$ *offenen Somen eine Teilklasse der für die Operation* $\underline{x}^{(2)}$ *offenen Somen bilden.*

Der Beweis folgt durch duale Umsetzung des Beweises eines Satzes von M. u. R., loc. cit. 3), S. 179, 180, über Operationen, welche abgeschlossene Hüllen liefern, die den Axiomen I, III und IV genügen.

Analoges gilt u.a. für die beiden folgenden Theoreme.

Theorem. *In einer Struktur, welche den Axiomen* 1° *und* 3° *genügt, und in welcher zwei Operationen* $\underline{x}^{(1)}$ *und* $\underline{x}^{(2)}$ *definiert sind, deren jede die Axiome I*, III* und IV* erfüllt, liefert Summenbildung:* $\underline{x}^{(1)} + \underline{x}^{(2)}$ *für jedes* $x \in S$ *eine neue Operation* $\underline{x}^{(3)}$, *welche die Axiome I*, III* und IV* erfüllt.*

Axiom $\overline{3}°$. Zu jeder Klasse von Somen gibt es eine Summe.

Theorem. *In einer Struktur S, welche den Axiomen* 1° *und* $\overline{3}°$ *genügt, und in welcher eine Klasse K von Operationen* $\underline{x}^{(\iota)}$ *definiert ist, deren jede die Axiome I*, III* und IV* erfüllt, liefert Summenbildung* $\sum_{\substack{\underline{x}^{(\iota)} \in K \\ x \text{ fest}}} \underline{x}^{(\iota)}$ *für jedes* $x \in S$ *eine neue Operation* $\underline{x}^{(K)}$, *welche die Axiome I*, III* und IV* erfüllt.*

III. Die dualen Grundbegriffe a^r und a_r.

§ 8 [1]). In einer BOOLEschen Algebra folgt aus

$$a^r = a \cdot \overline{a'} \text{ und } a \subset \bar{a},$$

dass

$$\bar{a} = a + a^r + (a')^r \text{ und } a^r \subset a$$

gilt. Umgekehrt, aus

$$\bar{a} = a + a^r + (a')^r \text{ und } a^r \subset a$$

folgt:

$$a^r = a \cdot \overline{a'} \text{ und } a \subset \bar{a}.$$

In S sind somit

$$T.A.I: a \subset \bar{a} \text{ und } T.A.I^r: a^r \subset a$$

acquivalent. Zwischen ihren Grundbegriffen $\bar{a}$ und a^r gelten in beiden Systemen die Relationen:

$$a^r = a \cdot \overline{a'} \text{ und } \bar{a} = a + a^r + (a')^r.$$

Die Aequivalenz bleibt bestehen nach Hinzufügung von

$$T.A.\,II\colon\ 0 = 0 \text{ bzw. } T.A.\,II^r\colon\ 1^r = 0,$$

nach weiterer Hinzufügung von

$$T.A.\,III\colon\ a \subset b \to \bar{a} \subset \bar{b} \text{ bzw. } T.A.\,III^r\colon a \subset b \to a \cdot (a^r)' \subset b \cdot (b^r)' \ ^4)$$

[oder: immer ist $ab^r \subset (ab)^r$] [5]),

nach abermaliger Hinzufügung von

$$T.A.\,IV\colon\ a = \bar{a} \text{ bzw. } T.A.\,IV^r\colon \{a \cdot (a^r)'\}^r = 0, \ ^6)$$

und schliesslich auch nach Hinzufügung von

$$T.A.\,V\colon\ \overline{a+b} \subset a + \bar{b} \text{ bzw. } T.A.\,V^r\colon\ (a \cdot b) \cdot \{(a \cdot b)^r\}' \supset a \cdot (a^r)' \cdot b\ (b^r)\ ^7)$$

[oder: immer ist $(ab)^r \subset ab^r + ba^r$] [8]).

§ 8bis. Durch duale Umsetzung erhält man:
In einer BOOLEschen Algebra folgt aus

$$a_r = a + \underline{a'} \text{ und } \underline{a} \subset a,$$

dass

$$\underline{a} = a \cdot a_r \cdot (a')_r \text{ und } a \subset a_r$$

gilt. Dies lässt sich umkehren.
In S sind somit

$$T.A.\,I^*\colon\ \underline{a} \subset a \text{ und } T.A.\,I_r\colon\ a \subset a_r$$

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4) Aus den Axiomen I, III folgt (siehe auch § 2): $a \subset b \to \underline{a} \subset \underline{b}$ oder $a \cdot (a^r)' \subset b \cdot (b^r)'$ da $x \,.\, (x^r)' = x\,[x' + (\overline{x'})'] = x \,.\, \underline{x} = \underline{x}$ ist. Umgekehrt liefern die Axiome I^r, III^r (erste Form): $a \subset b \to a \,.\, (a^r)' \subset b \,.\, (b^r)'$ oder $\underline{a} \subset \underline{b}$, somit auch $\bar{a} \subset \bar{b}$.

5) *T. A. III* liefert: $\overline{b'} \subset \overline{a' + b'}$, also auch $ab \,.\, \overline{b'} \subset ab \,.\, \overline{(ab)'}$, oder $a \,.\, b^r \subset (ab)^r$. Umgekehrt liefern *T. A. I*^{r} und *T. A. III*^{r} (zweite Form): $a \,.\, b \,.\, \overline{b'} \subset ab \,.\, \overline{(ab)'} = ab \,.\, \overline{a' + b'}$, somit $ab \,.\, \overline{b'} \subset \overline{a' + b'}$, und $(a' + b')' \,.\, \overline{b'} \subset \overline{a' + b'}$. Auch ist $(a' + b') \,.\, \overline{b'} \subset \overline{a' + b'}$, so dass $\overline{b'} \subset \overline{a' + b'}$. Aus $x \subset y$ folgt also immer $\bar{x} \subset \bar{y}$ (*T. A. III*).

6) Siehe J. ALBUQUERQUE, Portugaliae Math. 3 (1942), S. 186—187.

7) Aus den Axiomen I, V folgt (siehe auch RIDDER, loc. cit [1])): $\underline{ab} \supset \underline{a} \cdot \underline{b}$ oder $ab \,.\, \{(ab)^r\}' \supset a \,.\, (a^r)' \,.\, b \,.\, (b^r)'$. Umgekehrt liefern die Axiome I^r, V^r (erste Form): $(ab) \,.\, \{(ab)^r\}' \supset a \,.\, (a^r)' \,.\, b \,.\, (b^r)'$, oder $\underline{ab} \supset \underline{a} \cdot \underline{b}$, oder Axiom V.

8) *T. A. V* liefert: $\overline{a' + b'} \subset \overline{a'} + \overline{b'}$, also auch $ab \,.\, \overline{(ab)'} \subset ab \,.\, (\overline{a'} + \overline{b'})$, oder $(ab)^r \subset b \,.\, a^r + a \,.\, b^r$. Umgekehrt liefern *T. A. I*^{r} und *T. A. V*^{r} (zweite Form): $ab \,.\, \overline{(ab)'} \subset a \,.\, b \,.\, \overline{b'} + b \,.\, a \,.\, \overline{a'}$, somit $(a' + b')' \,.\, \overline{(a' + b')} \subset \overline{a'} + \overline{b'}$. Auch ist $(a' + b') \,.\, \overline{a' + b'} \subset \overline{a'} + \overline{b'}$, so dass $\overline{a' + b'} \subset \overline{a'} + \overline{b'}$. Also ist immer $\overline{x + y} \subset \bar{x} + \bar{y}$ (*T. A. V*).

aequivalent. Zwischen ihren Grundbegriffen $\underline{a}$ und a_r gelten in beiden Systemen die Relationen:

$$a_r = a + \underline{a'} \text{ und } \underline{a} = a \cdot a_r \cdot (a')_r.$$

Die Aequivalenz bleibt bestehen nach Hinzufügung von

$$T.A.\,II^*\colon \underline{1} = 1 \text{ bzw. } T.A.\,II_r\colon 0_r = 1,$$

nach weiterer Hinzufügung von

$$T.A.\,III^*\colon b \subset a \to \underline{b} \subset \underline{a} \text{ bzw. } T.A.\,III_r\colon b \subset a \to b + (b_r)' \subset a + (a_r)'$$

[oder: immer ist $a + b_r \supset (a + b)_r$],

nach abermaliger Hinzufügung von

$$T.A.\,IV^*\colon \underline{\underline{a}} = \underline{a} \text{ bzw. } T.A.\,IV_r\colon \{a + (a_r)'\}_r = 1,$$

und schliesslich auch nach Hinzufügung von

$$T.A.\,V^*\colon a \cdot \underline{b} \subset \underline{ab}\ ^{9}) \text{ bzw. } T.A.\,V_r\colon a + (a_r)' + b + (b_r)' \supset (a + b) + \{(a + b)_r\}'$$

[oder: immer ist $(a + b_r) \cdot (b + a_r) \subset (a + b)_r$].

§ 9 [1]). Im folgenden betrachten wir eine Struktur S_0, welche ausser den Axiomen 1°, 2°, 3°, 4°, 5°, 6°, 7° noch folgendem Axiom genügt:

Axiom 8°. Jedes nicht leere Soma ist entweder ein Primsoma oder enthält mehrere Primsomen; dabei ist *Primsoma* jedes nicht leere Soma, welches ausser dem leeren Soma und sich selbst kein weiteres Teilsoma enthält.

Eine derartige Struktur ist als ein Raum zu betrachten, dessen Punkte die Primsomen sind.

T. A. VI: Für jedes Primsoma p ist $\bar{p} = p$.

In einer Struktur S_0 gilt das Dualitätsprinzip von § 2 für die aus einem oder mehreren der topologischen Axiome I—VI ableitbaren Sätze, wenn man zu den angegebenen Ersetzungen hinzufügt: δ) falls p ein Primsoma ist, werde p durch $1 - p$, und umgekehrt, ersetzt.

Unter Annahme der Relationen (1) sind T.A. VI und

*T. A. VI**: Für jedes Primsoma p ist $\underline{1-p} = 1 - p$, aequivalent.

Die in § 8 betrachteten Axiomensysteme $\mathrm{I} - \mathrm{V}$ und $\mathrm{I}^r - \mathrm{V}^r$ bleiben in einer Struktur S_0 aequivalent nach Hinzufügung von

T. A. VI: für jedes Primsoma p ist $\bar{p} = p$, bzw. *T. A. VI^r*: für jedes Primsoma p ist $(1 - p)^r = 0$.

Duale Umsetzung des Beweises liefert:

Die in § 8bis betrachteten Axiomensysteme $\mathrm{I}^* - \mathrm{V}^*$ und $\mathrm{I}_r - \mathrm{V}_r$ bleiben in einer Struktur S_0 aequivalent nach Hinzufügung von

*T. A. VI**: für jedes Primsoma p ist $\underline{1-p} = 1 - p$, bzw. *T. A. VI_r*: für jedes Primsoma p ist $p_r = 1$.

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[9]) Die topologischen Axiome V und V* sind, unter Annahme der Relationen (1), aequivalent.

§ 10. Die Trennungsaxiome in ihrer gewöhnlichen Form enthalten alle den Begriff des offenen Kernes. Duale Umsetzung liefert die Form, bei der in allen die abgeschlossene Hülle als Grundbegriff auftritt.

So sind in einer Struktur S_0, in welcher die topologischen Axiome I — VI erfüllt sind, gleichwertig die topologischen Axiome:

A^*. zu jedem Paar p_1, p_2 von Primsomen mit $p_1 \cdot p_2 = 0$ gibt es Somen a, b mit $ab = 0$, $p_1 \subset \underline{a}$ und $p_2 \subset \underline{b}$;

und:

A. zu jedem Paar p_1, p_2 von Primsomen mit $(1 - p_1) + (1 - p_2) = 1$ gibt es Somen a, b mit $a + b = 1$, $1 - p_1 \supset \bar{a}$ und $1 - p_2 \supset \bar{b}$.

Bei Benutzung des Randes als Grundbegriff hat man als aequivalentes Axiom:

A^r. zu jedem Paar von Primsomen p_1, p_2 mit $p_1 \cdot p_2 = 0$ gibt es Somen a, b mit $ab = 0$, $p_1 \subset a(a^r)'$ und $p_2 \subset b(b^r)'$.

Duale Umsetzung liefert als mit A^r gleichwertig:

A_r. zu jedem Paar p_1, p_2 von Primsomen mit $(1 - p_1) + (1 - p_2) = 1$ gibt es Somen a, b mit $a + b = 1$, $1 - p_1 \supset a + (a_r)'$ und $1 - p_2 \supset$ $\supset b + (b_r)'$.

Wir überlassen dem Leser die Formulierung der übrigen Trennungsaxiome mittels offenen Kernes oder Randes [10], und der dualen Grundbegriffe.

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[10]) Siehe ALBUQUERQUE, loc. cit. 6), S. 193—196; auch RIDDER, loc. cit. 1), § 28.

Mathematics. — *Non-homogeneous binary quadratic forms.* IV. By H. DAVENPORT. (Communicated by Prof. J. G. VAN DER CORPUT.)

(Communicated at the meeting of May 31, 1947.)

1. The present paper is a continuation of paper II of this series [1]). We are concerned with the minimum of the product

$$|(\xi - a)(\xi' - b)|,$$

where ξ is an arbitrary integer of the field $k(\theta)$, say

$$\xi = x + \theta y, \qquad \theta = \tfrac{1}{2}(1 + \sqrt{5}),$$

and a, b are given real numbers. It was proved in II that if a, b are not of the form

$$a = \tfrac{1}{2}\tau + \xi_0, \quad b = \tfrac{1}{2}\tau' + \xi_0' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

nor of the form

$$a = \frac{1}{\sqrt{5}}\tau + \xi_0, \quad b = -\frac{1}{\sqrt{5}}\tau' + \xi_0', \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where τ is a unit and ξ_0 an integer of $k(\theta)$, then there exists an integer ξ satisfying

$$|(\xi - a)(\xi' - b)| < \frac{1}{6.34}.$$

The question of the existence of a third minimum was left unsolved, though it was proved that if such a third minimum exists, it cannot be less than

$$\frac{1}{4\theta} = \frac{1}{6.472\ldots}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

I have now established the existence of a third minimum, and indeed of an infinite sequence of minima having values greater than (3). These minima occur when

$$a = \frac{\tau}{\alpha_m} + \xi_0, \quad b = \frac{\tau'}{\alpha'_m} + \xi_0' \qquad \text{(or vice versa)}, \quad . \quad . \quad . \quad (4)$$

where τ is a unit and ξ_0 an integer of $k(\theta)$, and where m is an odd positive integer, and α_m is defined by

$$\alpha_m = \frac{2(\theta^{n+1} - 1)}{\theta^n + 1}, \qquad n = 3m. \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

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[1]) Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, **50** (1947) 378—389.

The case $m = 1$ gives

$$\alpha_1 = \frac{2(\theta^4 - 1)}{\theta^3 + 1} = 2\theta - 1 = \sqrt{5},$$

so that a, b are then of the form specified in (2). The case $m = 3$ gives

$$\alpha_3 = \frac{2(\theta^{10} - 1)}{\theta^9 + 1} = \frac{11}{4\theta - 3},$$

and the values of a, b defined by (4) are then those corresponding to the third minimum, the value of which is

$$\frac{1}{|\alpha_3 \alpha_3'|} = \frac{19}{121} = \frac{1}{6.368\ldots}.$$

The formal enunciation of the results is as follows.

Theorem 1. *Suppose that a, b are not of the form (1), nor of the form (4), where α_m is defined by (5) and $m = 1, 3, 5, \ldots$. Then the lower bound $M(a, b)$ of*

$$|(\xi - a)(\xi' - b)|$$

for integers ξ of $k(\theta)$ satisfies

$$M(a, b) \leqslant \frac{1}{4\theta}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Theorem 2. *If a, b are of the form (1), we have $M(a, b) = \frac{1}{4}$, and this minimum is then attained for an infinity of integers ξ. If a, b are of the form (4), we have*

$$M(a, b) = \frac{1}{|\alpha_m \alpha_m'|}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

and this minimum is then also attained for an infinity of integers ξ.

We may note that the value of the minimum corresponding to any m, which is specified in (7), may also be expressed in terms of the Fibonacci numbers. If we define these by

$$F_1 = F_2 = 1, \quad F_{n+2} = F_{n+1} + F_n \qquad (n = 1, 2, \ldots),$$

then $\theta^n = F_n\theta + F_{n-1}$, and we easily find that

$$\frac{1}{|\alpha_m \alpha_m'|} = \frac{F_{n+1} + F_{n-1}}{4(F_{n+2} + F_n - 2)}, \qquad n = 3m. \quad . \quad . \quad . \quad . \quad (8)$$

The greater generality of the present arguments, as compared with those of II, has the effect that the present paper supersedes a great deal of the former one. In fact, the only results which will be quoted from II are the comparatively simple Lemmas 1, 2, 3.

2. The assertion of Theorem 1 is that if

$$M = M(a, b) > \frac{1}{4\theta}$$

then a, b must be either of the form (1) or of the form (4). We may therefore write

$$\frac{1}{M} = 4\theta(1-\delta), \quad \ldots\ldots\ldots \quad (9)$$

where $\delta > 0$. Let ε_0 be an arbitrarily small positive number, which we can suppose to satisfy any desired inequality of the form $\varepsilon_0 < E(\delta)$, where $E(\delta)$ is any positive number depending only on δ. A finite number of such inequalities will be imposed on ε_0 in the course of the paper.

By the definition of M, there exists an integer ξ_0 of $k(\theta)$ such that

$$|(\xi_0 - a)(\xi_0' - b)| = \frac{M}{1-\varepsilon}, \qquad \text{where } 0 \leqq \varepsilon < \varepsilon_0. \quad \ldots \quad (10)$$

We define α, β by

$$\alpha = (a - \xi_0)^{-1}, \qquad \beta = (b - \xi_0')^{-1}, \quad \ldots\ldots \quad (11)$$

so that

$$|\alpha\beta| = \frac{1-\varepsilon}{M} = 4\theta(1-\varepsilon)(1-\delta). \quad \ldots\ldots \quad (12)$$

By the definition of M, and by (10) and (11), we have

$$|(\alpha\xi - 1)(\beta\xi' - 1)| \geqq 1 - \varepsilon \quad \ldots\ldots\ldots \quad (13)$$

for all integers ξ of $k(\theta)$.

By the operations of (i) replacing α, β by $\alpha\tau$, $\beta\tau'$, where τ is any unit, (ii) interchanging α, β, we can ensure, as in II, that

$$\alpha > 0, \qquad \frac{\alpha}{\theta} \leqq |\beta| \leqq \alpha. \quad \ldots\ldots\ldots \quad (14)$$

It follows from (12) and (14) that

$$\alpha^2 \leqq \alpha|\beta|\theta = 4\theta^2(1-\varepsilon)(1-\delta),$$
$$\alpha < 2\theta - \delta. \quad \ldots\ldots\ldots \quad (15)$$

Also

$$|\beta| \leqq (\alpha|\beta|)^{\frac{1}{2}} < (4\theta)^{\frac{1}{2}},$$
$$|\beta| < 2.55. \quad \ldots\ldots\ldots \quad (16)$$

3. **Lemma 1.** *If $\alpha < \sqrt{5} - \varepsilon$ then $\alpha = \beta = 2$.*

Proof. Lemmas 1, 2, 3 of II.

Lemma 2. *If* $\sqrt{5}-\varepsilon \leqq \alpha < 2\theta-\delta$ *then* $\beta < 0$.

Proof. By (13) with $\xi = \theta^{-2}$, we have

$$|(\theta^{-2}\alpha-1)(\theta^2\beta-1)| \geqq 1-\varepsilon.$$

Suppose $\beta > 0$, then $\beta \geqq \alpha/\theta \geqq (\sqrt{5}-\varepsilon)/\theta > 1$, by (14). Hence

$$|\alpha-\theta^2|(\beta-\theta^{-2}) \geqq 1-\varepsilon.$$

If $\alpha < \theta^2$, this gives

$$\begin{aligned}\beta &\geqq \theta^{-2}+(1-\varepsilon)(\theta^2-\alpha)^{-1} \geqq \theta^{-2}+(1-\varepsilon)(\theta^2-\sqrt{5}+\varepsilon)^{-1}\\ &= \theta^{-2}+(1-\varepsilon)(\theta^{-2}+\varepsilon)^{-1}.\end{aligned}$$

This last expression is almost 3, and so we have a contradiction to (16).

We may now suppose that $\alpha > \theta^2$. We have

$$\beta \geqq \theta^{-2}+(1-\varepsilon)(\alpha-\theta^2)^{-1},$$
$$\alpha\beta \geqq (1-\varepsilon)\{\theta^{-2}\alpha+1+\theta^2(\alpha-\theta^2)^{-1}\}.$$

This last expression decreases as α increases for $\theta^2 < \alpha < 2\theta^2$. Since $\alpha < 2\theta-\delta$, it follows that

$$\begin{aligned}\alpha\beta &\geqq (1-\varepsilon)\{2\theta^{-1}-\theta^{-2}\delta+1+\theta^2(2\theta-\theta^2-\delta)^{-1}\}\\ &= (1-\varepsilon)\{2\theta^{-1}-\theta^{-2}\delta+1+\theta^3(1-\delta\theta)^{-1}\}.\end{aligned}$$

Since $2\theta^{-1}+1+\theta^3 = 4\theta$, this gives $\alpha\beta > 4\theta$, provided ε is small compared with δ, and so gives a contradiction to (12). This proves Lemma 2.

Lemma 3. *If* $\sqrt{5}-\varepsilon \leqq \alpha < 2\theta-\delta$, *then*

$$2+\tfrac{1}{7}\delta < -\beta < \theta^2.$$

Proof. By Lemma 2, we have $\beta < 0$, and we write $\beta = -\bar{\beta}$, where $\bar{\beta} > 0$. In view of (16), it suffices to prove that $\bar{\beta} > 2+\frac{1}{7}\delta$.

By (13) with $\xi = \theta$, we have

$$(\alpha-\theta^{-1})|\bar{\beta}-\theta| \geqq 1-\varepsilon.$$

Now $\alpha-\theta^{-1} < 2\theta-\delta-\theta^{-1} = \theta^2-\delta$. Hence

$$|\beta-\theta| \geqq (1-\varepsilon)(\theta^2-\delta)^{-1} > (1-\varepsilon)\theta^{-2}(1+\delta\theta^{-2}) > \theta^{-2}+\tfrac{1}{7}\delta,$$

if ε is small compared with δ. If $\bar{\beta} < \theta$, this would give

$$\bar{\beta} < \theta-\theta^{-2}-\tfrac{1}{7}\delta,$$
$$\frac{\bar{\beta}}{\alpha} < \frac{\theta-\theta^{-2}-\frac{1}{7}\delta}{\sqrt{5}-\varepsilon} < \frac{1.237}{2.236} < \frac{1}{\theta},$$

contrary to (14). Hence $\bar{\beta} > \theta$, and

$$\bar{\beta} > \theta+\theta^{-2}+\tfrac{1}{7}\delta = 2+\tfrac{1}{7}\delta.$$

4. In virtue of (15), and Lemmas 1 and 3, we may suppose henceforth, in proving Theorem 1, that

$$\sqrt{5}-\varepsilon \leqq \alpha < 2\theta - \delta, \quad 2 + \tfrac{1}{7}\delta < \bar{\beta} < \theta^2, \quad . \quad . \quad . \quad . \quad (17)$$

where $\bar{\beta} = -\beta$.

We define integers ξ_m, η_m of $k(\theta)$, for $m = 1, 2, 3, \ldots$, as follows:

$$\xi_m = \frac{1+\theta^{-n+3}}{2\theta}, \quad \eta_m = \frac{1-\theta^{-n}}{2}, \qquad \text{where } n = 3m. \quad . \quad (18)$$

That these are integers follows from the fact that $\theta^3 = 2\theta + 1 \equiv 1 \pmod 2$. Since

$$\frac{1}{\eta_1} = \frac{2}{1-\theta^{-3}} = \frac{2\theta^3}{\theta^3 - 1} = \theta^2,$$

and $\dfrac{1}{\eta_m}$ decreases as m increases and has the limit 2, there will be exactly one value of m for which

$$\frac{1}{\eta_m} > \bar{\beta} > \frac{1}{\eta_{m+1}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

Moreover, by (17), m will have an upper bound depending only on δ. Hence, by a previous remark, we can suppose that ε_0 (and therefore also ε) is less than any positive number which depends only on m.

Lemma 4. *If* (19) *holds, then*

$$\frac{1}{\xi_m} < \alpha < \frac{1}{\xi_{m+2}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

Proof. One half of this is easily proved; by (12), (18), (19), we have

$$\alpha < \frac{4\theta}{\bar{\beta}} \leqq 4\theta\,\eta_{m+1} = 2\theta\,(1-\theta^{-n-3}) < \frac{2\theta}{1+\theta^{-n-3}} = \frac{1}{\xi_{m+2}}.$$

To obtain the other half, we first apply (13) with $\xi = \theta$. This gives

$$(\theta\alpha - 1)(\theta^{-1}\bar{\beta} - 1) \geqq 1 - \varepsilon, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

on noting that $\bar{\beta} > 2 > \theta$. By (19),

$$\theta^{-1}\bar{\beta} - 1 < \frac{1}{\theta\,\eta_m} - 1 = \frac{2 - \theta + \theta^{-n+1}}{\theta - \theta^{-n+1}}.$$

If we could neglect ε in (21), we should conclude that

$$\theta\alpha \geqq 1 + \frac{\theta - \theta^{-n+1}}{2 - \theta + \theta^{-n+1}} = \frac{2}{2 - \theta + \theta^{-n+1}} = \frac{2\theta^2}{1 + \theta^{-n+3}},$$

whence

$$\alpha \geqq \frac{2\theta}{1+\theta^{-n+3}} = \frac{1}{\xi_m}.$$

It is clear, therefore, that the corresponding deduction from (21) when we do not neglect ε will be of the form

$$\alpha \geqq \frac{1}{\xi_m} - \mu\varepsilon,$$

where μ is a positive number depending only on m. But from (13), with $\xi = \xi_m$, we have

$$|\alpha\xi_m - 1|(|\xi'_m|\bar{\beta}+1) \geqq 1-\varepsilon,$$

and if $\alpha \leq 1/\xi_m$ the last two inequalities give a contradiction if ε is less than some positive number depending only on m. Thus we have $\alpha > 1/\xi_m$.

Lemma 5. *The value of m determined by (19) cannot be even.*

Proof. We observe first that if m is even, then from (18)

$$\eta'_m = -\frac{\theta^n - 1}{2}, \qquad \eta'_{m+1} = \frac{\theta^{n+3}+1}{2}.$$

The proof is based on the two inequalities derived from (13) by taking $\xi = -\eta'_m$ and $\xi = -\eta'_{m+1}$. These are:

$$(|\eta'_m|\alpha - 1)(1-\eta_m\beta) \geqq 1-\varepsilon,$$

$$(\eta'_{m+1}\alpha + 1)(\eta_{m+1}\bar{\beta} - 1) \geqq 1-\varepsilon.$$

The expressions in brackets are all positive, by (19) and the fact that

$$|\eta'_m| \geqq \tfrac{1}{2}(\theta^3-1) = \theta, \quad \alpha \geqq 1.5 - \varepsilon.$$

We multiply these by η_{m+1} and η_m respectively and add, thus eliminating β. The result may be written

$$\frac{\eta_{m+1}-\eta_m}{1-\varepsilon} \geqq \frac{\eta_{m+1}}{|\eta'_m|\alpha - 1} + \frac{\eta_m}{\eta'_{m+1}\alpha + 1}. \qquad (22)$$

This inequality determines a lower bound for α. Hence, if we prove that the inequality is violated when α is replaced by $1/\xi_{m+2}$, we shall have reached a contradiction, by (20).

Since

$$\eta_{m+1} - \eta_m = \tfrac{1}{2}(\theta^{-n} - \theta^{-n-3}) = \theta^{-n-2},$$

it will suffice to prove that the sum on the right of (22), with the above value of α, exceeds θ^{-n-2} by a positive amount deending only on m. This sum is

$$\frac{\eta_{m+1}\,\xi_{m+2}}{|\eta'_m|-\xi_{m+2}}+\frac{\eta_m\,\xi_{m+2}}{\eta'_{m+1}+\xi_{m+2}}$$

$$=\frac{(1-\theta^{-n-3})(1+\theta^{-n-3})}{2\{\theta(\theta^n-1)-(1+\theta^{-n-3})\}}+\frac{(1-\theta^{-n})(1+\theta^{-n-3})}{2\{\theta(\theta^{n+3}+1)+1+\theta^{-n-3}\}}$$

$$=\frac{1-\theta^{-2n-6}}{2\{\theta^{n+1}-\theta^2-\theta^{-n-3}\}}+\frac{1-\theta^{-n}}{2\{\theta^{n+4}+1\}}$$

$$>\tfrac{1}{2}\theta^{-n-1}(1-\theta^{-2n-6})(1-\theta^{-n+1})^{-1}+\tfrac{1}{2}\theta^{-n-4}(1-\theta^{-n})(1+\theta^{-n-4})^{-1}$$

$$>\tfrac{1}{2}\theta^{-n-1}(1-\theta^{-2n-6})(1+\theta^{-n+1})+\tfrac{1}{2}\theta^{-n-4}(1-\theta^{-n})(1-\theta^{-n-4})$$

$$>\tfrac{1}{2}(\theta^{-n-1}-\theta^{-3n-7}+\theta^{-2n}-\theta^{-4n-6}+\theta^{-n-4}-\theta^{-2n-4}-\theta^{-2n-8})$$

$$=\theta^{-n-2}+\tfrac{1}{2}\theta^{-2n}(1-\theta^{-4}-\theta^{-8}-\theta^{-n-7}-\theta^{-2n-6}).$$

Since $n=3m\geq 6$, the sum in the bracket is greater than a positive absolute constant, and the result follows.

Lemma 6. *If m is odd, it is impossible that* (19) *holds and that*

$$\frac{1}{\xi_{m+1}}\leqslant\alpha<\frac{1}{\xi_{m+2}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

Proof. The proof is based on two inequalities, derived from (13) by taking $\xi=\xi_{m+1}$ and $\xi=-\eta'_{m+1}$. Since m is odd, we have, from (18),

$$\xi'_{m+1}=\tfrac{1}{2}\theta(\theta^n-1), \qquad \eta'_{m+1}=-\frac{\theta^{n+3}-1}{2}.$$

As we suppose that (19) and (23) hold, the inequalities, which are

$$|(\xi_{m+1}\alpha-1)(\xi'_{m+1}\bar{\beta}+1)|\geqslant 1-\varepsilon,$$
$$|(\eta'_{m+1}\alpha+1)(\eta_{m+1}\bar{\beta}-1)|\geqslant 1-\varepsilon,$$

take the form

$$(\xi_{m+1}\alpha-1)(\xi'_{m+1}\bar{\beta}+1)\geqslant 1-\varepsilon, \quad . \quad . \quad . \quad . \quad . \quad (24)$$
$$(|\eta'_{m+1}|\alpha-1)(\eta_{m+1}\bar{\beta}-1)\geqslant 1-\varepsilon, \quad . \quad . \quad . \quad . \quad . \quad (25)$$

where each factor is positive.

We use (24), together with the fact that $\bar{\beta}<4\theta/\alpha$ by (12), to obtain a lower bound for α. It will be clear from the nature of the calculations and result that we can neglect ε. Thus (24) gives

$$(\xi_{m+1}\alpha-1)\left(\frac{4\theta\,\xi'_{m+1}}{\alpha}+1\right)\geqslant 1.$$

It will be convenient to write $P=\xi_{m+1}\,\xi'_{m+1}$, so that P is a positive integer. Substituting for ξ'_{m+1}, the last inequality gives

$$4\theta P+\xi_{m+1}\alpha-\frac{4\theta P}{\xi_{m+1}\alpha}-1\geqslant 1, \quad . \quad . \quad . \quad . \quad . \quad (26)$$

whence

$$\{\xi_{m+1}\alpha+(2\theta P-1)\}^2 \geqslant 4\theta^2P^2+1,$$

$$\xi_{m+1}\alpha \geqslant |\sqrt{4\theta^2P^2+1}-2\theta P+1$$

$$>2\theta P\left\{1+\frac{1}{8\theta^2P^2}-\frac{1}{128\theta^4P^4}\right\}-2\theta P+1$$

$$=1+\frac{1}{4\theta P}-\frac{1}{64\theta^3P^3}.$$

Now

$$P=N(\xi_{m+1})=\tfrac{1}{4}N(1+\theta^n)=\tfrac{1}{4}(\theta^n-\theta^{-n}).$$

Hence

$$\xi_{m+1}\alpha>1+\theta^{-n-1}(1-\theta^{-2n})^{-1}-\theta^{-3n-3}(1-\theta^{-2n})^{-3}$$

$$>1+\theta^{-n-1}(1+\theta^{-2n})\ -\theta^{-3n-3}(1-\theta^{-2n})^{-3}$$

$$=1+\theta^{-n-1}+\theta^{-3n-3}\{\theta^2-(1-\theta^{-2n})^{-3}\},$$

$$\xi_{m+1}\alpha>1+\theta^{-n-1}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

since

$$(1-\theta^{-2n})^{-3}<1-<\theta^2.$$

We now use (25) in a similar way to obtain a lower bound for $\overline{\beta}$. Writing $Q=|\eta_{m+1}\eta'_{m+1}|$, the inequality analogous to (26) is

$$4\theta Q-\eta_{m+1}\overline{\beta}-\frac{4\theta Q}{\eta_{m+1}\overline{\beta}}+1\geqslant 1,$$

whence

$$(2\theta Q-\eta_{m+1}\overline{\beta})^2\leqslant 4\theta^2Q^2-4\theta Q.$$

Now

$$\overline{\beta}<\theta^2<2\theta^{\frac{1}{2}}(\theta^6-1)\leqslant 2\theta|\eta'_{m+1}|=2\theta Q/\eta_{m+1}.$$

Hence

$$\eta_{m+1}\overline{\beta}\geqslant 2\theta Q-\sqrt{4\theta^2Q^2-4\theta Q}$$

$$>2\theta Q-2\theta Q\left\{1-\frac{1}{2\theta Q}-\frac{1}{8\theta^2Q^2}\right\}$$

$$=1+\frac{1}{4\theta Q}.$$

We have

$$Q=|N(\eta_{m+1})|=\tfrac{1}{4}(1-\theta^{-n-3})(\theta^{n+3}-1)=\tfrac{1}{4}\theta^{n+3}(1-\theta^{-n-3})^2.$$

Hence

$$\eta_{m+1}\overline{\beta}>1+\theta^{-n-4}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

The inequalities (27) and (28) have been derived from (24) and (25) by neglecting ε. They should therefore be corrected by subtracting on the right terms depending only on m and ε, which tend to zero with ε for fixed m. We shall prove that (27) and (28) give a lower bound for $\alpha\bar{\beta}$ greater than 4θ. In view of (12), this gives a contradiction, since the correcting terms involving ε are negligible in comparison with δ, by a previous remark.

The product $\xi_{m+1}\,\eta_{m+1}$ is

$$\frac{1}{4\theta}(1+\theta^{-n})(1-\theta^{-n-3}) = \frac{1}{4\theta}(1+2\theta^{-n-2}-\theta^{-2n-3}).$$

The product of the expressions on the right of (27) and (28) is

$$(1-\theta^{-n-3})(1-\theta^{-n-4}) > 1-\theta^{-n-4}(\theta^3-1) = 1-2\theta^{-n-2}.$$

Hence $\alpha\bar{\beta} > 4\theta$, as was to be proved.

Mathematics. — *A characterization of the sub-manifold of* $C[a, b]$ *spanned by the sequence* $\{x^{n_k}\}$. By J. KOREVAAR. (Communicated by Prof. J. G. VAN DER CORPUT.)

(Communicated at the meeting of June 28, 1947.)

§ 1. *Introduction and results.* As usual, $C[a, b]$ denotes the space of all continuous functions of x, $a \leq x \leq b$. A set $\{\varphi_k(x)\} \in C[a, b]$ is said to be *closed* in $C[a, b]$, or to *span* this space, if it is possible to approximate uniformly over $[a, b]$ to every function $\in C[a, b]$ by linear aggregates $\sum a_k \varphi_k(x)$.

This note is concerned with closure-properties in $C[a, b]$ of sets

$$\{x^{n_k}\}, \qquad (k = 1, 2, \ldots) \tag{1.1}$$

where $\{n_k\}$ is an increasing sequence of non-negative integers. The classical result in this field is the theorem of MÜNTZ [3]) and SZÁSZ [5]) *which implies that the set* (1.1) *is closed in* $C[0, 1]$ *if, and only if,*

$$n_1 = 0, \quad \sum_{k=2}^{\infty} \frac{1}{n_k} = \infty . \tag{1.2}$$

It was shown recently by CLARKSON and ERDÖS [1]), and independently by L. SCHWARTZ [4]), that the sub-manifold of $C[0, 1]$ spanned by a sequence

$$\{x^{n_k}\}, \qquad \left(\sum_{n_k > 0} \frac{1}{n_k} < \infty \right) \tag{1.3}$$

is rather small: *a continuous function which is the uniform limit on* $[0, 1]$ *of a sequence of linear aggregates of functions* (1.3) *can be extended to be analytic in the interior of the unit-circle. The power series for this analytic extension contains only powers* x^{n_k}.

Conversely, the question arises whether it is possible to approximate uniformly over $[0, 1]$ by linear aggregates of functions (1.3) to every function

$$f(x) = \sum_{k=1}^{\infty} a_k x^{n_k} \tag{1.4}$$

where the series converges for $0 \leq x < 1$ and where

$$\lim_{x \to 1} f(x) \tag{1.5}$$

exists. This question was answered in part by CLARKSON and ERDÖS, viz. for the lacunary case $n_{k+1}/n_k > c > 1$. Here the existence of the limit (1.5) is known to imply the convergence of $\sum a_k$ and hence the uniform

convergence on [0,1] of the series (1.4). (See HARDY and LITTLEWOOD [2].) It is shown below (theorem 2) that $f(x)$ can be uniformly approximated by linear aggregates of functions (1.3) in the general case also. This yields a complete characterization of the sub-manifold of $C[0,1]$ spanned by the sequence (1.3).

In § 4 a similar result is reached for the interval $[a,b]$, $0 \leq a < b$. The establishment of this theorem 6 requires some more information than was known until now about the best approximation to powers x^m by linear aggregates $\Sigma a_k x^{n_k}$ on $[a,b]$. This information is obtained in § 3 by the method used by CLARKSON and ERDÖS to prove the extension to $[a,b]$ of the theorem of MÜNTZ and SZÁSZ for $[0,1]$. (See [1], § 3].)

§ 2. *The interval* $[0,1]$. The characterization-theorem for $[0,1]$ is

Theorem 1. *The set of continuous functions of* x, $0 \leq x \leq 1$, *spanned by the sequence*

$$\{x^{n_k}\}, \qquad \left(\sum_{n_k>0} \frac{1}{n_k} < \infty\right)$$

is identical with the set of all power series

$$\sum_{k=1}^{\infty} a_k x^{n_k}$$

convergent on $0 \leq x < 1$ *for which*

$$\lim_{x \to 1} \sum_{k=1}^{\infty} a_k x^{n_k}$$

exists.

The proof follows from the result of CLARKSON, ERDÖS and L. SCHWARTZ mentioned in § 1 combined with the following theorem, which may be of some interest in itself.

Theorem 2. *If the series*

$$g(x) = \Sigma b_k x^k$$

converges for $0 \leq x < 1$, *and if*

$$\lim_{x \to 1} g(x)$$

exists — name it $g(1)$ *— then it will be possible to approximate uniformly to* $g(x)$ *on* $[0,1]$ *by linear aggregates of the partial sums* $s_n(x)$ *of* $\Sigma b_k x^k$.

The proof of theorem 2 is exceedingly simple. It follows from the uniform continuity of $g(x)$ on $0 \leq x \leq 1$ that if ε is an arbitrary positive number, then

$$|g(x) - g(\theta x)| < \varepsilon/2 \qquad (0 \leqq x \leqq 1) \tag{2.1}$$

for θ sufficiently near to 1 $(\theta < 1)$. But the series for $g(\theta x)$ is convergent on $0 \leq x < \theta^{-1}$. Hence if N is sufficiently large,

$$|g(\theta x) - \sum_{1}^{N} b_k \theta^k x^k| < \varepsilon/2, \qquad (0 \leqq x \leqq 1). \tag{2.2}$$

A combination of (2.1) and (2.2) yields the theorem:

$$|g(x)-\sum_{1}^{N}\theta^k\{s_k(x)-s_{k-1}(x)\}|<\varepsilon, \qquad (0\leqq x\leqq 1).$$

The interval $[0,b]$. By the substitution $bx=x'$ theorem 1 yields a corresponding result for the interval $[0,b]$.

§ 3. *Approximation to powers* x^m *on* $[a,b]$. Starting from MÜNTZ's formula (see [3], [5]))

$$\text{l. b.}_{\{a_k\}}\int_0^1|x^m-\Sigma a_k x^{n_k}|^2\,dx=\frac{1}{2m+1}\prod_{k=1}^{\infty}\left(1-\frac{2m+1}{n_k+m+1}\right)^2$$

CLARKSON and ERDÖS proved an estimate (see [1], theorem 2]) which implies the following fundamental

Theorem 3. *Let* S *be an increasing sequence of non-negative integers* $\{n_k\}$ *satisfying the condition*

$$\sum_{n_k>0}\frac{1}{n_k}<\infty.$$

Let $0<\varepsilon<1$. *Then there exists an integer* $m_0(\varepsilon,S)$ *such that* $m>m_0(\varepsilon,S)$, m *not in* S, *implies*

$$\text{l. b.}_{\substack{\{a_k\}\\ \{n_k\}=S}}\max_{0\leqq x\leqq 1}|x^m-\Sigma a_k x^{n_k}|>(1-\varepsilon)^m.$$

Furthermore this m_0 *can be chosen independently of the particular sequence* S *for every family* F *of sequences* S *for which the functions*

$$\Phi(m,S)=\sum_{m<n_k}\frac{1}{n_k},\quad \Psi(m,S)=\frac{1}{m}\sum_{n_k\leqq m}1$$

tend to zero uniformly as $m\to\infty$ *for* $S\in F$.

Remark 1. Let the family F of sequences S satisfy the conditions of theorem 3. Now replace every sequence $S=\{n_k\}_{k=1,2,\ldots}\in F$ by the set of sequences $S_j=\{n_k+j\}_{k=1,2,\ldots}$, $j=0,1,2,\ldots$. Then the family F' formed by all sequences S_j will also satisfy the conditions of theorem 3. For

$$\Phi(m,S_j)=\sum_{m<n_k+j}\frac{1}{n_k+j}=\sum_{m<n_k}\frac{1}{n_k+j}+\sum_{m-j<n_k\leqq m}\frac{1}{n_k+j}\leqq$$

$$\leqq\sum_{m<n_k}\frac{1}{n_k}+\frac{1}{m}\sum_{n_k\leqq m}1=\Phi(m,S)+\Psi(m,S),$$

and

$$\Psi(m,S_j)=\frac{1}{m}\sum_{n_k+j\leqq m}1\leqq\frac{1}{m}\sum_{n_k\leqq m}1=\Psi(m,S).$$

Remark 2. Let the family F again satisfy the conditions of theorem 3. Now replace every sequence $S=\{n_k\}_{k=1,2,\ldots}\in F$ by the set of sequences

$S_j' = \{n_k\}_{k \neq j}$, $j = 1, 2, \ldots$. Then the family F'' formed by all sequences S_j' will again satisfy the conditions of theorem 3. For clearly

$$\Phi(m, S_j') \leqq \Phi(m, S), \quad \Psi(m, S_j') \leqq \Psi(m, S).$$

From theorem 3 a similar result will be derived for the interval $[a, b]$, $a \geq 0$.

Theorem 4. *Let S denote a sequence of integers $\{n_k\}$ satisfying the conditions*

$$0 \leqq n_1 < n_2 < \ldots < n_k < \ldots, \qquad \sum_{n_k > 0} \frac{1}{n_k} < \infty.$$

Let F be a family of sequences S satisfying the conditions that

$$\Phi(m, S) = \sum_{m < n_k} \frac{1}{n_k} \quad \text{and} \quad \Psi(m, S) = \frac{1}{m} \sum_{n_k \leq m} 1$$

tend to zero as $m \to \infty$ uniformly for $S \in F$. Let $0 < \varepsilon < b$. Then there exists an integer $m_0(\varepsilon, F)$ such that $m > m_0(\varepsilon, F)$ implies

$$\underset{\substack{S \in F \\ S \text{ not } \ni m}}{\text{l. b.}} \quad \underset{\{a_k\}\{n_k\}=S}{\text{l. b.}} \quad \max_{a \leq x \leq b} |x^m - \sum_k a_k x^{n_k}| \geqq (b - \varepsilon)^m.$$

There is no restriction in taking $b = 1$. (Substitution $x = bx'$.) Further let $a > 0$. Finally, let $1 - \varepsilon > a$. Now if there were no integer $m_0(\varepsilon, F)$ satisfying the above condition then there would be a sequence of integers $m_k \to \infty$ $(k \to \infty)$ and a corresponding sequence of linear aggregates

$$P_k(x) = \sum_j b_{kj} x^{n_{kj}}, \tag{3.1}$$

m_k not in $S_k = \{n_{kj}\}_{j=1,2,\ldots}$, $S_k \in F$, such that

$$\max_{a \leq x \leq 1} |x^{m_k} - P_k(x)| < (1 - \varepsilon)^{m_k}. \tag{3.2}$$

Now let $|f|$, $|f|'$, $|f|''$ denote the maximum of $|f|$ on $[a, 1]$, $[0, a]$ and $[0, 1]$ respectively. Let $0 < \theta < 1$. Then it will be possible to choose $k_1 = k_1(\theta)$ so large that

$$\| x^{2m_k} - x^{m_k} P_k(x) \|'' > \theta^{2m_k}, \qquad (k > k_1(\theta)). \tag{3.3}$$

This is a consequence of theorem 3, remark 1. For the set F_1 of all sequences $S_k' = \{n_{kj} + m_k\}_{j=1,2,\ldots}$ is a sub-set of the set F' considered there. Taking $\theta^2 = 1 - \varepsilon$ in (3.3) and comparing with (3.2) one sees that

$$\| x^{2m_k} - x^{m_k} P_k(x) \|' > (1 - \varepsilon)^{m_k}, \qquad (k > k_1).$$

Hence

$$\| x^{m_k} - P_k(x) \|' > \{(1 - \varepsilon)/a\}^{m_k} \qquad (k > k_1).$$

As $(1 - \varepsilon)/a > 1$, $\| P_k \|'$ must increase exponentially:

$$A_k = \| P_k \|'' > c^{m_k}, \qquad (k > k_2) \tag{3.4}$$

where $c > 1$.

Now let r_0 be so large that

$$\sum_{r_0<r} \{a(2-a)\}^r < \tfrac{1}{2},$$

and let n_0 be such that

$$\text{(3.5)} \qquad \text{l.b.}_{\substack{\{a_{ki}\{ \\ \{n_{ki}\{=S_k}} \| x^{n_{kj}} - \sum_{i \neq j} a_{ki}\, x^{n_{ki}} \|'' > (2-a)^{-n_{kj}}$$

for all k and j for which $n_{kj} > n_0$. This is possible by theorem 3, remark 2. By (3.5), if $b_{kj} \neq 0$, $n_{kj} > n_0$,

$$\left\| \frac{P_k(x)}{b_{kj}} \right\|'' = \| x^{n_{kj}} + \ldots \|'' > (2-a)^{-n_{kj}}.$$

Hence

$$|b_{kj}| < A_k (2-a)^{n_{kj}} \qquad (n_{kj} > n_0).$$

It follows that if $p_0 = \max(r_0, n_0)$,

$$\left\| \sum_{p_0 < n_{kj}} b_{kj}\, x^{n_{kj}} \right\|' < A_k \sum_{p_0 < r} \{a(2-a)\}^r < \tfrac{1}{2} A_k.$$

Hence if k_3 is so large that for $k > k_3$ $\| P_k \|'' = \| P_k \|'$, then $k > k_3$ must imply that

$$\sum_{n_{kj} \leq p_0} |b_{kj}| \geqq \left\| \sum_{n_{kj} \leq p_0} b_{kj}\, x^{n_{kj}} \right\|' > \tfrac{1}{2} A_k.$$

But then there exists, to every $k > k_3$, a $j(k)$ such that

$$\text{(3.6)} \qquad n_{k,j(k)} \leqq p_0$$

$$\text{(3.7)} \qquad |b_{k,j(k)}| > \frac{A_k}{2(p_0+1)}.$$

By (3.4) it is then possible to take k_4 so large that $k > k_4$ implies

$$\text{(3.8)} \qquad |b_{k,j(k)}| > 1.$$

Now for $k > k_{1,2,3,4}$ consider the expression

$$\text{(3.9)} \qquad |b_{k,j(k)}|^{-1} |x^{2m_k} - x^{m_k} P_k(x)| = |x^{\lambda_k} + Q_k(x)|.$$

Here $\lambda_k = n_{k,j(k)} + m_k$, and $Q_k(x)$ represents a linear aggregate of x^{2m_k} and $x^{n_{ki}+m_k}$, $i \neq j(k)$. If k is sufficiently large, the expression (3.9) will be less than a^{m_k}, where $0 < a < 1$, everywhere on $[0, 1]$. This follows from (3.2) and (3.8) for $[a, 1]$. For $[0, a]$, by (3.8) and (3.7),

$$|b_{k,j(k)}|^{-1} |x^{2m_k} - x^{m_k} P_k(x)| \leqq a^{2m_k} + 2(p_0+1)\, a^{m_k}.$$

Thus

$$\text{(3.10)} \qquad \| x^{\lambda_k} + Q_k(x) \|'' = \| x^{\lambda_k} - b_{k,j(k)}^{-1} x^{2m_k} + R_k(x) \|'' < a^{m_k}, \qquad (k > k_5).$$

The inequalities (3.10), (3.7) and (3.4) imply

$$\|x^{\lambda_k} + R_k(x)\|'' < \beta^{m_k}, \qquad (k > k_6)$$

where $0 < \beta < 1$. Finally, by (3.6),

$$\|x^{\lambda_k} + R_k(x)\|'' < \gamma^{\lambda_k}, \qquad (k > k_7), \tag{3.11}$$

$0 < \gamma < 1$. Here $R_k(x)$ contains only powers $x^{n_{ki}+m_k}$, $i \neq j(k)$. As the family of sequences $\{n_{ki} + m_k\}_{i \neq j(k)}$ $(k = 1, 2, \ldots)$ satisfies the conditions of theorem 3 the inequality (3.11) must be false. This completes the proof of theorem 4.

Corollary 1. (CLARKSON *and* ERDÖS) *The set*

$$\{x^{n_k}\} \tag{3.12}$$

of $C[a, b]$, *where* $0 \leq n_1 < n_2 < \ldots < n_k < \ldots \to \infty$, *is closed in* $C[a, b]$ $(0 < a < b)$ *if and only if the series*

$$\sum_{n_k > 0} \frac{1}{n_k} \tag{3.13}$$

diverges.

Proof. If the series (3.13) diverges the set (3.12) is closed in $C_0[0, b]$ by the theorem of MÜNTZ and SZÁSZ. Here $C_0[0, b]$ denotes the space of all continuous functions of x, $0 \leq x \leq b$, vanishing at $x = 0$. The set (3.12) is then a fortiori closed in $C[a, b]$, $a > 0$.

If the series (3.13) converges on the other hand, then it follows from theorem 4 by taking $F = S = \{n_k\}$ that it is impossible to approximate uniformly on $[a, b]$ to any function x^m, m not in S, as soon as $m > m_0$.

However, this is true for $m \leq m_0$ also:

Corollary 2. *If* m *does not belong to the increasing sequence of non-negative integers* $\{n_k\}$ *satisfying*

$$\sum_{n_k > 0} \frac{1}{n_k} < \infty$$

then it is impossible to approximate uniformly to x^m *on* $[a, b]$ *by linear aggregates*

$$\sum a_k x^{n_k}.$$

Proof. Take in theorem 4

$$F = \{S_j\}_{j=0,1,2,\ldots}, \qquad S_j = \{n_k + j\}_{k=1,2,\ldots}.$$

It follows that for $m + j > m_0$, m not in $\{n_k\}$,

$$\text{l. b.}_{\{a_k\}} \max_{a \leq x \leq b} |x^{m+j} - x^j \sum a_k x^{n_k}| \geqq (b - \varepsilon)^{m+j}.$$

Hence, taking $j = m_0 + 1$,

$$\text{l. b.}_{\{a_k\}} \max_{a \leq x \leq b} |x^m - \sum a_k x^{n_k}| \geqq (b - \varepsilon)^{m+m_0+1} b^{-m_0-1}$$

whenever m is not an n_k.

The following corollary will also be used in § 4.

Corollary 3. *Let $\{n_k\}$ be an increasing sequence of non-negative integers satisfying the condition*

$$\sum_{n_k>0} \frac{1}{n_k} < \infty .$$

Let $0<\varepsilon<b$. Then there exists an integer $k_0 = k_0(\varepsilon)$ such that $k>k_0$ implies

$$\text{l. b.} \max_{|a_j|} \max_{a\le x\le b} |x^{n_k} - \sum_{j\neq k} a_j x^{n_j}| \geqq (b-\varepsilon)^{n_k}.$$

Proof. See theorem 4, and theorem 3, remark 2.

§ 4. *The principal theorem.* The above corollaries 2, 3 yield the analogue for $[a, b]$ of the result of CLARKSON, ERDÖS and L. SCHWARTZ for $[0, 1]$ mentioned in § 1.

Theorem 5. *Let the increasing sequence of non-negative integers $\{n_k\}$ satisfy the condition*

$$\sum_{n_k>0} \frac{1}{n_k} < \infty .$$

Let the sequence of linear aggregates

$$P_j(x) = \sum_k a_{jk} x^{n_k} \qquad (j = 1, 2, \ldots)$$

converge uniformly to $f(x)$ on $a \le x \le b$, $a \geq 0$, as $j \to \infty$. Then $f(x)$ can be extended to be analytic in the interior of the circle $|x| < b$. The power series for this analytic extension is

$$\sum_{k=1}^{\infty} A_k x^{n_k},$$

where

$$A_k = \lim_{j\to\infty} a_{jk} \qquad (k = 1, 2, \ldots).$$

The proof is essentially the same as that given by CLARKSON and ERDÖS for the interval $[0, 1]$. It follows here for the sake of completeness, however.

(i) a_{jk} tends to a limit as $j \to \infty$. For to any $\varepsilon > 0$ there exists a j_0 such that $j, j' > j_0$ implies

$$\varepsilon > \max_{a\le x\le b} |P_j(x) - P_{j'}(x)| = |a_{jk} - a_{j'k}| \max_{a<x<b} |x^{n_k} - Q(x)|,$$

where $Q(x)$ is a linear aggregate of powers x^{n_i}, $i \neq k$. Now by theorem 4, corollary 2,

$$\max_{a\le x\le b} |x^{n_k} - Q(x)| \geqq c_k > 0.$$

Hence if $j, j' > j_0$,

$$|a_{jk} - a_{j'k}| < \varepsilon c_k^{-1}.$$

Let

$$\lim_{j\to\infty} a_{jk} = A_k \qquad (k = 1, 2, \ldots). \tag{4.1}$$

(ii) The next step is to find a proper estimate for the A_k. Let $\varepsilon > 0$. Let $Q_{jk}(x)$ be defined by the equation

$$P_j(x) = a_{jk}\{x^{n_k} + Q_{jk}(x)\}$$

whenever $a_{jk} \neq 0$. By theorem 4, corollary 3,

$$\max_{a\leq x\leq b} |x^{n_k} + Q_{jk}(x)| \geq (b-\varepsilon)^{n_k}$$

for $k > k_0(\varepsilon)$, $j = 1, 2, \ldots$. The $P_j(x)$ will be uniformly bounded:

$$|P_j(x)| < M \qquad (a \leq x \leq b;\ j = 1, 2, \ldots).$$

Hence for all j

$$M > |a_{jk}|(b-\varepsilon)^{n_k}, \qquad (k > k_0) \tag{4.2}$$

and by (4.1),

$$|A_k| \leq M(b-\varepsilon)^{-n_k}, \qquad (k > k_0). \tag{4.3}$$

The power series $\Sigma A_k x^{n_k}$ will thus at least converge for $|x| < b$. Let the sum of the series be $g(x)$.

(iii) It remains to be proved that $f(x) = g(x)$ on $a \leq x < b$. This will follow from the relation

$$\lim_{j\to\infty} |P_j(x) - g(x)| = 0, \qquad (a \leq x < b).$$

To prove it, let x be fixed on $a \leq x < b$.

$$\begin{aligned} |P_j(x) - g(x)| &= \Big|\sum_{k=1}^{\infty} A_k x^{n_k} - \sum_{k=1}^{\infty} a_{jk} x^{n_k}\Big| \leq \\ &\leq \sum_{1}^{N} |A_k - a_{jk}|\, x^{n_k} + \sum_{N+1}^{\infty} (|A_k| + |a_{jk}|)\, x^{n_k}. \end{aligned} \tag{4.4}$$

Now first choose N so large that

$$|a_{jk}|,\ |A_k| \leq M\left(\frac{b+x}{2}\right)^{-n_k}$$

for $k > N$. (See (4.2) and (4.3).) Next let N increase until the last term of (4.4) is sufficiently small. Finally take j so large that the first term of the third member of (4.4) is also small enough.

A combination of theorems 5 and 2 now yields the principal theorem.

Theorem 6. *The set of continuous functions of x, $a \leq x \leq b$, $a \geq 0$, spanned by the sequence*

$$\{x^{n_k}\} \qquad \left(\sum_{n_k>0} \frac{1}{n_k} < \infty\right)$$

is identical with the set of all power series

$$\sum_{k=1}^{\infty} a_k x^{n_k}$$

convergent on $a \leq x < b$ *for which*

$$\lim_{x \to b} \sum_{k=1}^{\infty} a_k x^{n_k}$$

exists.

Another corollary to theorem 5 is

Theorem 7. *Let the increasing sequence of non-negative integers* $\{n_k\}$ *satisfy the condition*

$$\sum_{n_k > 0} \frac{1}{n_k} < \infty .$$

Let the sequence of linear aggregates

$$P_j(x) = \sum_k a_{jk} x^{n_k} \qquad (j = 1, 2, \ldots)$$

converge uniformly to $f(x)$ *on* $a \leq x \leq b$, $a \geq 0$. *Then the sequence* $\{P_j(x)\}$ *is uniformly convergent in every circle* $|x| \leq b - \delta$, $\delta > 0$. *Its limit is the analytic extension of* $f(x)$.

In particular, let $\{P_j(x)\}$ *converge uniformly to zero on* $a \leq x \leq b$, $a \geq 0$. *Then it will do so in every circle* $|x| \leq b - \delta$, $\delta > 0$.

Proof. See (4.4).

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Mathematics. — *Inequalities for the coefficients of trigonometric polynomials.* II. By R. P. BOAS Jr. (Communicated by Prof. J. G. VAN DER CORPUT.)

(Communicated at the meeting of May 31, 1947.)

1. Let $F(t) = \sum_{-n}^{n} a_j e^{ijt}$ be a real trigonometric polynomial. The inequality

$$|a_0| + \tfrac{2}{3}|a_k| \leqq \tfrac{1}{4}\int_0^{2\pi} |F(t)|\,dt, \qquad k > \tfrac{1}{2}n. \quad . \quad . \quad . \quad (1)$$

was given by VAN DER CORPUT and VISSER [1]). The constant $\frac{1}{4}$ in (1) was improved [2]) to $\frac{1}{2}(1+\frac{1}{3}\sqrt{2})/\pi = .234\ldots$. Here I shall obtain the best possible result

$$|a_0| + \tfrac{2}{3}|a_k| \leqq C\int_0^{2\pi} |F(t)|\,dt, \qquad k > \tfrac{1}{2}n. \quad . \quad . \quad . \quad (2)$$

with

$$C = 1/(2\pi - 4\delta), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$\sin\delta + \tfrac{1}{3}\delta = \tfrac{1}{6}\pi, \quad 0 < \delta < \pi/2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

We have $.2136 < C < .2137$.

More generally, for any positive γ,

$$|a_0| + 2\gamma|a_k| \leqq C_\gamma\int_0^{2\pi} |F(t)|\,dt, \qquad k > \tfrac{1}{2}n. \quad . \quad . \quad . \quad (5)$$

where C_γ is given by (3) and δ is the smallest positive root of $\sin\delta = \frac{1}{2}\gamma(\pi - 2\delta)$; equality occurs in (5) for some $F(t) \not\equiv 0$. For example, $C_1 = .338$; the value given before [2]) was $\frac{1}{2}(1+\sqrt{2})/\pi = .384\ldots$. Thus we have

$$|a_0| + 2|a_k| < 2.126 \cdot \frac{1}{2\pi}\int_0^{2\pi} |F(t)|\,dt, \qquad k > \tfrac{1}{2}n,\ F(t) \not\equiv 0,$$

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[1]) J. G. VAN DER CORPUT and C. VISSER, Inequalities concerning polynomials and trigonometric polynomials, Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, **49**, 383—392 (1946).

[2]) R. P. BOAS Jr., Inequalities for the coefficients of trigonometric polynomials, Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, **50**, 492 (1947).

which it is interesting to compare with the inequality [1])

$$|a_0| + 2|a_k| \leqq \max |F(t)|, \qquad k > \tfrac{1}{2} n.$$

2. To prove (5), let

$$h(x) = 1 + 2\gamma \cos x + \sum_{m=2}^{\infty} c_m \cos m x, \quad . \quad . \quad . \quad . \quad . \quad (6)$$

where the coefficients c_m are to be determined later; if

$$g(x) = \frac{1}{2\pi} \int_0^{2\pi} F(x-t)\, h(k\,t)\, d\,t, \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

then for $k > n/2$

$$|g(x)| = |a_0 + \gamma (a_k e^{ikx} + \bar{a}_k e^{-ikx})| \leqq \max |h(t)| \cdot \frac{1}{2\pi} \int_0^{2\pi} |F(t)|\, dt. \quad . \quad (8)$$

By choosing x appropriately, we have

$$|a_0| + 2\gamma |a_k| \leqq \max |h(t)| \cdot \frac{1}{2\pi} \int_0^{2\pi} |F(t)|\, d\,t \quad . \quad . \quad . \quad . \quad (9)$$

and it remains to choose $h(x)$ to make max $|h(x)|$ as small as possible. We shall take $h(x)$ to be an even step function of period 2π, of the form

$$h(x) = a, \qquad 0 < x < \pi - \delta,$$
$$h(x) = -a, \qquad \pi - \delta < x < \pi,$$

and try to determine δ so that (6) is true. Then we must have

$$\frac{1}{\pi} (\pi - 2\delta)\, a = 1, \qquad \frac{2}{\pi} \cdot 2\, a \sin \delta = 2\gamma,$$

and so $\gamma = \dfrac{2 \sin \delta}{\pi - 2\delta}$. With this choice of δ, max $|h(x)| = a = \pi/(\pi - 2\delta)$. Using this in (9), we have (5).

To prove that (5) is best possible, consider the polynomial

$$F(x) = 1 + \sec \delta \cos x,$$

with $n = 1$. Then by (8) with $x = 0$

$$1 + 2\gamma \sec \delta = \frac{1}{2\pi} \int_0^{2\pi} h(t)\, F(-t)\, d\,t;$$

since $F(x)$ changes sign at $x = \pi/\delta$, $F(x)$ has the sign of $h(x)$ and so,

since $|h(x)|$ is constant,

$$1 + 2\gamma \sec\delta = \max|h(t)| \cdot \frac{1}{2\pi}\int_0^{2\pi} |F(t)|\,dt.$$

This shows that equality in (5) is attained for this $F(x)$.

3. In the same way we can prove

$$|a_k| \leqq \tfrac{1}{8}\int_0^{2\pi} |F(t)|\,dt, \qquad k > \tfrac{1}{3}n \quad . \quad . \quad . \quad . \quad . \quad (10)$$

which was given by FEJES [3]) and VAN DER CORPUT and VISSER [1]), and is best possible. Here we take $h(x)$ even, $h(x) = \pi/4$ in $0 < x < \pi/2$, $h(x) = -\pi/4$ in $\pi/2 < x < \pi$. Then

$$h(x) = \sum_{m=0}^{\infty} \frac{(-1)^m \cos(2m+1)x}{2m+1}$$

and $g(x)$, defined by (7), satisfies, for $k > n/3$,

$$|g(x)| = \tfrac{1}{2}|a_k e^{ikx} + \bar{a}_k e^{-ikx}| \leqq \max|h(t)| \cdot \frac{1}{2\pi}\int_0^{2\pi} |F(t)|\,dt = \tfrac{1}{8}\int_0^{2\pi} |F(t)|\,dt.$$

Choosing x appropriately, we have (10).

4. Inequalities for $k \leqq n/2$ or $k \leqq n/3$ can be obtained by the method used previously [2]) for inequalities involving $\max|F(x)|$. To generalize (5), we take $h(x)$ as in (6) and consider

$$H(x) = \frac{1}{2\pi k}\int_0^{2\pi} h(k(x-t))\,d\alpha(kt),$$

where $\alpha(t)$ is nondecreasing and has FOURIER–STIELTJES coefficients $b_0 = 1$, $b_1 = b_{-1} = \lambda$, $b_j = b_{-j} = 0$, $j = 2, 3, \ldots, p$, $p = [n/k]$, $0 < \lambda < \frac{1}{2}\sec\pi/(p+2)$. Then

$$H(x) = 1 + 2\lambda\gamma\cos kx + \sum_{m=p+1}^{\infty} b_m c_m \cos mkx,$$

$$|H(t)| \leqq \max|h(t)|.$$

Now replace (7) by

$$g(x) = \frac{1}{2\pi}\int_0^{2\pi} F(x-t)\,H(t)\,dt;$$

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[3]) L. FEJES, Two inequalities concerning trigometric polynomials, J. London Math. Soc., **14**, 44—46 (1939).

then (8) is replaced by

$$|a_0 + \lambda\gamma(a_k e^{ikx} + \bar{a}_k e^{-ikx}| \leqq \max|h(t)| \cdot \frac{1}{2\pi}\int_0^{2\pi} |F(t)|\,dt.$$

Hence, letting $\lambda \to \frac{1}{2}\sec \pi/(p+2)$, we obtain

$$|a_0| + |a_k|\gamma \sec\frac{\pi}{[n/k]+2} \leqq C_\gamma \int_0^{2\pi} |F(t)|\,dt$$

with C_γ as in (5).

The corresponding generalization of (10) is

$$|a_k| \leqq \tfrac{1}{4}\cos\frac{\pi}{p+2} \cdot \int_0^{2\pi} |F(t)|\,dt,$$

where $p = 2[\frac{1}{2}(n-k)/k] + 1$ is the largest odd integer with $pk \leqq n$.

Chemistry. — *Elektrochemisch gedrag van ionen-wisselende stoffen. Potentiaalmetingen aan plantenwortels.* III. *Metingen aan Sinapis alba.* By W. H. VAN DER MOLEN and H. J. C. TENDELOO. (Communicated by Prof. J. W. BIJVOET.)

(Communicated at the meeting of April 26, 1947.)

Door de theorie van het Donnan-evenwicht uit te breiden en toe te passen op systemen met niet-diffusabele anionen van een zwak zuur, kon in een vorige mededeling [1]) worden aangetoond, dat het mogelijk was de potentiaal van een planten-wortel in verdunde oplossingen van KCl te beschrijven. Uit metingen van de potentiaal is de verhouding, V, der concentraties van de diffusabele ionen in en buiten de wortel te berekenen; uit de formule:

$$V^3 + \left(K' C_2 + \frac{A}{C_2 + C_1}\right) V^2 - V - K' C_2 = 0 \quad . \quad . \quad . \quad (1)$$

waarin:

$K' = \frac{1}{K}$, K = dissociatie constant
C_2 = waterstofionenconcentratie der buitenoplossing
C_1 = kaliumionenconcentratie der buiten oplossing
A = concentratie van het zwakke zuur in de wortel

volgt:

$$K' C_2 + A \frac{V^2}{(V^2 - 1)(C_2 + C_1)} = -V \quad . \quad . \quad . \quad . \quad . \quad (2)$$

waaruit te zien is, dat er een lineair verband moet zijn tussen

$$\frac{V^2}{(V^2 - 1)(C_2 + C_1)} \text{ en } -V \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Voor C_2 werd tevoren gebruik gemaakt van de pH van de buitenoplossing. Het is echter waarschijnlijk, dat veel meer bepalend is de pH van het milieu in de onmiddellijke omgeving van de wortel, die niet te bepalen is. Daarom werd bij metingen aan wortels van Sinapis alba voor verschillende waarden van C_2 de betrekking (3) grafisch nagegaan. Hierbij bleek, dat de rechte lijn het best benaderd werd met $C_2 = 3{,}2 \cdot 10^{-4}$.

De wortel-potentialen bij verschillende KCl-concentraties zijn vermeld in tabel 1, als gemiddelden van daarin genoemde aantallen metingen. Voorts zijn in deze tabel de berekende waarden van V vermeld.

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[1]) Versl. Ned. Akad. v. Wetensch., Afd. Natuurkunde, Vol. 53, 169 (1944).

TABEL 1.

Conc. KCl C_1	2.10^{-1}	10^{-1}	10^{-2}	10^{-3}	10^{-4}	10^{-5}
Aantal metingen	5	30	30	30	30	30
E gemiddeld	+16.0	+12.47	−7.87	−29.37	−44.27	−51.43
V berekend	—	0.802	0.356	0.151	0.0832	0.0627

Het geringe aantal metingen bij de hoogste concentraties is het gevolg van de onbetrouwbaarheid dezer metingen, vermoedelijk ten gevolge van beschadiging van de wortel door de hoge KCl concentratie.

Door lineaire vereffening van (2) werd berekend:

$$K' = 2{,}36 \cdot 10^3$$

$$A = 0.0653$$

Met deze waarden werden vervolgens de waarden van V berekend met (1), en vervolgens de potentialen volgens:

$$E = 0.577 \log V.$$

In tabel 2 zijn de aldus berekende potentialen verzameld met de bij de verschillende KCl-concentraties gemeten waarden.

TABEL 2.

Conc. KCl C_1	$>10^0$	2.10^{-1}	10^{-1}	10^{-2}	10^{-3}	10^{-4}	10^{-5}
E gev.	—	+16.0	+12.47	− 7.87	−29.37	−44.27	−51.43
E ber.	+18.0	+15.8	+13.7	− 5.7	−32.6	−47.7	−51.0

Samenvatting.

Volgens een reeds eerder gegeven theorie blijkt, dat Sinapis alba gekarakteriseerd wordt door

$$K' = 2{,}36 \cdot 10^3 \text{ en } A = 0.0653.$$

Laboratorium voor Physische- en Kolloidchemie.

Wageningen, Maart–April 1947.

Applied Mechanics. — *On the plastic stability of thin plates and shells.* By P. P. BIJLAARD. (Communicated by Prof. F. A. VENING MEINESZ.)

(Communicated at the meeting of May 31, 1947.)

About nine years ago we published in these Proceedings our theory on the plastic stability of thin plates [1]). Some time ago KOLLBRUNNER communicated very extensive and systematic tests on the same subject [2]). It appeared that these tests confirm our theory completely.

The tests were effected with thin plates of avional, an aluminium alloy, whilst the stress-strain graph of the material was determined by compression of short plates.

We will compare here the tests as given by KOLLBRUNNER in his figures 33, 34 and 35 with the results of our theory. All these tests relate to plates, compressed in longitudinal direction and supported at the unloaded sides.

Fig. 33 refers to plates of which the unloaded sides are simply supported. According to our theory [3]) the buckling force of such plates is, if the entire plate deforms plastically

$$h\sigma_x = (2\pi^2 EI/b^2)(\sqrt{AD} + B + 2F). \quad . \quad . \quad . \quad . \quad (1)$$

The modulus of elasticity E of avional is 715000 kg/cm^2. The thickness h and the breadth b of the plates were 0,2 cm and 6,2 cm respectively, so that $\pi^2 EI/b^2 = \pi^2 Eh^3/12\,b^2 = 122{,}8$ kg/cm. Furthermore, as in this case the second principal stress is zero, so that $\beta = \varrho_2/\varrho_1 = 0$ and $\eta^2 = \beta^2 - \beta + 1 = 1$, we have [4])

$$A = \varphi_1/\varphi_4,\ B = \varphi_2/\varphi_4,\ D = \varphi_3/\varphi_4,\ F = m/(2m + 2 + 3em),$$

in which

$$\left.\begin{aligned} \varphi_1 &= m^2\{E + (4 + 3e)\tan\varphi\} \\ \varphi_2 &= 2m(mE + 2\tan\varphi) \\ \varphi_3 &= 4m^2(E + \tan\varphi) \\ \varphi_4 &= m(5m - 4 + 3em)E + \{4(m^2 - 1) + 3em^2\}\tan\varphi \end{aligned}\right\} \quad (2)$$

in which $m = 1/\nu = 10/3$, value ν being POISSON's ratio, $e = E\varepsilon_p/\sigma$ and $\tan\varphi = d\sigma/d\varepsilon_p$, the latter two values having to be measured from the stress-strain graph with pure compression at a stress σ being equal to the

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1) BIJLAARD, Proc. Kon. Akad. v. Wetensch., Amsterdam, Nrs. 5 and 7 (1938).
2) KOLLBRUNNER, Mitt. a. d. Institut f. Baustatik, Zürich, Nr. 17 (1946).
3) BIJLAARD, loc. cit. eq. (53).
4) BIJLAARD, loc. cit. eqs. (22)—(24).

buckling stress σ_x. The plastic strain ε_p is equal to the total strain ε minus the elastic strain $\varepsilon_e = \sigma/E$, so that

$$\tan \varphi = \frac{d\sigma}{d\varepsilon_p} = \frac{d\sigma}{d\varepsilon - d\varepsilon_e} = \frac{1}{1/E_t - 1/E} = \frac{EE_t}{E - E_t} \quad . \quad . \quad . \quad (3)$$

$E_t = d\sigma/d\varepsilon$ being called the tangent modulus.

With a stress $\sigma_x = 2200$ kg/cm² we find from the stress-strain graph of avional $E_t = 350000$ kg/cm² $= 0{,}49\,E$, $e = 0{,}065$, $\tan \varphi = 0{,}96\,E$, by which eqs. (2) yield $A = 0{,}655$, $B = 0{,}41$, $D = 1{,}02$, $F = 0{,}36$, so that it follows from eq. (1)

$$\sigma_x = 3{,}886\,\pi^2\,EI/b^2h = 2385 \text{ kg/cm}^2$$

being more than the stress 2200 kg/cm² we started from.

Assuming now a stress $\sigma_x = 2300$ kg/cm² we find in the same way $E_t = 300000$ kg/cm² $= 0{,}42\,E$, $e = 0{,}11$, $\tan \varphi = 0{,}725\,E$, $A = 0{,}59$, $B = 0{,}41$, $D = 0{,}985$, $F = 0{,}34$, $\sigma_x = 2276$ kg/cm².

Interpolating linearly between assumed as well as between resulting values σ_x we finally find a buckling stress $\sigma_x = 2288$ kg/cm².

With these tests the eccentricity of the load was certainly such, that with buckling practically no discharge occurred, so that we will have to assume that the plates showed no elastic region. Hence the buckling stress is indeed determined by eq. (1) only, in the same way as also in practice, in connection with small eccentricities, the critical stress with a plate, assumed to be plastic all over, is determinant for the strength of the plate, as we stated already previously [5]).

According to our theory [6]) the plate should buckle in waves with a half wave length $a/p = (A/D)^{\frac{1}{4}}\,b$. With a buckling stress 2288 kg cm² we find $A = 0{,}60$ and $D = 0{,}99$, so that the half wave length is 0,882 b. This wave length will occur if the plate is free in selecting its most favourable wave length, i.e. if it is infinitely long. With finite length and if the loaded edges of the plate are simply supported, the half wave length will have to be an integer part of the total length, so that it will deviate to both sides from the most favourable wave length. With the tests, however, the loaded edges were not simply supported, but somewhat clamped, so that the number of half waves will have been the same as with a plate with simply supported extremities, that may be assumed to be about one third of a half wave length shorter. Hence the half wave length, calculated by dividing the total length a of the plates by the number of half waves p, will give a somewhat too high value for the real wave length, whilst moreover these lengths will deviate to both sides from this too high value. The longer the plates, however, the better the optimum wave length will be approximated. According to fig. 33 of KOLLBRUNNER's publication

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[5]) BIJLAARD, Publications Int. Association for Bridge and Structural Engineering. Zürich, Vol. 6 (1940/1941), p. 54, footnote 10.

[6]) BIJLAARD, lit. footnote 1, p. 739.

we give in the following table I the lengths a of the plates and the calculated half wave lengths a/p, the numbers in brackets indicating the number of tests.

TABLE I.

a	3,2 b	4,85 b	6,45 b	8,1 b
a/p	1,07 b (3)	0,97 b (3)	0,92 b (3)	0,90 b (3)

It is clear that with longer plates the half wave length converges indeed towards value 0.882 b according to our theory. That for example value 1,07 b, with a plate length 3,2 b, is also in accordance with our theory, may be seen as follows. With a free supported plate length of 3 times 0,882 b = 2,646 b we could expect a buckling in 3 half waves of the optimum wave length 0,882 b, yielding the minimum buckling stress. But from eq. (52), given in our paper mentioned in footnote 1, it may be computed easily, that the same number of waves will occur with freely supported plate lengths between $0{,}882\,b\,\sqrt{6} = 2{,}16\,b$ and $0{,}882\,b\,\sqrt{12} = 3{,}06\,b$. According to our statement above here, the freely supported plate length of the plate with $a = 3{,}2\,b$ will be somewhat less than 3,0 b, so that according to our theory it should indeed buckle in 3 waves, yielding the calculated wave length $a/p = 3{,}2\,b/3 = 1{,}07\,b$, in accordance with the tests.

The buckling stresses of the 12 testplates vary between 2040 and 2190 kg/cm², except one, that yields the low value 1950 kg/cm². Hence, disregarding the latter value, the lowest buckling stress 2040 kg/cm² is only 11 % below the theoretical value 2288 kg/cm² according to our theory. As this discrepancy is not more than the percentage the experimental values in the elastic domain remain underneath the theoretical values, owing to unavoidable inaccuracies, and the theoretical values in the elastic domain being undoubtedly right, we may conclude also to the exactness of our buckling stresses for the plastic domain.

We now consider the buckling of plates of which the unloaded sides are fixed, the test results of which are given in KOLLBRUNNER's fig. 34. The plates have a thickness 0,2 cm and a breadth $b = 4{,}4$ cm. According to our theory the buckling condition for these plates in the plastic domain is [7])

$$\alpha_1 \tanh(\alpha_1 b/2) + \alpha_2 \tan(\alpha_2 b/2) = 0 \quad . \quad . \quad . \quad . \quad . \quad (4)$$

in which

$$\left.\begin{aligned} \alpha_{1,2} &= \sqrt{\pm G\lambda^2 + \lambda\sqrt{H\lambda^2 + K\varphi^2}} \\ G &= \frac{B+2F}{D},\ H = \frac{(B+2F)^2 - AD}{D^2},\ K = 1/D \\ \varphi^2 &= \frac{h\sigma_x}{EI},\ \lambda = \pi p/a \end{aligned}\right\} \quad . \quad . \quad . \quad (5)$$

[7]) BIJLAARD, lit. footnote 5, p. 57.

After assuming σ_x to be 3100 and 3150 kg/cm², it appears that its real value is about 3140 kg/cm². With $\sigma_x = 3150$ kg/cm² we found, in the same way as before: $E_t = 0{,}04\,E$, $e = 1{,}49$, $\tan\varphi = E/24$, $A = 0{,}157$, $B = 0{,}238$, $D = 0{,}485$, $F = 0{,}141$, and by using eqs. (5): $G = 1{,}07$, $H = 0{,}825$, $K = 2{,}06$, $\varphi^2 = 1{,}32$. Assuming now $\lambda = 1{,}55$ we obtain $\alpha_1 = 2{,}435$, $\alpha_2 = 0{,}89$, so that eq. (4) yields

$$\alpha_1 \tanh(\alpha_1 b/2) + \alpha_2 \tan(\alpha_2 b/2) = 0{,}27$$

instead of zero.

Assuming now $\sigma_x = 3140$ kg/cm² we find $G = 1{,}072$, $H = 0{,}827$, $K = 2{,}01$, $\varphi^2 = 1{,}315$, yielding, with $\lambda = 1{,}55$, values $\alpha_1 = 2{,}43$ and $\alpha_2 = 0{,}872$, by which eq. (4) yields

$$\alpha_1 \tanh(\alpha_1 b/2) + \alpha_2 \tan(\alpha_2 b/2) = 0{,}02$$

so that σ_x is round **3140** kg/cm² with $\lambda = \pi p/a = 1{,}55$ and a half wave length $a/p = \pi/1{,}55 = 2{,}02$ cm $=$ **0,46 b**.

Calculating in the same way the buckling stresses with other values λ it appeared that with $\lambda = 1{,}55$ the critical stress is about a minimum. In the following table the plate lengths a and the half wave lengths $a\ p$ according to the tests are given.

TABLE II.

a	4,55 b	6,80 b	9,10 b	11,40 b
a/p	0,495 b (3)	0,52 b (2) 0,485 b (1)	0,535 b (1) 0,505 b (2)	0,475 b (3)

It is clear that with longer plates the half wave length converges indeed to our theoretical value 0,46 b. The latter plate, of which the free supported length may be assumed as 11,20 b, buckled in 24 half waves, whilst with the theoretical buckling length 0,46 b this number of waves should occur between freely supported plate lengths of about 23,5 and 24,5 times 0,46 b, or 10,81 b and 11,27 b, so that the 24 half waves are indeed in accordance with our theory.

One extraordinary low value, being 2580 kg/cm², excepted, the buckling stresses for the other 11 tests were between **2830** and **3115** kg/cm², the lowest value 2830 kg/cm² being only 10 % underneath our theoretical value.

Finally considering fig. 35, referring to plates that are simply supported at one unloaded side and fixed at the other, with $h = 0.2$ cm and $b = 5.3$ cm, the buckling condition is given by [7])

$$\alpha_1 \coth \alpha_1 b - \alpha_2 \cot \alpha_2 b = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

whilst α_1 and α_2 follow from eqs. (5).

In the same way as before we find here that the buckling stress σ_x acquires a minimum value with about $\lambda = 0{,}865$, hence with a half wave length $a/p = \pi/\lambda = 3{,}63$ cm $=$ **0,685 b**, whilst $\sigma_x =$ **2882** kg/cm².

The following table gives the experimental values a/p, converging again in an excellent way to our theoretical value 0,685 b. The buckling stresses of all thirteen tests are between **2660** and **2850** kg/cm², the lowest value being only 8 % underneath our theoretical value.

TABLE III.

a	3,8 b	5,65 b	7,55 b	9,45 b
a/p	0,635 b (3) 0,760 b (1)	0,705 b (3)	0,755 b (1) 0,685 b (2)	0,727 b (1) 0,675 b (2)

Hence we may conclude, that KOLLBRUNNER's tests have proved the applicability of our theory, as the number of waves shows, that the anisotropic behaviour of the material is exactly such as predicted by our theory, whilst the discrepancies of the buckling stresses are not more than in the elastic domain.

Under these circumstances it may be of interest to give here a short indication of the application of our theory to the buckling of shells.

Our fundamental equations are those giving the relation between the excess stresses and the excess strains with buckling, being [8])

$$\left.\begin{aligned} \sigma'_x &= E(A\,\varepsilon'_x + B\,\varepsilon'_y) \\ \sigma'_y &= E(B\,\varepsilon'_x + D\,\varepsilon'_y) \\ \tau'_{xy} &= EF\gamma'_{xy} \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (7)$$

values A, B, D and F being given by eqs. (22) and (23) of our publication mentioned in footnote 1.

Using the same notations as TIMOSHENKO [9]), except our primes, indicating infinitely small stresses and strains occurring with buckling, we have

$$\left.\begin{aligned} \varepsilon'_x &= \varepsilon'_1 - \chi'_x\, z \\ \varepsilon'_y &= \varepsilon'_2 - \chi'_y\, z \\ \gamma'_{xy} &= \gamma' - 2\chi'_{xy}\, z \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (8)$$

in which ε'_1, ε'_2 and γ' are the excess strains of the middle surface in X- and Y-directions, χ'_x and χ'_y are the changes of curvature and χ'_{xy} is the twist, whilst z is the distance from the middle surface. Substituting eqs. (8) in eqs. (7) we get

$$\left.\begin{aligned} \sigma'_x &= E\{A\,\varepsilon'_1 + B\,\varepsilon'_2 - z(A\,\chi'_x + B\,\chi'_y)\} \\ \sigma'_y &= E\{B\,\varepsilon'_1 + D\,\varepsilon'_2 - z(B\,\chi'_x + D\,\chi'_y)\} \\ \tau'_{xy} &= EF(\gamma' - 2\,\chi'_{xy}\, z) \end{aligned}\right\} \quad \ldots\ldots \quad (9)$$

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[8]) BIJLAARD, lit. footnote 1, eqs. (21)—(24).

[9]) TIMOSHENKO, Theory of elastic stability. Chapters VIII and IX

Hence we find [10])

$$\left.\begin{aligned} N'_x &= \int_{-h/2}^{+h/2} \sigma'_x \, dz = Eh\,(A\,\varepsilon'_1 + B\,\varepsilon'_2) \\ N'_y &= \int_{-h/2}^{+h/2} \sigma'_y \, dz = Eh\,(B\,\varepsilon'_1 + D\,\varepsilon'_2) \\ N'_{xy} &= N'_{yx} = \int_{-h/2}^{+h/2} \tau'_{xy} \, dz = EFh\,\gamma' \\ M'_x &= \int_{-h/2}^{+h/2} \sigma'_x \, z\,dz = -EI\,(A\,\chi'_x + B\,\chi'_y) \\ M'_y &= \int_{-h/2}^{+h/2} \sigma'_y \, z\,dz = -EI\,(B\,\chi'_x + D\,\chi'_y) \\ M'_{xy} &= -M'_{yx} = -\int_{-h/2}^{+h/2} \tau'_{xy} \, z\,dz = 2EIF\,\chi'_{xy} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (10)$$

As an example we will consider the buckling of a cylindrical shell under the action of uniform axial pressure $h\sigma_x$ per unit breadth. If buckling symmetrical to the axis of the cylinder occurs, the equilibrium of an element hdx of a strip OP of unit width (fig. 1) requires, if compressive stresses

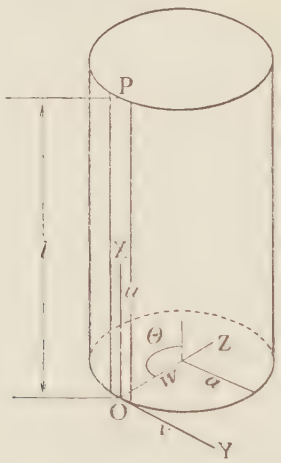


Fig. 1.

are denoted as positive and denoting displacements with buckling in Z-direction by w

$$\frac{dQ'_x}{dx} - h\sigma_x \frac{d^2 w}{dx^2} - \frac{N'_y}{a} = 0. \quad . \quad . \quad . \quad . \quad . \quad (11)$$

Value $Q'_x = dM'_x/dx$. Owing to impediment of distortion of the cross section, value χ'_y may be equated to zero, so that eqs. (10) yield

$$M'_x = -EIA\,\chi'_x = -EIA\,d^2 w/dx^2$$

[10]) Cf. same values in elastic domain in TIMOSHENKO, l.c. p. 421, 422.

and, as $\varepsilon_2' = -w/a$

$$N_x' = E h (A \varepsilon_1' - B w/a).$$

As, however, $h\sigma_x$ does not increase with buckling, N_x' must be zero, yielding $\varepsilon_1' = (B/A)(w/a)$, by which we obtain

$$N_y' = E h (B^2/A - D) w/a.$$

Substitution of these values in eq. (11) yields, as in (11) N_y' is a compression, the differential equation

$$E I A \frac{d^4 w}{dx^4} + h \sigma_x \frac{d^2 w}{dx^2} - \left(\frac{B^2}{A} - D\right) E h w/a^2 = 0. \quad . \quad . \quad (12)$$

With $w = w_0 \sin p \pi x/l$ eq. (12) yields, after ranging

$$h \sigma_x = E I A p^2 \pi^2/l^2 + (D - B^2/A)(E h/a^2) l^2/p^2 \pi^2. \quad . \quad . \quad (13)$$

With sufficiently long cylinders the wave length can establish itself in such a way as to make $h\sigma_x$ a minimum. Differentiation shows that then

$$\frac{p \pi}{l} = \sqrt[4]{\frac{A D - B^2}{A^2} \frac{h}{a^2 I}}$$

insertion of which in eq. (13) yields the critical stress

$$\sigma_x = \frac{E h}{a} \sqrt{\frac{A D - B^2}{3}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

whilst the length of the half waves in X-direction is

$$\frac{l}{p} = \pi \sqrt[4]{\frac{A^2}{12 (A D - B^2)}} \sqrt{a h}. \quad . \quad . \quad . \quad . \quad . \quad (15)$$

In the elastic domain $e = 0$ and $\tan \varphi = \infty$, so that eqs. (2), which also apply to this case, yield $A = D = m^2/(m^2-1)$ and $B = m/(m^2-1)$, by which eqs. (14) and (15) transform in

$$\sigma_x = \frac{E h}{a} \sqrt{\frac{m^2}{3 (m^2-1)}} \quad \text{and} \quad \frac{l}{p} = \pi \sqrt[4]{\frac{m^2 a^2 h^2}{12 (m^2-1)}}$$

in accordance with the values obtained directly for this case [11]).

Considering the more general case of buckling of a cylindrical shell under axial compression and denoting the displacements with buckling in X-, Y- and Z-direction by u, v and w respectively, our equations (10) yield, after

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[11]) TIMOSHENKO, loc. cit. p. 440—441.

expression of values $\varepsilon_1', \varepsilon_2', \gamma', \chi_x', \chi_y'$ and χ_{xy}' in terms of the displacements [12]),

$$\left.\begin{aligned}
N_x' &= Eh\left\{A\frac{\partial u}{\partial x}+B\left(\frac{\partial v}{a\,\partial\theta}-\frac{w}{a}\right)\right\}\\
N_y' &= Eh\left\{B\frac{\partial u}{\partial x}+D\left(\frac{\partial v}{a\,\partial\theta}-\frac{w}{a}\right)\right\}\\
N_{xy}' &= N_{yx}' = EFh\left(\frac{\partial u}{a\,\partial\theta}+\frac{\partial v}{\partial x}\right)\\
M_x' &= -EI\left\{A\frac{\partial^2 w}{\partial x^2}+\frac{B}{a^2}\left(\frac{\partial v}{\partial\theta}+\frac{\partial^2 w}{\partial\theta^2}\right)\right\}\\
M_y' &= -EI\left\{B\frac{\partial^2 w}{\partial x^2}+\frac{D}{a^2}\left(\frac{\partial v}{\partial\theta}+\frac{\partial^2 w}{\partial\theta^2}\right)\right\}\\
M_{xy}' &= -M_{yx}' = 2EFI\frac{1}{a}\left(\frac{\partial v}{\partial x}+\frac{\partial^2 w}{\partial x\,\partial\theta}\right)
\end{aligned}\right\} \quad \ldots \quad (16)$$

value θ being indicated in fig. 1.

The conditions of equilibrium are in our notations and neglecting second order terms [13])

$$\left.\begin{aligned}
&a\frac{\partial N_x'}{\partial x}+\frac{\partial N_{yx}'}{\partial\theta}=0\\
&\frac{\partial N_y'}{\partial\theta}+a\frac{\partial N_{xy}'}{\partial x}-ah\sigma_x\frac{\partial^2 v}{\partial x^2}+\frac{\partial M_{xy}'}{\partial x}-\frac{\partial M_y'}{a\,\partial\theta}=0\\
&-ah\sigma_x\frac{\partial^2 w}{\partial x^2}+N_y'+a\frac{\partial^2 M_x'}{\partial x^2}+\frac{\partial^2 M_{yx}'}{\partial x\,\partial\theta}+\frac{\partial^2 M_y'}{a\,\partial\theta^2}-\frac{\partial^2 M_{xy}'}{\partial x\,\partial\theta}=0
\end{aligned}\right\} \quad \ldots \quad (17)$$

Substitution of eqs. (16) in eqs. (17) yields

$$\left.\begin{aligned}
&A\frac{\partial^2 u}{\partial x^2}+\frac{B+F}{a}\frac{\partial^2 v}{\partial x\,\partial\theta}-\frac{B}{a}\frac{\partial w}{\partial x}+\frac{F}{a^2}\frac{\partial^2 u}{\partial\theta^2}=0\\
&(B+F)\frac{\partial^2 u}{\partial x\,\partial\theta}+aF\frac{\partial^2 v}{\partial x^2}+\frac{D}{a}\left(\frac{\partial^2 v}{\partial\theta^2}-\frac{\partial w}{\partial\theta}\right)+\\
&\alpha\left[\frac{D}{a}\left(\frac{\partial^2 v}{\partial\theta^2}+\frac{\partial^3 w}{\partial\theta^3}\right)+a(B+2F)\frac{\partial^3 w}{\partial x^2\partial\theta}+2aF\frac{\partial^2 v}{\partial x^2}\right]-\frac{ah\sigma_x}{Eh}\frac{\partial^2 v}{\partial x^2}=0\\
&-\frac{ah\sigma_x}{Eh}\frac{\partial^2 w}{\partial x^2}+B\frac{\partial u}{\partial x}+\frac{D}{a}\frac{\partial v}{\partial\theta}-\frac{D}{a}w-\alpha\left[a(B+4F)\frac{\partial^3 v}{\partial x^2\partial\theta}+\frac{D}{a}\frac{\partial^3 v}{\partial\theta^3}+\right.\\
&\left.a^3A\frac{\partial^4 w}{\partial x^4}+2a(B+2F)\frac{\partial^4 w}{\partial x^2\partial\theta^2}+\frac{D}{a}\frac{\partial^4 w}{\partial\theta^4}\right]=0
\end{aligned}\right\} \quad (18)$$

in which $\alpha = h^2/12\,a^2$.

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[12]) TIMOSHENKO, loc. cit. p. 434.

[13]) Cf. TIMOSHENKO, loc. cit. eqs (c), p. 454.

Assuming

$$\left.\begin{aligned} u &= u_0 \cos n\theta \cos p\pi x/l \\ v &= v_0 \sin n\theta \sin p\pi x/l \\ w &= w_0 \cos n\theta \sin p\pi x/l \end{aligned}\right\} \quad \ldots\ldots \quad (19)$$

eqs. (18) yield

$$\left.\begin{aligned} &(A\lambda^2 + Fn^2)\,u_0 - (B+F)\,n\lambda\,v_0 + B\lambda\,w_0 = 0 \\ &-(B+F)\,n\lambda\,u_0 + [Dn^2(1+\alpha) + \\ &\qquad F\lambda^2(1+2\alpha) - \sigma_x\lambda^2/E]\,v_0 - n\,[D(1+\alpha n^2) + (B+2F)\,\alpha\lambda^2]\,w_0 = 0 \\ &B\lambda\,u_0 - n\,[D(1+\alpha n^2) + (B+4F)\,\alpha\lambda^2]\,v_0 + \\ &\qquad [A\alpha\lambda^4 + 2(B+2F)\,\alpha n^2\lambda^2 + D(1+\alpha n^4) - \sigma_x\lambda^2/E]\,w_0 = 0 \end{aligned}\right\} \quad (20)$$

in which $\lambda = p\pi a/l$.

Hence the critical stress σ_x follows by equating the determinant of these equations to zero. Further computations may be effected along the same lines as in the elastic domain.

In order to check eqs. (20) we assume again buckling symmetrical to the axis of the cylinder, so that in eqs. (19) we have to equal n to zero, so that also v becomes zero. Hence the second equation (20) vanishes and eqs. (20) transform in

$$\left.\begin{aligned} &A\,\lambda\,u_0 + B\,w_0 = 0 \\ &B\,\lambda\,u_0 + (A\,\alpha\,\lambda^4 + D - \sigma_x\lambda^2/E)\,w_0 = 0 \end{aligned}\right\} \quad \ldots\ldots \quad (21)$$

yielding $$\sigma_x = E\,[A\alpha\lambda^2 + (AD - B^2)/A\lambda^2]$$

or $$h\sigma_x = EIAp^2\pi^2/l^2 + (D - B^2/A)\,(Eh/a^2)\,l^2/p^2\pi^2$$

in accordance with eq. (13).

As a matter of fact thick tubes, buckling in the plastic domain, do this usually in a symmetrical way, whilst with thin tubes buckling which is non-symmetrical with respect to the axis usually occurs [14]). This behaviour is in good accordance with our theory, because non-symmetrical buckling causes twisting stresses, against which, if e is small, as with steel, the resistance is only slightly diminished according to our theory. We can proof this directly with our eqs. (20).

For the elastic domain TIMOSHENKO proofs, that, if λ^2 is a large number, the critical stress with non-symmetrical buckling is equal to that with symmetrical deformation [15]). Taking into account the same terms as he

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[14]) TIMOSHENKO, loc. cit. p. 443.

[15]) TIMOSHENKO, loc. cit. p. 456.

does, we find for the plastic domain, by equating the determinant of eqs. (20) to zero, the following equation

$$[(A\lambda^2+Fn^2)(F\lambda^2+Dn^2)-(B+F)^2n^2\lambda^2]\lambda^2\sigma_x/E=$$
$$(AD-B^2)F\lambda^4+$$
$$\alpha[A\lambda^4+2(B+2F)n^2\lambda^2+Dn^4][(A\lambda^2+Fn^2)(F\lambda^2+Dn^2)-(B+F)^2n^2\lambda^2]$$

or, after some transformation

$$\frac{\sigma_x}{E}=\frac{(AD-B^2)\lambda^2}{A\lambda^4+Dn^4+[(AD-B^2)/F-2B]n^2\lambda^2}+\frac{\alpha[A\lambda^4+Dn^4+2(B+2F)n^2\lambda^2]}{\lambda^2}. \quad (22)$$

In the elastic domain we have[16] $A = D = m^2/(m^2-1)$, $B = m/(m^2-1)$, $F = m/2(m+1)$, so that $(AD-B^2)/F-2B = 2(B+2F) = 2m^2/(m^2-1)$, by which the denominator of the first fraction of the second member of eq. (22) is equal to the term in brackets of the numerator of the second fraction. If in the plastic domain this would be so too, we could write eq. (22) as follows

$$\sigma_x/E=(AD-B^2)\psi+\alpha/\psi \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

value ψ being a function of values λ and n, that determine the number of waves in axial and circumferential direction. In order to make σ_x a minimum, we then would have the condition

$$\psi=\sqrt{\frac{\alpha}{AD-B^2}}$$

by which eq. (23) would yield

$$\sigma_x=2E\sqrt{\alpha(AD-B^2)} \quad \text{or} \quad \sigma_x=(Eh/a)\sqrt{(AD-B^2)/3}$$

in accordance with the buckling stress given by eq. (14) for symmetrical buckling. In the plastic domain, however, considering for example buckling at the yield stress with mild steel, value $(AD-B^2)/F-2B$ will be much less than value $2(B+2F)$. In this case we have[17] $A = 0{,}421$, $B = 0{,}426$, $D = 0{,}938$, $F = 0{,}322$, by which $(AD-B^2)/F-2B = -0{,}19$, whilst $2(B+2F) = 2{,}14$. Hence the first fraction of the second member of eq. (22) has a much higher value than with our assumption that gave eq. (23). Hence we may conclude that with higher values λ^2 in the plastic domain the critical stress with non-symmetrical buckling is higher than with symmetrical buckling, which may explain why short and thick tubes usually buckle symmetrically. In another paper we will consider these questions more in detail.

As follows also from the good agreement, obtained in this way with the tests of KOLLBRUNNER, with a given stress-strain graph of the material,

[16] BIJLAARD, lit. footnote 1, p. 739.
[17] BIJLAARD, lit. footnote 1, p. 736.

the buckling stresses of thin plates will have to be computed under the assumption that the entire plate deforms plastically. If, however, not the stress-strain graph is given, but for example the relation between slenderness ratio and buckling stress of columns, it evidently makes little difference in the resulting buckling stresses whether both columns and plates are assumed to show elastic regions or not, the latter way being, however, the most simple one. Although with shells conditions are in several respects different from those with plates, we think that here too, with a given stress-strain graph, the most logical way is to assume that no elastic regions occur with buckling.

Botany. — *Heterocaryosis and the symbiosis-concept*. By H. G. DERX. (Communicated by Prof. A. J. KLUYVER.)

(Communicated at the meeting of June 28, 1947.)

The work of HANSEN and SMITH (6) on *Botrytis cinerea* has drawn the attention to a phenomenon which proved to be of very general occurrence. It showed that pure cultures of this mould could be separated, by continuous selection, into three distinct types: two "constant" types *a* and *b* (subsequently called *C* (conidial) and *M* (mycelial) and an "inconstant" type *x* (or *MC*), in general intermediate between *M* and *C* in its production of conidia and aerial mycelium.

According to HANSEN and SMITH this "dual phenomenon" is due to the presence of two genetically distinct types of nuclei in the original culture. The conidia of *Botrytis cinerea* are polynuclear. By chance, however, some of them will only contain one of the genetical types of nuclei; these will give rise to the constant *C* type, whereas the conidia containing exclusively the other nuclear type will grow out to the constant *M* type. The inconstant *MC* type originates from conidia containing both types of nuclei. HANSEN (5) later demonstrated the same phenomenon in a series of very different genera of Fungi Imperfecti and thus showed the universality of the phenomenon known as *heterocaryosis*: the presence in one cell or in one mycelium of genotypically distinct nuclei; nuclei with a different genome, and consequently provided with unequal potencies.

The number of nuclear types in a mycelium need not be restricted to two; therefore, the denomination "dual phenomenon" might lead to confusion. For example, LINDEGREN (9) described the changes having occurred in a stock-culture of a (—) strain of *Neurospora crassa* which — though grown from one (mononuclear) ascospore and therefore undoubtedly homocaryotic originally — contained three distinct types of nuclei when examined genetically some years later. Obviously some nuclei of the original "normal" (—) strain had mutated in the course of time in two ways and this mutation caused the appearance of two new phaenotypes which were called "peach" and "clump". In other words the homogenous set of nuclei had become heterogenous, i.e. the culture heterocaryotic.

The variability of moulds in pure culture is a well known phenomenon. Their pleiotropy and their tendency to "degenerate", often irreversibly, on culture-media is a nuisance to mycologists (WESTERDIJK (13)). It is true that in the multinuclear haploïd fungus-mycelium, mutation in a single nucleus will not show itself instantly; a large excess of normal nuclei being still present beside the mutated one. Initially these will dominate.

But the unnatural culture-medium as well as other unnatural circumstances of development (humidity, temperature, way of transplanting, etc.), constantly exerting their selective action, ultimately decide whether the original or the mutated nuclei will predominate and henceforth determine the phaenotype.

It is usual, when isolating a mould from nature in view of its pure culture, to transplant conidia from a colony grown from a single conidium. If such a conidium is mononuclear, as in the case of Penicillia, the strain will be homocaryotic (at least for a short time after its isolation). If the conidia are multinuclear, as in the case of *Neurospora* or *Botrytis* sp., the isolation of a single-conidium strain does not give any certainty about the homogeneity of the nuclear material of the pure culture.

A strain of *Neurospora sitophila* freshly introduced from Java [1]), where this ubiquitous mould constantly occurs on an indigenous relish named "ontjom beureum", equally showed distinct heterocaryosis: single conidium isolations provided two phaenotypically different strains. These, however, were still heterocaryotic, for a crossing of one of them with an albino (+) strain of this mould furnished ascospores, from which several phaenotypically different (—) strains were obtained [2]).

Beside HANSEN, LINDEGREN (8; 10) has shown that in nature the occurrence of heterocaryonts is a rule rather than an exception. This was demonstrated convincingly by his experiments with five strains of the heterothallic Ascomycete *Neurospora crassa* which were collected directly from nature. Three of these five strains were shown to contain (+) as well as (—) nuclei! Nevertheless they did not form fertile perithecia, presumably as a result of the action of modifying- or "sterility"-factors.

Such self-sterility factors, preventing the formation of zygotes, will easily be preserved in fungi when their presence shows no disadvantage from a selective point of view. This is especially the case with fungi possessing another, more efficient, mode of reproduction beside the sexual one; i.e. conidia. In fact, in most Mucoraceae and in many lower Ascomycetes the asexual conidia are often formed much easier, in greater abundance and, above all, much earlier than the sexual fruitbodies. Sterility factors as well as other mutational changes will easily be preserved in multinuclear mycelia because neither of them will be of sufficient disadvantage to be eliminated from the coencytium by selection. On the contrary, the preservation of several genotypically different nuclei in one coencytium may even prove to constitute a direct advantage, when by chance a favourable combination of some mutated nuclei gives rise to a haploïd mycelium of exceptional vigour, which eventually may outgrow the original coencytium (heterosis!).

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[1]) I have to thank my friend Dr. S. POSTMUS, Director of the Eykman Institute, Batavia, for the kindness of sending me a bit of Ontjom.

[2]) Ir. A. KNETEMAN, bacteriologist of the Unilever Works at Rotterdam was so kind to carry out the isolations and the crossings for me.

In this connection it should be mentioned that, apart from this mutation of nuclei by chance in a mycelium or in a conidium, the heterocaryotic condition may be created in another way, i.e. synthetically (DODGE (4)).

When two homocaryotic strains are grown together, hyphal anastomoses between them may occur, followed by migration of nuclei through the perforated mycelial septa. These synthetic heterocaryonts, combining the biochemical potencies of the original strains, may prove to be an association more viable and less dependent on external circumstances and will now be considered from the standpoint of symbiosis.

According to the classical definition (DE BARY (2)) symbiosis signifies the obligatory occurrence together of different kinds of organisms, the relations between the partners being either antagonistic (parasitic) or mutualistic. Usually, however, the term symbiosis is used in a narrower sense and has been confined to the mutualistic relations only of two (or more) *different* kinds of organisms.

Examples of mutualistic symbiosis in nature are invariably examples of mutual dependence of the symbionts. The organisms *must* constantly occur together, because their interrelations have the character of mutual dependencies. The original idea that the interdependence only consisted of the mutual exchange of nutritious matter or of mineral elements (as was formerly thought to be the case with Lichens) has been abandoned. It is now realized that a restriction to such exchanges is not justified and that symbiosis is more frequently due to exchange of auxiliary substances: growth hormones, auximones, nutrilites, or in general *ergones*, which one of the symbionts is able to synthesize, while the other is not. This is undoubtedly the case in symbioses between micro-organisms growing together in a common medium, rich in nutritious matter (Kefir, Kombucha, etc.).

Classical examples are formed by the so-called synthetic symbioses: KÖGL and FRIES (7) showed that *Nematospora Gossypii* and *Polyporus adustus* can grow together on a medium free from biotine and aneurine, although *N. Gossypii* wants biotine and *P. adustus* aneurine (or at least pyrimidine) for its growth. The possibility of the growing together of both fungi on a biotine- and aneurine-free medium is brought about by the circumstance that biotine is secreted into the medium by *P. adustus*, while aneurine is furnished by *N. Gossypii*. The synthetical symbiosis between *Mucor Ramannianus* producing the pyrimidine component of the aneurine-molecule and *Rhodotorula rubra* producing the thiazole component of the said molecule is physiologically speaking still stricter (MÜLLER and SCHOPFER (12), MÜLLER (11)). Both fungi are capable of building up aneurine from its components, but *M. Ramannianus* is unable to synthesize the thiazole-half and *Rh. rubra* unable to synthesize the pyrimidine-half of the aneurine molecule.

In the cases mentioned of natural and synthetic symbiosis, i.e. in mutualistic ecto-symbiosis, the establishment of the symbiosis is due to

secretion of ergones *into a common external medium*; in endo-symbiosis between micro-organisms and higher plants or animals *the internal medium of the higher organism forms the external medium of the micro-organism*. The third possibility: production of ergones *in a common internal medium* is realized in heterocaryotic organisms.

Natural science has not yet fully outgrown the anecdotic-descriptive phase: thence the emphasis on the phenomenal character of symbiosis and the restriction of the concept to organisms of a different kind seem justifiable. However, as has been demonstrated, the essence of symbiosis lies in the interdependence of the symbionts rather then in their dissimilarity. The mutual dependence of symbionts is the consequence of the dissimilarity of their biochemical potencies. In symbiotic organisms these potencies are complementary, thus permitting the partners to subsist together, but not separately.

Similar considerations have induced BAAS BECKING (1) to distinguish, beside the classical hetero-symbiosis, the symbiosis between equal but unequivalent organisms. These were called *homoio-symbioses* or *auto-symbioses*.

When two Neurospora mutants, one of which is unable to synthesize pantothenic acid (= "pantothenic-less", BEADLE), the other unable to synthesize tryptophane (= "tryptophane-less", BEADLE), are brought together, not only will they grow together beside each other, but they will actually unite by means of hyphal anastomoses and migrations of nuclei through the perforated mycelial septa to form a heterocaryont. Not depending upon the presence in the medium of either pantothenic acid or tryptophane, such heterocaryonts do not differ in aspect or in physiological capacities from a normal homocaryotic strain (BEADLE and COONRADT (3)). Both types of nuclei grow, divide and multiply beside each other; the non-mutated gene of one type provides in the want of pantothenic acid, the non-damaged gene of the second type in the want of tryptophane, of the common cytoplasm. But in contrast to the normal homocaryotic strain it will be possible, theoretically at least, to separate the heterocaryont into its component pantothenic-less resp. tryptophane-less homocaryonts.

BEADLE and his collaborators (3) have proved experimentally that not only mutants which need *different* ergones, but equally two strains, for example nicotinic acid-less ones, may form heterocaryonts which grow in a normal way on a medium free from nicotinic acid. This is the case when the mutants are genotypically different; i.e. in each strain a different step in the synthesis of the same ergon has been blocked by mutation. Normal growth of the heterocaryont in such a case proves the non-allelism of the mutated genes.

The preceding exposition clearly justifies the extension proposed by BAAS BECKING. Heterocaryosis forms a new class of homoiosymbiosis. Indeed, any scruple to admit the symbiotic character of heterocaryosis

between two different "loss-mutants" of the same organism should disappear, when it is realized that the components of a classical hetero-symbiosis may equally be regarded as loss mutants. They too have once lost by mutation the power of synthesizing definite ergones, and are since dependent for their subsistence on another organism or on a medium which provides the missing ergones. This medium may be either a mutual external one or a one-sided internal one, or it may be a common cytoplasm; the latter is the case in heterocaryotic fungi.

In other words, since the genotypical difference between the nuclei co-operating in symbiosis determines their interdependence, it should be this genotypical difference — and not any different phaenological expression — which delimits the symbiosis concept.

BEADLE and COONRADT (3) in their recent work propose a very interesting extension of the symbiosis-concept by suggesting that the heterocaryotic condition of haploïd organisms, which these authors denominate: "intracellular, internuclear symbiosis", might actually lead to fusion of the genotypically different nuclei, and thus induce the diploïd state or "intracellular, intranuclear symbiosis". By such a fusion, which must be followed by meiosis, the efficiency of the complementary gene actions would be still further increased. Heterocaryosis would thus lie at the base of sexuality. This form of homoio-symbiosis has indeed also been considered by BAAS BECKING, who has given it the name of *gamosymbiosis.*

Apart from these, still another and last extension of the symbiosis idea has been proposed by BAAS BECKING in the section of mutualistic homoio-symbioses. These are the so-called "histosymbioses" and comprise the collaboration of the different tissues, which together constitute the living organism. Ontogenesis shows that, notwithstanding genotypical identity of the nuclei, differentiation occurs in the course of the development of the organism from embryo to adult. This differentiation occurs under the influence of chemical entities (H. SPEMANN's organisers or evocators) secreted into the protoplasm in due time by different groups of cells, the result being a complex of tissues and organs, each limited in its potencies and collaborating in "hormonal equilibrium". However, the nuclei in these different tissues, salivary glands as well as liver cells, are genotypically alike and this identity appears for example in the behaviour of tissue cultures. In the course of time these commonly de-differentiate and revert to ordinary connective tissue. Clearly then, differentiation of tissues, their diversity and thence their interdependence is *medium-induced*. Consequently histo-symbiosis forms a class fundamentally distinct from homoio-symbiosis as delimited before. It should therefore be treated accordingly, if one feels inclined to consider the collaboration of tissues in a single organism as a case of symbiosis at all. If so, the "collaboration" of genes in a nucleus, of enzymes in a single cell, each effecting a single but indispensable step in the biochemical synthesis of any physiological

product, equally constitutes a symbiosis; of a different order perhaps, but quite certainly belonging to the same class as the histo-symbioses. It seems therefore questionable, if we may call this "symbiosis" without extending the symbiosis idea "ad infinitum" and thereby rendering the concept itself meaningless.

However, PAUL VALÉRY says: "Tout concept (comme ils disent) est expédient". (L'idée fixe.)

It should indeed be realized that it might well be, that strict autotrophy exists only as a mental extrapolation. Though green plants are generally supposed to be "self supporting" even these are frequently found to be dependent in one way or another on the co-operation of microbes.

Living matter is symbiotic in essence; mutual dependences in every degree, in every imaginable form are to be found everywhere. Consequently symbiosis in some form *must necessarily* pervade the numberless manifestations of that single autonomous phenomenon: Life.

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Astronomy. — *Egyptian "Eternal tables"*. II. By B. L. VAN DER WAERDEN. (Communicated by Prof. A. PANNEKOEK.)

(Communicated at the meeting of May 31, 1947.)

In the preceding paper I have discussed the dates of entrance of planets into the signs of the ecliptic, contained in 3 Egyptian texts, published by O. Neugebauer:

P: Berlin Papyrus P 8279

S: Stobart tablets A, $C_1 + C_2$ and E

T: Papyrus Teptunis II 274

and I have shown that the dates for Venus, Mars and Jupiter are calculated by means of Babylonian methods.

In this paper I shall compare the motion of Saturn and Mercury in these texts with Babylonian ideas, and I shall discuss the reduction of Babylonian dates to the Alexandrian calendar in the texts S and T.

Saturn.

According to the Babylonian procedure text AO 6477, analyzed by KUGLER (Sternkunde II, p. 578) the velocities of Saturn are as follows:

Daily velocity of Saturn	"slow motion" 10° ♌ to 30° ♒	"fast motion" 30° ♒ to 10° ♌
Near the sun	5′	6′
After heliacal rising 30^D	5′	6′
Until 1st station 90^D	3′ 20″	4′
Until opposition $52\frac{1}{2}{}^D$	− 4′ 13″ 40‴	− 5′ 4″ 24‴
Until 2nd station $69\frac{2}{3}\frac{1}{6}{}^D$	− 3′ 20″	4′
After 2nd station 90^D	3′ 35″ 30‴	4′ 18″ 40‴
Until heliacal setting 30^D	5′	6′
Total retrograde course	7° 33′ 7″ 30‴	9° 3′ 45″
Total synodic course	11° 43′ 7″ 30‴	14° 3′ 45″

The text does not tell us, how long the time "near the sun" lasts. In order to obtain the prescribed synodic arcs (11°43′7″30‴ or 14°3′45″) the time ought to be $46\frac{1}{2}{}^D$, but in order to obtain the right synodic period it ought to be only 24^D. Probably the time interval was intentionally not prescribed, in order to avoid errors in either date or position, which would sum up to large errors after many synodic periods. I suppose that at the beginning of a new synodic period the position and date were determined anew from tables of well-known type giving date and position at heliacal rising [1]). This would explain also the irregular date differences which we found in the case of Mars in the neighbourhood of the con-

[1]) See B. L. VAN DER WAERDEN, Babylonische Planetenrechnung, Eudemus I (1941).

junction. Possibly such corrections were also applied at the stationary points; this would explain certain irregularities for Jupiter in P and S.

Our text S is not calculated according to the procedure text AO 6477, but the division of the ecliptic into a slow and a fast region is the same. In the fast region the time needed for traversing a sign is always less than in the slow region, as is seen from the following table:

Slow region: ♍ ♎ ♏ ♐ ♑ ♒					Fast region: ♓ ♈ ♉ ♊ ♋				
Year	Date		Sign	Difference	Year	Date		Sign	Difference
Vespasian 6	2	7	♐	802d	Trajan 13	7	24	♓	771
Vespasian 8	4	18	♑		Trajan 15	[9]	4	♈	790
Trajan 9	3	5	♑	800	Trajan 17	11	4	♉	
Trajan 11	5	14	♒	800	Trajan 16	6	7	♈	783
Trajan 13	7	24	♓		Trajan 18	7	30	♉	774
Hadrian 11	2	1	♎	801	Hadrian 1	9	13	♊	778
Hadrian 13	[4]	11	♏		Hadrian 3	11	1	♋	
Hadrian 14	2	1	♏	811					
Hadrian 16	4	22	♐						

The normal, (i.e. minimal) difference in the slow region is apparently $800^{d} = 27^{M}3^{D}$, in the fast region $771^{d} = 26^{M}4^{D}$. This means: The time necessary for traversing 2 complete synodic arcs, that is 23°26′15″ or 28°7′30″, and the additional 6°3′45″ or 1°52′30″ necessary to make up 30°, is $27^{M}3^{D}$ in the slow region, and $26^{M}4^{D}$ in the fast region. (The larger differences 811, 790, 783, 774 and 778 are apparently due to the fact that in these cases the additional arc does not entirely belong to the "linear part", where the velocity is highest).

Now 2 synodical periods are $25^{M}16^{D}$ or $25^{M}21^{D}$, for this is the time which the sun needs to traverse 720° + 23°26′15″ or + 28°7′30″; hence the time necessary to traverse the remaining 6°3′45″ or 1°52′30″ at maximum speed is

$$27^{M}3^{D} - 25^{M}16^{D} = 47^{D}$$

or

$$26^{M}4^{D} - 25^{M}21^{D} = 13^{D}$$

So the maximum velocity of Saturn is approximately

$$364' : 47^{D} = 7\tfrac{3}{4}\ (' \text{ per } ^{D})$$

in the slow region, and

$$112\tfrac{1}{2} : 13 = 8\tfrac{1}{3}\ (' \text{ per } ^{D})$$

in the fast region. These figures are only approximate, especially the last one. Most probably the exact velocities are 7′30″ in the slow region and 9′ in the fast region. In any case they are higher than those of the procedure text, just as in the case of Jupiter.

Mercury.

For Mercury the date differences are, if we restrict ourselves to the main part of text S and to non-retrograde motion

(~ means: retrogradation in this sign).

Text C_1+C_2		♍	♎	♏	♐	♑	♒	♓	♈	♉	♊	♋	♌
Trajan	8				20	16	17	~	17	16	19	~	18
	9	19	20	~	17	18	~	16	14	18	~	28	18
	10	18	~	17	19	18	~	17	16	19	~	18	16
	11	23	~	19	18	~	13	20	44	~	23	15	18
	12	~	19	19	16	~	17	16	24	~	15	15	~
	13	19	20	19	~	18	17	17	~			20	~
	14	18	20	20	~	17	17	~	19	15	17	~	18
	15	21	20	~	34	17	18	~	17	16	22	~	21
	16	17	~	20	17	18	~	16	16	17	~	19	15
	17	20	~	17	19	19	~	16	17	23	~	16	16
	18	~	18	19	20	~	18	16	28	~	14	15	20
	19	~	19	20	~	21	16	17	~	25	13	15	~
Hadrian	1	21	19	19	~	17	16	18	~		15	23	~
	2	19	20	~	19	18	15	~	16	16	18	~	20
	3	20	13	~	20	18	20						

In some synodic periods we have 2, in most cases 3 successive date differences, of which the middle one is generally smallest. This could be expected, because for all planets the motion is fastest in the middle part of the direct course. Now if we want to single out this maximum speed, we have to restrict ourselves to those cases where the middle number is really the smallest of the three, or where there are only two nearly equal numbers. We also exclude the cases in which the middle number seems too small. The remaining middle numbers are, if the method is applied to the whole text S:

	Text A	Text C_1+C_2	Text E
♍	19	19, 19, 20	20, 17
♎	19, 19, 19	19, 19, 20	19
♏	20	19, 19, 20	20
♐	17	17, 18, 17	18
♑	17	16, 18, 17, 17, 18	17
♒	16, 15	17, 17, 16, 16	
♓	15	16, 16	17
♈		14, 16, 16, 17	16
♉	16	16, 15, 16, 16	16, 16
♊		15, 13, 15	14, 15
♋		15, 15, 16, 15	15, 15
♌		18, 16, 15, 16	17, 18

It appears from this list, that the ecliptic was divided into a *slow region*

(containing the signs ♍ ♎ ♏), where the time for traversing a sign lies between 19 and 20^{d}, one or two *middle regions*, where the time lies between 16 and 17$\frac{1}{2}^{d}$, and a *fast region* (containing ♐ ♑ ♋), where the time is about 15^{d}. In Babylonian units, the speeds might be 1°30′, 1°48′(?) and 2° per 1^{D}.

Now in Babylonian texts, the ecliptic is also divided into 3 parts. For instance, in Sp II 57 + 59 (KUGLER, Sternkunde I, p. 188) the synodic arc from one heliacal rising of Mercury in the morning to the next one is:

106° between 1° ♌ and 16° ♓
141°20′ between 16° ♓ and 30° ♉
94°13′20″ between 30° ♓ and 1° ♌

Similarly, the synodic arc from one heliacal rising of Mercury in the evening to the next one is

160° between 6° ♋ and 26° ♎
106°40′ between 26° ♎ and 10° ♈
96° between 10° ♈ and 6° ♋

I do not, however, see any direct connection between these synodic arcs and the maximum velocities of Mercury previously found. In order to establish such a connection, it would be necessary to study the retrograde motion as well.

Calendar reduction.

The dates given in T are in close agreement with those of text S. NEUGEBAUER writes: "Comparisons between the positions given in T and S show a slight tendency to place the entrance of a planet into a sign earlier than T, but this might be purely accidental in our little fragments (in 10 instances it is earlier, averaging one day; in 3 instances later; in 12 there is exact agreement)" [2].

The most natural explanation of this agreement and of these divergences seems to be, that S and T used the same Babylonian planetary text, but that the reduction to the Alexandrian calendar was made by a different scheme. The means to perform this reduction are given by the text T itself, for it contains the following dates of New Moons and subsequent crescents:

(Trajan) year 11 VIII 4
X 3 6
XII 2 5

The New Moons of months IX and XI are left indetermined. I suppose that they were tacitly assumed to take place either 30 days after the

[2] O. NEUGEBAUER, Egyptian Planetary Texts p. 242 (Trans. Amer. Philos. Soc. 32, 1942).

preceding New Moon, or 30 days before the next one. Adopting the first supposition, we get the complete scheme used for date reduction:

	New Moon	Crescent	Carlsberg crescent
Year 11	VIII 4	VIII 7	VIII 7
Trajan	IX 4	IX 7	IX 6
	X 3	X 6	X 6
	XI 3	XI 6	XI 5
	XII 2	XII 5	XII 5
	epag 2	epag 5	epag 4

The last column "Carlsberg crescent" is obtained as follows: In Pap. Carlsberg 9 a periodical scheme for calculating new moons is exposed. NEUGEBAUER and VOLTEN, who have published this papyrus [3]) suppose that the dates given in Carlsberg 9 mean crescent, i.e. first visibility of the moon, instead of new moon. This supposition is confirmed by their calculations for the first year of the last cycle (AD 144/145), but in most other years the dates given by the Papyrus are nearer to the new moon than to the crescent, and quite near to the mean new moon. Two instances will illustrate this:

Year AD	Cycle year	Carlsberg date	Mean New Moon	New Moon	Crescent
108	14	X 6 = April 28	April 28, 12^h	April 28, 23^h	April 29 or 30
146	3	VI 7 = Dec. 21	Dec. 21, 6^h	Dec. 21, 2^h	Dec. 23

For the year Trajan 11 the Carlsberg text gives the following dates:

(Egyptian) VIII 7 (= Alexandrian VII 4)
X 6 (= IX 3)
XII 5 (= XI 2)

Adding (as in T) 3 days to get the crescent, although this would seem a little bit too much [4]), and interpolating the Alexandrian months VIII, X and XII as before by the addition of 30 days, I obtained the dates given in the last column "Carlsberg crescent".

The near agreement of this column with the preceding one derived from text T is another strong argument in favour of our interpretation of the dates in Carlsberg 9 as new moon dates. It is seen that in half of the cases Carlsberg leads to the same crescent as T, in the other half to a crescent 1 day earlier. Hence in applying these or similar schemes to the reduction of Babylonian dates, half of the dates will result 1 day earlier

[3]) Quellen u. Studien, Gesch. Math. B 4, p. 383.

[4]) According to SCHOCH (in Langdon–Fotheringham, Ammizaduga) the lapse of time between New Moon and the noon of the day after crescent in Babylon (day 1 of the Babylonian Month) varies between 1½ and 3 days.

according to the second scheme, which is just what NEUGEBAUER found in comparing the dates of S and T.

Still, I am not quite sure that the reduction of dates in S was performed exactly by means of the scheme "Carlsberg Crescent". The only date for which this hypothesis can be directly checked is the entrance of Mercury into Leo in the year 11 of Trajan, for which S and T give the dates

T: XI 21 ♌ S: [XI] 20 ♌

If the original Babylonian date is supposed to be Duzu 16, the dates in T and S would be obtained by assuming Duzu 1 = XI 6 and = XI 5 respectively, as in our columns "Crescent" and "Carlsberg Crescent". However, in the beginning of the same year 11, months 2—3, the dates of T coincide exactly with those of S for 2 months in succession, which would mean that here the reduction schemes of S and T coincide. In the years 14 and 15 we find again a regular alternation of months in which S and T give the same data, and months in which the difference is 1[d], e.g.:

Planet	Year	Month	Text T	Text S
Mars	14	8	3	3 ♒
Mars	14	9	13	12[5]) [♓]
Jupiter	15	8	6 ♓	5 ♓
Saturn	15	9	4 ♈	4 ♈

A closer examination shows that in these years the "Carlsberg crescent" scheme does not give a satisfactory explanation of the text dates, unless the new moons not mentioned in Carlsberg are chosen 30 days before the next ones instead of 30 days after the preceding ones. This point requires further investigation.

Program for further investigation.

By lack of time, I am not able to carry the investigation beyond this point, but I hope that others will continue it. It seems desirable

1) to reduce all dates of text S to the Babylonian calendar with the aid of Carlsberg 9 or a similar scheme, and to see whether the regularity of the differences (e.g. for Venus) is increased by this reduction,

2) to investigate whether P is calculated by the same rules as S,

3) to establish conclusively the law of motion of Venus in P and S,

4) to determine the stationary points, especially of Jupiter, and to compare their positions with those given by Babylonian Jupiter tables,

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5) NEUGEBAUER writes: Traces only compatible with 12 or 17. But only 12 gives the correct difference 39 (see our table of differences of Mars).

5) to see whether the Babylonian planetary ephemerides, published by KUGLER (see my paper I, footnote [13]), follow the same rules as our texts P and S.

6) to give a more complete description of the motion of Jupiter, Mars, Mercury and, if possible, also of Saturn.

I hope that this investigation will lead to a better understanding of Babylonian astronomy in its highest development, and of its influence on Hellenistic astronomy and astrology.

Zoology. — *Viscosity changes during cleavage in the eggs of Limnaea stagnalis.* By MIEKE C. HEIKENS. (From the Zoological Laboratory, University of Utrecht.) (Communicated by Prof. CHR. P. RAVEN.)

(Communicated at the meeting of May 31, 1947.)

Introduction.

The observations of FRY & PARKS (1934) show that in eggs of *Nereis, Cumingia* and *Arbacia* the viscosity changes in cycles during the first period of development. These cyclical variations in viscosity correspond to the mitotic cycles. The following cycle has been observed: prophase: viscosity low; metaphase: viscosity low or rising; anaphase and telophase: viscosity high. Viscosity is low when the structures concerned are being organised, rising when the cell processes are in action and remains high at the conclusion of the cycle, until the next cycle makes its appearance.

Similar viscosity changes have been observed by LEHMANN (1938) in the eggs of *Tubifex*.

The same investigations were carried out by RAVEN (1945) in eggs of *Limnaea stagnalis*. He also observed viscosity changes during the period of the uncleaved egg, but, whereas FRY & PARKS found a high viscosity at the formation of the first polar body, the formation of the second polar body, the fusion of the pronuclei and the first cleavage, RAVEN observed an increased viscosity at the fusion of the pronuclei and the first cleavage only. During the maturation divisions viscosity shows no significant cyclic changes in *Limnaea*. There is an increase in viscosity, beginning shortly before the extrusion of the second polar body and reaching its maximum when the sperm-aster is at its height, a rapid drop to a minimum at the prophase of the first cleavage and a rise during cleavage mitosis.

The purpose of this investigation was to determine whether cyclic changes in viscosity can also be observed during the next cleavages.

Methods.

By means of centrifugation a stratification is brought about in the egg consisting of a fat zone, hyaloplasm zone, zone of granules, zone of proteid yolk. The degree of stratification is used as a means of estimating the relative viscosity changes. The viscosity is low if the egg is fully stratified, and relatively high if stratification is less distinct. Comparing the stratification of eggs centrifuged at different successive stages, one gets an impression of viscosity changes occurring in the egg.

Experiments.

Eggs of *Limnaea stagnalis*, belonging to one egg mass, were centrifuged, at intervals of 15 minutes, for 5 minutes at a velocity of 3800 revolutions per minute (1860 × gravity). Immediately afterwards the eggs were taken from the centrifuge, examined by means of a horizontal microscope and drawn or photographed after 3 or 4 minutes.

The following series gives an example of an experiment (cf. fig. 1 and 2).

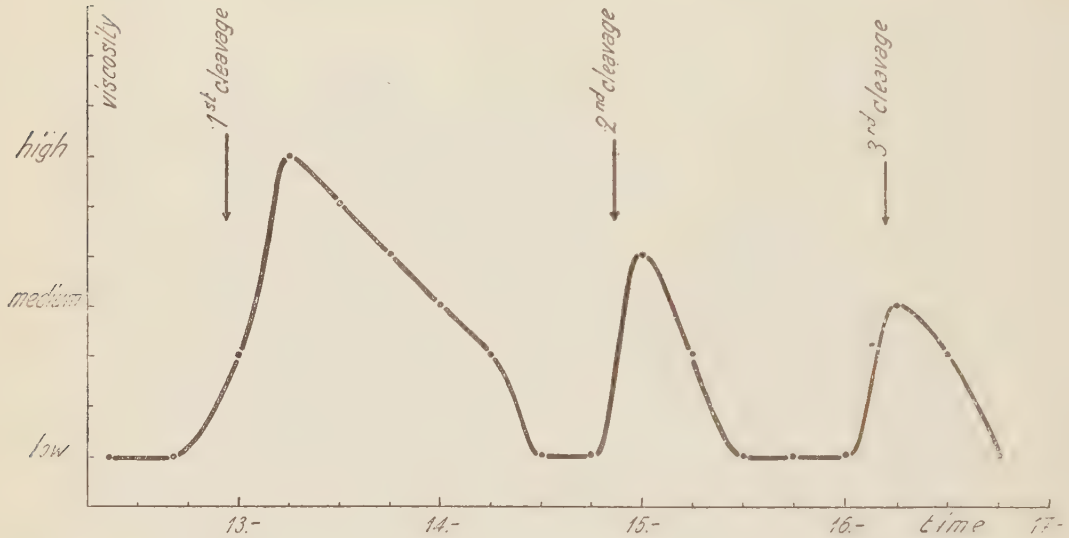


Fig. 2. Diagram showing variations of viscosity during 1st, 2d and 3d cleavage of *Limnaea stagnalis*. Abscissae: time. Ordinates: estimated relative value of viscosity.

Nr. of sample	Time	Viscosity	Nr. of sample	Time	Viscosity
1	12.20	low	10	14.45	low
2	12.40	low	11	15.—	rather high
3	13.—	low	12	15.15	medium
4	13.15	very high	13	15.30	low
5	13.30	high	14	15.45	low
6	13.45	less high	15	16.—	low
7	14.—	medium	16	16.15	rather high
8	14.15	medium	17	16.30	medium
9	14.30	low	18	16.45	low

The first cleavage took place between 12.40 and 13.—, the second between 14.45 and 15.—, the third between 16.— and 16.15.

The results indicate that

1. Viscosity is low immediately before each cell division;

2. it begins to rise at the moment at which the cleavage furrow makes its appearance, and reaches a maximum about 10—15 minutes afterwards;

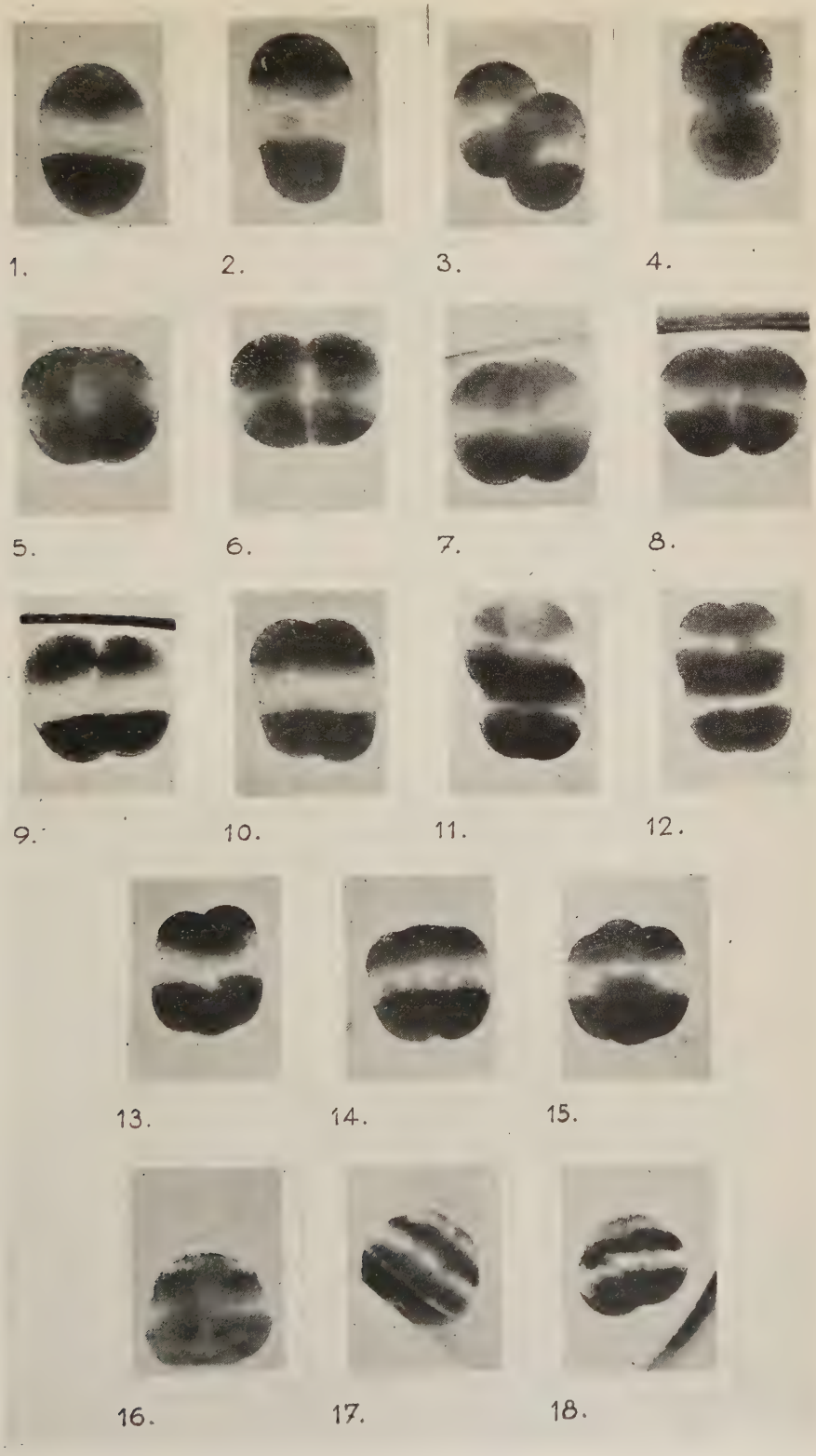


Fig. 1. Eggs of *Limnaea stagnalis*, centrifuged at various moments during 1st, 2d and 3d cleavage. Numbers cf. text.

3. after this maximum there is a gradual drop of viscosity. Corresponding to the longer interval between 1st and 2nd cleavage (about 2 hours at 20° C against 1¼—1½ hours between 2nd and 3rd cleavage), this drop is slower after the 1st cleavage than after the 2nd and 3rd divisions.

The results of the other experiments are in full agreement to these observations.

Discussion.

It appears therefore that there is a relationship between the viscosity of the egg and the mechanism of cleavage. It can be concluded from the experiments that the viscosity of the protoplasm is maximum during the period of cell division. It remains low till the cleavage furrows begin to cut through the egg; at this moment or some minutes later a sudden rise of viscosity sets in, leading to a maximum about 10—15 minutes after the beginning of cleavage. When we compare these observations with the data given by RAVEN (1946) on the time schedule of cleavage, it is clear that the viscosity is low during prophase, metaphase and early anaphase, and that the rise of viscosity corresponds to late anaphase or early telophase. The maximum coincides with late telophase, the drop of the curve with the fusion of the karyomeres to a polymorphic resting nucleus.

According to HEILBRUNN (1920), the increase of viscosity of the protoplasm is caused by gelification, which is necessary for spindle formation and therefore for cleavage. In other words, an increase of viscosity is necessary before spindle formation can take place. From his investigations of the eggs of *Arbacia, Cumingia* and *Nereis,* HEILBRUNN (1921) concluded that viscosity is high during prophase and then drops; it rises again just before cleavage, presumably about the time of early telophase. FRY & PARKS (1934) have made probable, however, that what HEILBRUNN considered as the prophase stage was actually a middle phase in the history of the spermaster. They conclude from their observations that viscosity is low during the prophase stage. This conclusion is corroborated by my observations on the viscosity changes during the 1st, 2nd and 3rd cleavage of *Limnaea*. On the other hand, with reference to the viscosity at the end of the cycle, my results are in accordance with HEILBRUNN's statement that viscosity rises at the time of early telophase; according to FRY & PARKS, it rises somewhat earlier, usually during early anaphase.

Summary.

1. Viscosity changes during 1st, 2nd and 3rd cleavage in *Limnaea stagnalis* have been studied by means of centrifugation.

2. Viscosity is low immediately before each cell division, begins to rise at the moment of appearance of the cleavage furrows, and reaches a maximum about 10—15 minutes afterwards; then, there is a gradual drop of viscosity.

3. Viscosity is low during prophase, metaphase and early anaphase of the mitotic cycle; it rises during late anaphase and telophase, and drops again during the fusion of the karyomeres.

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Zoology. — *The first illustrations of stick-lac and their probable origin.* By S. MAHDIHASSAN. (Department of Biochemistry, Osmania University, Hyderabad Deccan.) (Communicated by Prof. CHR. P. RAVEN.)

(Communicated at the meeting of May 31, 1947.)

VON GERNET (1) draws attention to the fact that the work of the Dutch Herbalist, C. CLUSIUS, contains illustrations of stick-lac, which are the earliest to be found in the literature. Through the kindness of the Librarian of the Royal Dutch Library, at the Hague, I was fortunate enough to borrow a copy of this work, entitled "Aromatum et Simplicium", being a Latin translation of GARCIA DA HORTA's Portugese book, "Colloquios dos Simplices e Drogas", printed at Goa, in 1563. Sir GEORGE WATT (2) says, in this connection, that GARCIA, "as physician to the Portugese Governor of India visited that country in 1534. His volume of Colloquies was published at Goa in 1563 and was the second book printed in India".

Mrs GARLAND JANE of the Lisbon Branch of the Historical Society kindly sent me a photostatic copy of the article where GARCIA deals with lac. She did me the additional favour of kindly translating into English the entire article on lac so that I could compare GARCIA'S text with that of CLUSIUS. The last named author gives three pictures of stick-lac whereas none is found in the Portugese original. CLUSIUS printed his translation at Antwerp, the first edition appearing in 1567, only four years after GARCIA'S book had appeared at Goa. This is a remarkably short period considering the facilities of communication as they existed then. The question is being raised as to the origin of the three illustrations offered by CLUSIUS but absent in GARCIA'S original.

Had there been only one species of the lac insect the problem would have been simple, for only one biological source could have represented these pictures. In fact this would still be the official opinion in India today. However, even as early as 1780, SWAGERMAN (3) found at least three sorts of lac. He differentiated them, according to their geographical source. From Siam was the reddest of all, from the Coromandel coast was intermediate, while from Bengal, as it was understood then, was the palest and yellow in colour. In 1786, Dr CHAMBERS, F.R.S. (4), also mentions three localities which export lac; "Stick lac is brought from Malabar, Bengal and Pegu", — the text has been condensed. Now the localities mentioned by SWAGERMAN and by CHAMBERS coincide with each other in so far as the species of lac insect found in Siam and in Pegu is *Lakshadia chinensis,* on the Coromandel coast and in Malabar it is *Lakshadia communis,* while in Bengal, as it was understood then, it is *Lakshadia nagoliensis,* an insect which

SWAGERMAN also illustrates. Stick-lac from *L. chinensis* has been previously illustrated by me to show its special features (5); likewise I have illustrated the encrustations of *L. communis* (6) and of *L. nagoliensis* (6). The specific differences among them have been already brought out. Since these represent the most common sorts of stick-lac it is probable that CLUSIUS also illustrated one of these three varieties.

From ancient times upto the medieval ages there has been a great scarcity of a suitable red dye for wool and for silk. The one that excelled all was the lac dye. Even books on ALCHEMY (7) contain recipes, with lac as the main ingredient, for dyeing wool scarlet-red. Being thus a very valuable product there had been a natural desire to know its origin which apparently remained a mystery. According to some it was a plant secretion and the name gumlac is current to this day. According to others it was collected by ants. B. WITSEN (8), Master of Amsterdam, writes as follows, to RICHARD WALLER, F.R.S., in 1698, "Gum lack is the house of a larger sort of ants which they make on boughs of trees which serve to keep them from the weather". Even upto 1789 there seems to have been no recognised explanation, for SAUNDERS (9) writes, "Many vague and unauthenticated reports concerning lac have reached the public and though among the multiplicity of accounts the true history of this substance has been nearly hit on, little credit is given in Europe to any description of it hitherto published". GARCIA in 1563 must have realised this even more forcibly.

Conscious of his position in the academic world and of the opportunities he had of incorporating first hand information in a book that was to become a classic, he seems anxious of pronouncing the last word on a very important subject. The discovery of the real nature of a substance both valuable and mysterious would excite any author to write as exhaustively as one can. In similar situations ROXBURGH (10) in 1781 and CARTER (11) in 1861 have missed no opportunity both of describing the object and of illustrating it. Their enthusiasm has induced them to offer even superfluous illustrations for some of them are not intelligible even to this day. GARCIA is equally profuse in his text but refrains from adding any illustration of lac. Even CLUSIUS is liberal in offering three pictures of stick-lac, one of which is quite unintelligible to me even after several years of its study. How could GARCIA refrain from a temptation to have the object illustrated when he writes, "Here in Goa a boy brought me a branch of a tree which we call the apple tree and the Deccanis Ber, of which I have already spoken and which are here in this orchard for my eyes to see. And because it bears little lac they do not value it, the soil not being suited to produce it. (However) many have told me that they have seen it on apple trees". In similar situations ROXBURGH and CARTER have had illustrations made from the material while still fresh. How did GARCIA miss such a chance and leave it to CLUSIUS, his own translator, to offer the illustrations?

It has been pointed out that GARCIA'S book was the second that was printed in India. The condition of the press then was probably not advanced enough to undertake the publication of any pictures, be it from wood engravings. Moreover it is hard to imagine that CLUSIUS, within four years, could have got samples of lac such as he felt were good enough to be illustrated. CLUSIUS must have kept himself busy with the work of translation in addition to his other occupations. It is most likely that the pictures were executed in India, under GARCIA's guidance, from the fresh material he saw for the first time, and these drawings must have been passed on to CLUSIUS, since they could not be reproduced in India for technical reasons. These pictures must have formed the basis for the wood engravings which were executed and finally printed at Antwerp. ROXBURGH likewise had his pictures made from fresh material which were copied on a copper plate and the illustrations reproduced in England as engravings.

Reproductions, as they appear in CLUSIUS, represent the lac encrustations natural size, also the most convenient size for an artist to make. The drawings must have been executed with ink and a feather pen as was the custom then. When a drawing is transferred onto wood fine details are lost and this must be remembered in interpreting the pictures offered by CLUSIUS particularly as the shading there mars the details very much. Fig. 1 offered here from CLUSIUS shows one end of the twig cut with a slant. When twigs of *Zizyphus jujuba* are cut to compare them with Fig. 1 a fresh twig alone bears resemblance while a dry one does not; indicating that the original picture was drawn from fresh material.

GARCIA and thus also CLUSIUS devote a chapter to the Ber tree, Zizyphus species and mention it as thorny. In Figs. 1 and 2, however, no thorn is evident. These pictures, as they appear in CLUSIUS, have the identical length. Along their greatest lengths, taken longitudinally, they measure 7.6 cm, an observation I cannot explain. The absence of thorns within 8 cm, on twigs of Zizyphus, however, is quite possible, although not probable. Now Zizyphus species form favourite host plants of *L. communis*.

Of all species of lac insects, *L. communis* secretes the thinnest layer of lac. Fig. 1 shows that the rows of insects on either side of the encrustation leave a narrow margin in between them. The upper part of Fig. 2 has a corresponding narrow space which has not been bridged over by the secretion of lac produced from insects on either side. Fig. 3 is a natural size photograph of *L. communis* encrustation forming, as it were, two laps of secretion which meet and yet do not fuse. I have seen nothing like this in any other material excepting *L. communis*. It is a specimen preserved in the Natural History branch of the British Museum, London and I have to thank Mr LAING for his kind permission to reproduce the picture here. The host plant is probably *Butea frondosa*. Fig. 4 is another speciment of

L. communis kept at the British Museum and certainly on *Butea frondosa*. The encrustation of lac is so papery that it has been damaged in handling. I have previously (12) illustrated how stick-lac from *L. communis*, merely on storing, can show cracks due to the quality of secretion being so very poor. Just as there are poor breeds of milk cattle in *L. communis* we have a lac insect which occupies a similar position among insects of its class.

GARCIA mentions that Goa produces little lac as the soil there is not suited for its production. This would be a typical reply, even today, from a forest officer who has to deal with L. communis, be it in Goa or anywhere in South India where it occurs wild. CARTER (11) in 1861 found this species in Bombay, which is north of Goa and so did I in 1916 in the same city. Samples sent to me from Travancore also indicated the presence there of *L. communis*. I am therefore certain that the insect found in Goa is *L. communis*. Why lac from this biological source is scanty is due to its inherent constitutional defect and not to any fault of the soil. The insect all of a sudden gives to such a preponderance of males in a generation that the colony becomes extinct and the species is able to continue only when the male larvae become bisexual and this happens with difficulty. These rare hermaphrodites, unlike normal cells, which are round, have been called crown shaped cells on account of their shape. The crown shaped cells are mostly found with *L. communis* which also shows the greatest variation in sex ratios. Now Fig. 1 from CLUSIUS shows the presence of such a crown shaped cell. This would confirm the original specimen was derived from Goa.

Fig. 5 gives a picture of a crown shaped cell, marked, K, at the lower end of the twig, towards the left; it is seen isolated and hence a conspicuous object by itself. The generation of larvae that emerged from this mother cell has formed the encrustation on the upper portion of the twig. The encrustation is not fully developed. It was collected during September in Bangalore on *Ficus mysorensis* while it would have taken another four weeks for it to be fully formed. The encrustation seen in Fig. 1 likewise seems rather immature, comparable with Fig. 5 here. Perhaps for this reason the crown shaped cell, in Fig. 1, is relatively conspicuous.

Fig. 6 shows an ideal full grown encrustation of *L. communis* derived from one crown shaped cell marked, K. It was growing at Bangalore, on *Pithecolobium saman* and was collected early in November. Imagine such a photograph being transferred onto wood and the picture reproduced as wood engraving. This has to be remembered in appreciating the presence of crown shaped cell in Fig. 1 which is a wood cut.

In Fig. 1 the crown shaped cell is seen projecting from the main mass of encrustation. In a previous article (13) on the probable occurrence of sex reversal, in Fig. 3 there, I have shown a crown shaped cell still more conspicuous. In the communication on sex ratio variability (14) in Fig. 2 there,

EXPLANATION OF FIGURES.

Fig. 1. Stick lac illustrated by CLUSIUS in 1567. At the extreme basal end, to the left, there is a crown shaped cell, seen in profile and as silhouette. It is further shown in Fig. 7. It belongs to *Lakshadia communis.*

Fig. 2. Stick lac as illustrated by CLUSIUS; the picture here is seen a little enlarged. Here and in Fig. 1 resin secretion from the sides does not fuse due to the insect being a poor producer of a resin. It represents L. communis.

Fig. 3. A chunk of lac from *L. communis,* typical of this insect, with secretion from the sides not fusing together. Original is in British Museum. The host plant is *Butea frondosa.*

Fig. 4. A chunk of lac from *L. communis,* on *Butea frondosa,* in the British Museum, showing a papery secretion of lac, fragile and broken.

Fig. 5. A living colony of immature insects of *L. communis* on *Ficus mysorensis.* At the basal end, a crown shaped cell, *K,* represents the mother of the entire generation. Magnification 11 : 10.

Fig. 6. An encrustation of *L. communis,* from a single crown shaped cell, *K,* on *Pithecolobium saman,* full grown. Magnification 11 : 10.

Fig. 7. The crown shaped cell incorporated in Fig. 1 is seen by itself.



Fig. 1.



Fig. 2.



Fig. 3.



Fig. 4.



Fig. 5.



Fig. 6.



Fig. 7.

it is seen in profile, comparable with Fig. 1 here. In both these illustrations the crown shaped cell occupies the lowest portion of the colony, a position typical for it. In Fig. 1 the crown shaped cell is seen not merely in profile but also as silhouette. Fig. 7 shows the crown shaped cell of Fig. 1 by itself. It is comparable with Fig. 2 C of the previous communication (14). The object has been to interpret what is found rather than complain what has been omitted which should have been the task of contemporary critics.

Summary.

CLUSIUS in 1567 gives three illustrations of stick lac. They are printed from wood engravings. Probably the drawings were made by GARCIA in India whose book CLUSIUS translated. GARCIA'S work was printed in 1563 at Goa where the press was not advanced to enable the illustrations to appear. GARCIA probably passed on these drawings to CLUSIUS.

The insect species represented in these illustrations is probably *L. communis*, a common lac insect found all over south India, including Goa. It has some special features which the illustrations in CLUSIUS bear out. It secretes a very poor layer of lac. The insect produces crown shaped cells which are hermaphrodites, being male larvae which have become females. Fig. 1 from CLUSIUS shows a crown shaped cell attached to it.

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Genetics. — *De mendelistische verklaring van de erfelijkheid door polymere factoren in haar beteekenis voor ons inzicht in de zelfstandigheid en de gebondenheid van de afmetingen van de zaden van Phaseolus vulgaris.* By G. P. FRETS. (Communicated by Prof. J. BOEKE.)

(Communicated at the meeting of June 28, 1947.)

De studie van de erfelijkheid, door kruising van eenige eigenschappen van twee zuivere lijnen van Phaseolus vulgaris bracht mee, dat nagegaan werd, in welken vorm de polymerietheorie hier voor de verklaring van de erfelijkheid dienst kon doen. Ik onderzocht de volgende eigenschappen, waarvan ik door uitgebreide onderzoekingen van de niet-erfelijke variabiliteit de gemiddelden vaststelde.

	I-lijn	II-lijn
lengte	150 [1])	110
breedte	90	80
dikte	60	70
gewicht	55 [2])	45
LB-index	60	70
LD-index	40	60
BD-index	70	85

We zien, dat er tusschen de lengten van de boonen van de I-lijn en van de II-lijn een groot verschil is; het verschil tusschen de breedten is kleiner, dat tusschen de dikten is ook kleiner en in omgekeerde richting. De boonen van de I-lijn zijn lang, breed en dun, die van de II-lijn zijn kort, minder breed en dik. In den loop van het onderzoek heb ik met het oog op de verschillende grootte der verschillen van de afmetingen van de boonen van de I-lijn en de II-lijn verschillende vormen van polymerie overwogen. De verschillende grootte van dit verschil moet in de erfelijkheidsformule van de boonen tot uitdrukking komen.

1. Met het oog op het veel grootere verschil van de lengten van de boonen van de I-lijn en de II-lijn dan van de breedten, respect. de dikten, nemen we een aantal polymere factoren aan, die alle 3 afmetingen hebben, bv. 3, dus L_1, L_2 en L_3, B_1, B_2 en B_3, en D_1, D_2 en D_3, en bovendien een tweede aantal polymere factoren, die alleen aan de lengte toekomen, bv. eveneens 3, dus L_4, L_5 en L_6. Op grond van onze kennis van de gemiddelden en de variatiebreedte van de afmetingen kunnen we de grens tusschen

[1]) Afmetingen in 0.1 mm.
[2]) Gewicht in cg.

de lengte van de boonen van de I-lijn en de II-lijn op 13.0 mm stellen en bepalen, dat de lengtefactoren L_1—L_3 de lengte doet toenemen van 13.1—15.5 mm, terwijl door de werking van de factoren L_4—L_6 de lengtetoeneming der boonen van de I-lijn boven de 15 mm verklaard wordt. De werking van de factoren L_1—L_3 voor de lengte valt samen met een evenredige werking van de factoren B_1—B_3 voor de breedte en van de factoren D_1—D_3 voor de dikte. Door deze voorstelling wordt dus rekening gehouden met het veel grootere verschil der lengten van de boonen van de I-lijn en de II-lijn dan van de breedten en de dikten. Ter vereenvoudiging van het betoog en van de schrijfwijze vatten we de lengtefactoren L_1—L_3 samen als L_1, L_4—L_6 als L_2, B_1—B_3 als B en D_1—D_3 als D; L_1 is dus gelijkwaardig met B en met D en we schrijven de formule voor de boonen van de I-lijn als $L_1\ L_2\ B\ d$ en die van de II-lijn als $l_1\ l_2\ b\ D$. Volgens deze voorstelling hebben we bij kruising te doen met het tetrahybride schema. De op elkaar volgende generaties maken een overwegend intermediairen indruk.

De hypothese van de erfelijkheid door polymere factoren in dezen vorm leidt tot moeilijkheden. Na kruising kunnen in volgende generaties verschillende nieuwe genencombinaties ontstaan; bv. van de lengte $l_1\ l_1\ L_2\ L_2$. In dit geval ontbreken de genen voor de lengte van 13.1—15.5 mm, terwijl de genen voor de lengtevermeerdering van 15.6 mm aanwezig zijn. Het is duidelijk, dat we hier een onbestaanbaarheid voor ons hebben. We moeten aan de hypothese een anderen vorm geven.

1*b*. Iedere L-factor geeft eenzelfde vermeerdering van de lengte, de lengtefactoren kunnen voor elkaar in de plaats treden, de 2 aantallen der lengtefactoren, L_1—L_3 en L_4—L_6 zijn identiek. We nemen aan, dat bij de boonen met de formule $l_1\ l_1\ L_2\ L_2$ op een oogenblik in den groei de L_2-genen hun lengtevermeerdering op dezelfde wijze geven als bij boonen met de form. $L_1\ L_1\ l_2\ l_2$ de L_1-genen. Ook moet het levensproces van den groei, dus de werking van de groeistof, het mogelijk maken, dat bij de boonen met de formule $l_1\ l_1\ L_2\ L_2$, de L_2-genen hun werkzaamheid zullen kunnen volbrengen, evenzeer als bij de boonen met de form. $L_1\ L_1\ L_2\ L_2$.

Bij de splitsing ontstaan er volgens deze hypothese boonen met de form. $L_1\ L_2\ B\ D$, d.z. boonen met de factoren L_1, L_2, B en D (in homozygoten en heterozygoten vorm) en boonen met de form. $L_1\ l_2\ B\ D$ of $l_1\ L_2\ B\ D$, d.z. boonen met de factoren L_1 of L_2, B en D (in homo- en heterozygoten vorm). Er ontstaat hier dus een klasse boonen met een zeer groote lengte, een matig groote breedte en een matig groote dikte en een klasse boonen met een matig groote lengte en eveneens een matig groote breedte en een matig groote dikte. Het onderzoek levert geen bevestiging van deze aanneming (tab. 1 en 2): de boonen met de grootste lengte hebben ook de grootste breedte en de grootste dikte. We moeten echter met de duiding van dit resultaat voorzichtig zijn. Er is naast de erfelijke een zeer groote niet-erfelijke variabiliteit.

Het is de vraag, of in ons F_2-materiaal de erfelijke variabiliteit, als ze

zou verschillen van de niet-erfelijke, tot uitdrukking kon komen. Alleen door individueel voortkweeken is de erfelijke variabiliteit aan te toonen (vgl. ook de regelmatige kromme van het materiaal eener boonen-populatie van JOHANNSEN). Toch aanvaarden we deze hypothese 1*b* niet en gaan de mogelijkheid van de volgende hypothese na.

2. We nemen een even groot aantal erffactoren aan voor het verschil van de lengten, de breedten en de dikten van de boonen van de I- en de II-lijn. Hierbij geeft een factor voor de lengte een relatief veel grootere toeneming van de lengte dan een factor voor de breedte van de breedte en een factor voor de dikte van de dikte.

Met deze hypothese is in overeenstemming, dat we in het materiaal van onze F_2-zaadgeneratie van 1933 vinden, dat onder boonen met alle 3 groote afmetingen de boonen met de grootste lengte ($L_1 L_2$) ook veelal de grootste breedte en de grootste dikte hebben en dat van boonenopbrengsten, die een groot aantal boonen met een zeer groote lengte bevatten ($L_1 L_2$ B D) de gemiddelde breedte en de gemiddelde dikte grooter zijn dan van de boonen-opbrengsten met een groot aantal boonen met een minder groote lengte (L_1 B D; zie tab. 1). Uit onze hypothese vloeit voort,

TABEL 1. De gemiddelde afmetingen, het gemiddelde gewicht en de gemiddelde indices van boonenaantallen in de klassen 1*a* met de form. $L_1 L_2$ B D en 1*b* met de form. L_1 B D der classificatie volgens het gewijzigde tetrahybride schema voor een drietal boonenopbrengsten van de F_3-zaadgeneratie van 1934.

	Kl. 1*a*, form. $L_1 L_2$ B D Pl. 272, n = 33			Kl. 1*b*, form. L_1 B D Pl. 268, n = 67			Kl. 1*b*, form. L_1 B D Pl. 274, n = 58		
	M	Gr. Var.	Kl. Var.	M	Gr. Var.	Kl. Var.	M	Gr. Var.	Kl. Var.
L	166	181	156	147	155	135	141	155	131
B	102	103	95	91	97	86	91	98	86
D	71	79	67	71	76	66	70	75	66
Gew.	85	106	72	66	81	56	63	73	53
L B	62	68	58	62	68	59	65	71	58
L D	43	47	39	48	53	43	50	56	45
B D	70	77	63	77	83	67	77	85	68

dat als we een groot aantal L B D-boonen indeelen volgens stijgende lengteklassen (tab. 2), we zullen vinden, dat met stijgende lengte, ook de gemiddelde breedten en de gemiddelde dikten der op elkaar volgende klassen zullen stijgen. Dit vinden we inderdaad.

Het resultaat van tab. 2 berust niet alleen op erfelijke variabiliteit; integendeel, er is ook een groote niet-erfelijke variabiliteit. De erfelijke, naast de niet-erfelijke variabiliteit blijkt, als we de erfelijkheid als correlatie berekenen van twee op elkaar volgende generaties. We vinden voor de correlatie, berekend met de form. van BRAVAIS, van de lengte der boonen van de F_3- en de F_2-generatie van 1934 en 1933, $r = + 0.44 \pm 0.09$, voor die van 1935 en 1934, $r = + 0.42 \pm 0.1$. Voor de breedte en voor de dikte vinden we een slechts zeer kleine positieve correlatie. In het materiaal van zuivere lijnen, met dus alleen niet-erfelijke variabiliteit, vinden we deze

TABEL 2. Boonen met een groote lengte, een groote breedte en een groote dikte ($L_1 L_2 B D$ en $L_1 B D$) van de F_2-zaadgeneratie van 1933 gerangschikt volgens stijgende lengteklassen.

Lengteklasse	Gem. breedte	Gem. dikte	Aantal boonen	Gem. breedte	Gem. dikte	Aantal boonen	Gem. breedte	Gem. dikte	Aantal boonen
	Na I × II-kruisingen Kasmateriaal n = 67			Na I × II- en II × I-kruis. Buitenmateriaal n = 141			Het geheele materiaal n = 269		
131—135[1])	—	—	0	89.22	70.56	9	89.22	69.44	9
136—140	86.5	68.0	4	88.36	69.67	33	88.18	69.39	39
141—145	87.38	68.88	16	90.72	70.05	42	89.42	69.58	69
146—150	88.05	70.16	19	90.78	70.36	36	89.15	69.79	78
151—155	88.74	69.89	19	93.07	70.36	14	90.06	69.53	47
156—160	89.2	69.2	5	92.43	71.14	7	90.95	69.80	20
151—165	89.3	73.3	3	—	—	—	90	72.83	6
166—170	91	72	1	—	—	—	91	72	7

[1]) In 0.1 mm.

correlatie niet. Zoo vinden we voor de lengte van de boonen van de I-lijn van 1937 en haar uitgangs-(ouder-)boonen van 1936, $r = + 0.08 \pm 0.1$. De erfelijke variabiliteit moet ook blijken uit de selectie. Uit de vergelijking van F_3-materiaal van 1934 met zijn F_2-uitgangsmateriaal blijkt de regressie (1937, fig. 11): er is hier eenige selectie mogelijk. In materiaal van zuivere lijnen, bv. van de I- en de II-lijn, is de regressie volkomen (1934, fig. 1—7 en deze meded. fig. 1).

De hypothese 2 voor de erfelijkheid van de afmetingen der boonen, waartoe we door onze kruisingsonderzoekingen komen, houdt dus in, dat de drie afmetingen een even groot aantal erffactoren hebben. D.w.z. dat bij den groei steeds tegelijk een factor voor de lengte, de breedte en de dikte werkzaam zijn (als „aanwezig" of „afwezig"). Hieruit zien we de mate van gebondenheid en zelfstandigheid van de afmetingen. De studie van den groei en van de variabiliteit (1938—1941) leerden ze ons reeds. Hier, bij de studie van de kruisingen, treft ze ons opnieuw: het mendelisme leert ons een groote mate van zelfstandigheid der afmetingen kennen bij de uitsplitsingen, doch we zien nu, volgens hypothese 2, dat niet aparte genenparen voor een der afmetingen voorkomen: altijd zijn er factoren voor de lengte, de breedte en de dikte tegelijk werkzaam. Dit resultaat voert ons weer tot de opvatting, dat de boon in grootte en vorm een eenheid is, een binnenste heeft; er is een reguleerend principe voor de afmetingen, dat vorm en grootte van de boon bepaalt, de werking der genen leidt.

Waar we aannemen, dat de polymere factoren voor ieder der afmetingen in een even groot aantal aanwezig zijn en steeds een factor voor de lengte, een voor de breedte en een voor de dikte tezamen bij den groei werkzaam zijn, — en elkaars werkzaamheid reguleeren —, zou het kunnen zijn, dat steeds een factor voor de lengte, een voor de breedte en een voor de dikte in eenzelfde chromosoom gelegen zijn. Deze mogelijkheid kan door onderzoek op koppeling worden nagegaan.

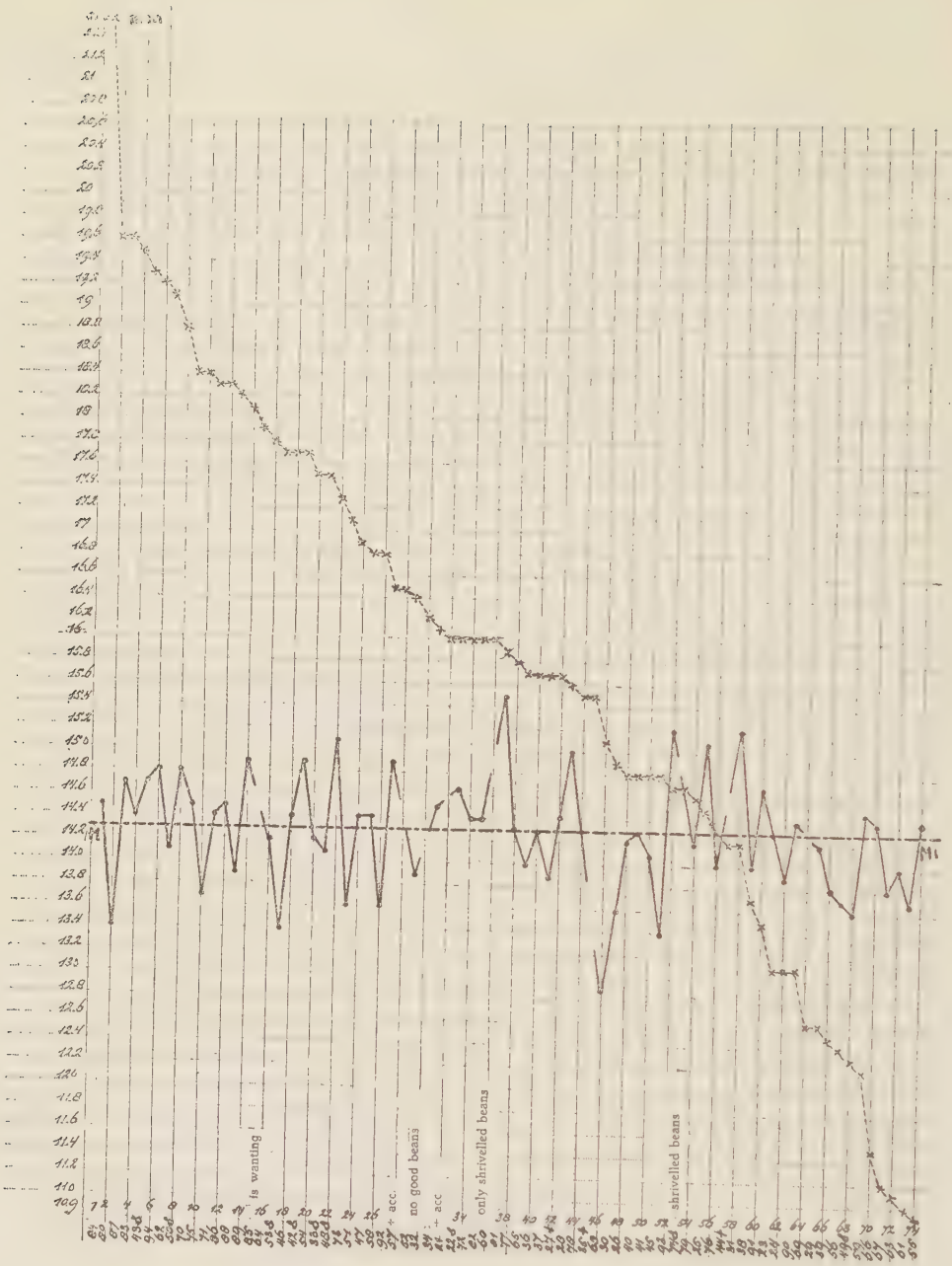


Fig. 1. I-lijn 1936. De lengte van 62 van 74 uitgezaaide uitgangsboonen van de boonenopbrengst van de I-lijn van 1935 en de gemiddelde lengten van hun boonenopbrengsten van 1936. Op de horizontale is de nummering van de boonen aangegeven volgens dalende lengte; er onder zijn de nrs van de boonen aangegeven in de volgorde, waarin ze uitgezaaid zijn. Eenige nrs zijn uitgevallen, doordat enkele boonen niet zijn opgekomen, of de plant verongelukt, gestorven, of verloren gegaan is, of geen goede boonenopbrengst opgeleverd heeft. Op de verticale is de lengte der boonen in mm aangegeven. Gestippelde lijn: lengte van de uitgangsboonen; getrokken lijn: gemiddelde lengte der boonenopbrengsten; gestreept en gestippeld: horizontale ter hoogte van de gemiddelde lengte van alle boonenopbrengsten.

Een moeilijkheid van de hypothese 2 is, dat ze niet zonder meer algemeen geldig is. Hoe moeten we de genen kiezen bij de kruising van bv. boonen van de I-lijn met de boonen van een zuivere lijn, waarvan de lengte minder groot is dan die van boonen van de I-lijn, doch grooter dan die van de II-lijn? Verschillende kruisingen stellen verschillende eischen voor de toeneming der afmetingen door een gen. De volgende hypothese behoudt het voordeel van hypothese 2, dat er voor de 3 afmetingen een even groot aantal genen is, doch verbindt er aan, dat de genen eenzelfde toeneming der afmetingen geven.

3. Het mendelistisch gezichtspunt brengt mee, dat we bij het opstellen van mendelistische formules de konsekwentie van het mendelisme aanvaarden, die eischt, dat als we 2 varieteiten kruisen, die in enkele eigenschappen verschillen, ieder van die 2 varieteiten de genen moet hebben (als aanwezig of als afwezig) voor al deze eigenschappen. In het geval van de afmetingen der boonen moeten we aannemen, dat van de twee elkaar geheel vreemde boonenrassen, de I- en de II-lijn, de I-lijn de genen voor alle polymere factoren van de II-lijn bezit (als „afwezig") en de II-lijn die van de I-lijn (eveneens als „afwezig"). We schrijven daarom de formule van de boonen van de I-lijn als L L B B d d en die van de II-lijn als l l b b D D. De mogelijkheid der kruising — mendelistisch opgevat — vooronderstelt deze situatie. We kunnen daarom — in mendelistischen zin — de kruising van de boonen van de I-lijn en de II-lijn nog als volgt zien. We behoeven, om het zooveel grootere verschil der lengten van de boonen van de I- en de II-lijn dan van de breedten en de dikten, in formule te brengen, niet aan te nemen, dat er evenveel L-genen als B- en D-genen zijn en dat de lengte-vermeerdering door het L-gen belangrijk grooter is dan door het B-gen en het D-gen, doch kunnen ook aannemen, ten 1e, dat een L-gen een relatief even groote toeneming van de lengte geeft als het B-gen van de breedte en het D-gen van de dikte, dat ten 2e er een even groot aantal L-genen als B- en D-genen is, doch dat ten 3e eenige B- en D-genen, zoowel in de I-lijn als in de II-lijn alleen als b b en d d, dus als „afwezig" voorkomen. Bij onze aanneming van 6 L-factoren en 3 B- en 3 D-factoren wordt de formule voor de boonen van de I-lijn dan $L_1 L_1 — L_6 L_6 B_1 B_1 \ldots B_3 B_3 b_4 b_4 \ldots b_6 b_6 d_1 d_1 \ldots d_3 d_3 d_4 d_4 \ldots d_6 d_6$ en die voor de II-lijn $l_1 l_1 \ldots l_6 l_6 b_1 b_1 \ldots b_6 b_6 D_1 D_1 \ldots D_3 D_3 d_4 d_4 \ldots$ $\ldots d_6 d_6$. De formule voor F_1-boonen is dan $L_1 l_1 \ldots L_6 l_6 B_1 b_1 \ldots$ $\ldots B_3 b_3 b_4 b_4 \ldots b_6 b_6 D_1 d_1 \ldots D_3 d_3 d_4 d_4 \ldots d_6 d_6$ en in de F_2-generatie zal er een klasse boonen zijn met $L_1 — L_6$, $B_1 — B_3$ en $D_1 — D_3$ (in homozygoten en heterozygoten vorm) in de formule en ook een klasse met $L_1 — L_3$ (of $L_4 — L_6$), $B_1 — B_3$ en $D_1 — D_3$ (in homozygoten en heterozygoten vorm) in de formule. De eenvoudigere schrijfwijze, waarbij we het aantal polymere factoren, dat alle 3 afmetingen hebben, voor de lengte door L_1 voorstellen en het aantal, dat alleen de lengte heeft door L_2 — in de breedte en de dikte zijn overeenkomstige breedte- respect. diktefactoren „afwezig" — is overzichtelijker. Volgens deze vereenvoudigde schrijf-

wijze is de formule voor de boonen van de I-lijn $L_1 L_1 L_2 L_2 B_1 B_1 b_2 b_2 d_1 d_1 d_2 d_2$ en die voor boonen van de II-lijn $l_1 l_1 l_2 l_2 b_1 b_1 b_2 b_2 D_1 D_1 d_2 d_2$ en de formule van de F_1-kruisingsboonen is dan $L_1 l_1 L_2 l_2 B_1 b_1 b_2 b_2 D_1 d_1 d_2 d_2$.

In deze formule van 6 factorenparen zijn er 4 in heterozygoten en 2 in homozygoten vorm. In de F_2-generatie is er dus splitsing van 4 factoren: de splitsing verloopt volgens het tetrahybride schema. (Volgens de hypothese 2 is er splitsing volgens het trihybriede schema.) En we treffen er een klasse boonen bij aan, die zeer lang, matig breed en matig dik zijn en een 2e klasse boonen, die matig lang, matig breed en matig dik zijn.

Deze klassen ontstaan bij volledige dominantie van de groote over de kleine afmetingen; van de boonen van de I- en de II-lijn vinden we slechts een onvolledige dominantie. Ook werkt de zeer groote niet-erfelijke variabiliteit storend (blz. 800). Het is de vraag, of het met behulp van statistische methoden mogelijk zal zijn, om de hypothesen 2 en 3, die, volgens deze hypothesen, in F_2 verschillend samengestelde boonenopbrengsten geven, aan ons materiaal te toetsen. Door individueel voortkweeken (blz. 800), — en dit is belangrijk —, kan het voorkomen van $L_1 L_2 B_1 b_2 D_1 d_2$ en van $L_1 l_2 B_1 b_2 D_1 d_2$ varianten, d.i. van erfelijke variaties met een zeer groote lengte, een matig groote breedte en een matig groote dikte ev. van erfelijke variaties met een matig groote lengte en eveneens een matig groote breedte en matig groote dikte worden vastgesteld, en dus de beslissing over de hypothesen 2 en 3 bereikt worden. Een voordeel van de hypothese 3 is, dat ze algemeen voor de erfelijkheid van de afmetingen van boonen van zuivere lijnen toepassing kan vinden. Het schijnt me toe, dat de erfelijkheid van den hoofdvorm (afmetingen en indices) bij den mensch ook volgens deze hypothese de beste verklaring vinden kan. We kunnen voor de verklaring van de erfelijkheid van afmetingen naast de werking van polymere factoren ook nog nevenwerkingen van andere genen aannemen.

Samenvatting. De erkenning van de gebondenheid van de afmetingen bij den groei, door het reguleerende principe, bleek tot grondslag genomen te kunnen worden voor de mendelistische verklaring van de erfelijkheid door polymere factoren van de afmetingen en het gewicht van de boonen van ons onderzoek. We nemen aan, dat er een even groot aantal genen voor het grootteverschil van alle 3 afmetingen is, die ieder eenzelfde groottetoeneming geven, zoowel dus voor de lengte, de breedte, als de dikte. Steeds werken bij den groei een gen voor de lengte, een voor de breedte en een voor de dikte samen en ze zijn in eenzelfde chromosoom gelegen. Bij de door ons onderzochte zuivere lijnen I en II is voor de lengte in de I-lijn het heele aantal aangenomen factoren in den dominant homozygoten vorm aanwezig, voor de breedte is een kleiner aantal aanwezig, evenals voor de dikte in de II-lijn.

Samenvatting. In deze mededeeling wordt de mendelistische verklaring van de erfelijkheid door polymere factoren besproken in haar beteekenis voor ons inzicht in de zelfstandigheid en de gebondenheid van de afmetingen van de zaden van Phaseolus vulgaris.

De schrijver bestudeerde de erfelijkheid bij de kruising van boonen van 2 zuivere lijnen. De boonen van de I-lijn zijn lang, breed en dun, die van de II-lijn zijn kort, smal en dik. Het verschil van de lengten van de boonen van de I- en de II-lijn is veel grooter dan dat van de breedten en de dikten; van de dikten is het verschil in omgekeerde richting. De formules zijn dus L L B B d d en l l b b D D voor de oudervormen en L_1 l_1 B_1 b_1 D d_1 voor de F_1-boonen.

Er worden achtereenvolgens een 4-tal hypothesen opgesteld (1, 1a, 2 en 3). Schr. aanvaardt hypothese 3. Om het zooveel grootere verschil van de lengten van de boonen van de I- en de II-lijn dan van de breedten en de dikten in de formules uit te drukken, wordt bij hypothese 3 aangenomen, dat er een even groot aantal polymere factoren voor de lengte, de breedte en de dikte zijn, die ieder een even groote toeneming van de lengte, de breedte en de dikte geven. Bovendien wordt aangenomen, dat deze polymere factoren voor de lengtevermeerdering aanwezig zijn in den homozygoten dominanten vorm, dus als L L, terwijl van de factoren voor de breedte-, resp. de diktevermeerdering er eenige in den homozygoten recessieven vorm aanwezig zijn dus als b b en d d.

Als 6 polymere factoren worden aangenomen en als ter vereenvoudiging van de schrijfwijze de 3 factoren van de eerste groep, die alle 3 afmetingen in den homozygoten dominanten vorm hebben als L_1 L_1, B_1 B_1 en D_1 D_1 geschreven worden en die van de 2de groep als L_2 L_2, b_2 b_2 en d_2 d_2, dan krijgen we voor de boonen van de I- en van de II-lijn de volgende formules L_1 L_1 L_2 L_2 B_1 B_1 b_2 b_2 d_1 d_1 d_2 d_2 en l_1 l_1 l_2 l_2 b_1 b_1 b_2 b_2 D_1 D_1 d_2 d_2 en voor F_1-boonen is dan de formule L_1 l_1 L_2 l_2 B_1 b_1 b_2 b_2 D_1 d_1 d_2 d_2. We zien, dat we in deze formule met 4 heterozygote factorenverbindingen te doen hebben. De kruising verloopt dus volgens het tetrahybride schema.

De beteekenis van deze hypothese is, dat bij de werkzaamheid van de genen, dus bij den groei, steeds een factor voor de lengtetoeneming, een voor die van de breedte en een voor die van de dikte, tegelijk werkzaam zijn. We hebben bij vroegere onderzoekingen van den groei en de variabiliteit gevonden, dat er een zekere mate van zelfstandigheid en van gebondenheid is van de afmetingen. Bij de hypothese 3 is deze gebondenheid en zelfstandigheid ook in de erfelijkheid, die hier de mendelistische erfelijkheid is, verwerkelijkt. Het ligt voor de hand om aan te nemen, dat steeds een factor voor de lengte, een voor de breedte en een voor de dikte in hetzelfde chromosoom gelegen zijn. De studie van de splitsing biedt de mogelijkheid, om de hypothese nader te toetsen.

Summary. In this communication the mendelian interpretation of heredity by polymere factors is given in its value for our opinion of the measure of independance of the dimensions of the seeds of Phaseolus vulgaris.

The author studied the heredity with the crossing of beans of 2 pure lines. The beans of the I-line are long, wide and thin, those of the II-line are short, narrow and thick. The difference of the lengths of the beans of the I- and the II-line is much larger than that of the breadths and the thicknesses; the latter is in inverted direction. The formulas of the parent forms are L L B B d d and l l b b D D and of the F_1-beans $L_1\, l_1\, B_1\, b_1\, D_1\, d_1$.

Four hypotheses are successively drawn up (1, 1*a*, 2 and 3). The author accepts hypothesis 3. There is here an equal number of polymere factors for the length, the breadth and the thickness and the increase of the length that a factor for the length gives is relatively as great as the increase of the breadth that a factor for the breadth gives and the increase of the thickness by a factor for the thickness.

All polymere factors for the increase of the length are in the homozygous and dominant form (L L); some factors for the increase of the breadth are in the homozygous and dominant form (B B), other of these factors are in the homozygous recessive form (b b); the same is true for the factors for the thickness (D D and d d). The much greater difference of the lengths of the beans of the I- and the II-line than of the breadths and the thicknesses is taken into consideration in this way.

According to hypothesis 3 the formulas for the beans of the I-line and of the II-line are written as $L_1\, L_1\, L_2\, L_2\, B_1\, B_1\, b_2\, b_2\, d_1\, d_1\, d_2\, d_2$ and $l_1\, l_1\, l_2\, l_2\, b_1\, b_1\, b_2\, b_2\, D_1\, D_1\, d_2\, d_2$, in which L_1, B_1 and D_1 represent the polymere factors of the 1st group of factors and L_2, B_2 and D_2 those of the 2nd group. The formula of F_1-beans is $L_1\, l_1\, L_2\, l_2\, B_1\, b_1\, b_2\, b_2\, D_1\, d_1\, d_2\, d_2$. We see that 4 factors in the heterozygous form occur in this formula. It is a cross of the tetrahybrid scheme.

The sense of this hypothesis is, that a factor for the increase of the length, a factor for the increase of the breadth and a factor for the increase of the thickness are always active at the same time. We found with earlier investigations of the growth and of variation of beans, that the dimensions are dependent of each other in some degree; there is a compensational growth, a principle of regulation. The principle of regulation is also realized in the mendelian heredity, as it is expressed in hypothesis 3. It is obvious to assume that a factor for the length, a factor for the breadth, and a factor for the thickness always lie in the same chromosome.

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Anatomy. — *Variation- and correlationcoefficient.* By A. DE FROE, J. HUIZINGA and J. VAN GOOL. (Communicated by Prof. M. W. WOERDEMAN.)

(Communicated at the meeting of June 28, 1947.)

Introduction.

In anthropology it has gradually become customary to study mass-phenomenons with the help of statistical methods.

It is very remarkable that, among anthropologists there are two groups which regarding statistics are taking up very extreme positions.

On the one side a group, which following KARL PEARSON is actually prospering in composing an ever more refined and subtler, statistical apparatus and which thereby uses the anthropological facts merely as paradigmata to its mathematical theories: the biometrical group.

On the other side a group of anthropologists, which opposes the use of statistics or applies some out of date or superficial methods to save appearances and who are actually looking down with contempt upon the mathematical caprices of their colleagues.

This latter group may be found among the investigators in palaeontology and praehistory wherein up to now no mass-phenomenons have occurred. Our point of view now is, that reflecting the essentials of anthropology we find both groups entirely in the wrong.

In the first place anthropology is a biological science, not a mathematical one. Essential are the biological facts, their ordination and interpretation. In statistical-operation of anthropological data the biological outlook should never be disregarded.

On the other hand, the anthropologist is examining large groups of people. He has to ordinate the yielded data and also this ordination actually is a statistical one.

Statistics are to the anthropologist nothing else but the ordination and allowable simplification of the obtained data.

The anthropologist of today should therefore apply statistical methods on his obtained stock of facts.

Our opinion is, that with respect to this problem, the task of the anthropologist is not yet finished.

It is our conviction that the anthropologist ought to test the used statistical methods for tolerance, efficiency and usefullness.

One of the methods that can be applied is to make use of different methods and to compare the results afterwards. Problems will arise such as could not have been anticipated beforehand and which possess a great instructive value to the anthropologist in making him perceive pitfalls constantly that would have remained hidden through superficial application.

One of the problems that will fit into this conception immediately, is the one concerning the relation between indices and correlations.

An index in anthropology is the name for the proportional relation of two detail-measures whereby the smallest is expressed as a percentage of the largest. So the index is a way to express a form or structure which could also be determined by expressions like long, short, broad, narrow, high, low etc.

Better expressions would be round, elliptical, slender, plump etc. for they indicate the specific forms.

However much these singular expressions tend to suggest singular quality or character, the index still remains an arithmetical quotient of two metrical data. This leads to the problem of the so-called spurious correlation.

The relation between variability of an index and the correlation between variables (parameters) out of which the index is calculated and the problem of spurious correlation will be studied hereafter.

§ 1. *The relation between variability of an index and the correlation of the variables from which the index has been calculated.*

We may consider the correlationcoefficient to be an expression of the degree to which the points we have represented in a two-axial rectangular system of coordinates, are inclined to arrange themselves in a straight line.

This view lead to a proposal from statistically minded men to change the name correlationcoefficient into linearitycoefficient according to FRECHET.

When a high correlationcoefficient is involved (to indicate a greater liniar-connection of the variables) one can, superficially thinking, expect the quotients (indices) of the variables to fluctuate between more narrow limits than in case of absence of tendency toward linearity i.e. with very high correlationcoefficients the variationcoefficients of indices of variables between which the correlation is calculated will be small and vice versa: with a very low correlationcoefficient a high variationcoefficient of the index has to be expected.

With regard to this problem we find DE FROE's statement [4]), that at positive correlation the correlationcoefficient of two measures and the variationcoefficient of the index of those measures, would show a relation, which could be represented grafically as an orthogonal hyperbola.

Let us examine this opinion closer with the aid of data adopted from the thesis of DE FROE [3]) from which we were able to collect the following figures of correlationcoefficients, variationcoefficient, etc. etc. (Table 1.)

As abbreviations are being used:

V_i for variationcoefficient of an index

r for correlationcoefficient of the variables, realised in the additional index.

If V_i and r should show a functional connection, which can be represented by an orthogonal hyperbola, than this can be indicated as $V_i \cdot r = \text{constant}$. If we check this up with our data, we get table 1.

TABLE 1.

Nr.	V_i	r	$V_i \, r$
1	5.25	0.320	1.667
2	6.19	0.228	1.411
3	6.50	0.232	1.508
4	8.82	0.134	1.182
5	6.92	0.249	1.723
6	7.02	0.508	3.566
7	11.47	0.269	3.086
8	6.65	0.103	0.685
9	6.39	0.198	1.265
10	14.04	0.028	0.393
11	4.69	0.674	3.161
12	5.30	0.512	2.714
13	5.54	0.528	2.925

There appears to be no question of constancy in $V_i \cdot r$; the supposed hyperbolic relation is not present.

In accordance with the said supposition is the fact (and may be its cause), that with the lowest value of r the highest value of V_i is found (number 10), while with the highest value of r corresponds the lowest V_i (number 11).

For the intermediate values this connection is missing in most cases. The same symptom occurs with data adopted from a not yet published investigation of femura, the extremes are inversely proportional to each other, the intermediate values are not affected in this way.

So an extremely low correlationcoefficient causes a greater variation of the index than a high correlationcoefficient.

We must point out however, that, when we find a low correlationcoefficient we cannot always expect that the points we get in our system of coordinates will be represented entirely chaotically.

A poor linearity by no means implicates disorder and every kind of functional relation except linearity may be present.

Considering more closely the variation of the index we can stipulate that as we are dealing with a quotient both the numerator and the denumerator will influence the variability of the index.

A great variability (among other things) can be established as well by an ample variation of the numerator (ev. denumerator) combined with lesser variability of the denumerator (ev. numerator), as by a great variability of both factors at the same time.

The connection between correlationcoefficient and variability of the measures concerned, we find expressed in a formula which has been derived theoretically by PEARSON (formula 1)

$$r = \frac{v_n^2 + v_t^2 - v_i^2}{2\, v_n\, v_t} \quad (1)$$

In this formula the said influence of the variability of numerator (V_t) and denumerator (V_n) becomes obvious and to such an extent that V_n and V_t are taking up entirely equivalent places in the formula.

PEARSON has been neglecting certain things.

With respect to the following it is important to be more exact.

Suppose we are considering the value of two variables taken from a group of individuals. The mean values of those two variables we denote M_t and M_n.

If we select one individual from the group and assign the values of each of those variables than these values will practically never be equal to the mean-values but will deviate from it more or less; these deviations we denote resp. α and β.

Now PEARSON neglects in his deduction the quadratics and all higher powers of $\frac{\alpha}{M_n}$ and $\frac{\beta}{M_t}$. This may only be permitted if these values are small; if they are great, differences may occur between the correlationcoefficient calculated with the said formula of PEARSON and the one computed in the usual way. Hence PEARSON's formula is a theoretical-approximation.

In practice however it generally turns out to suffice very well; to this we will return later.

Regarding the following it is also important that with PEARSON's omissions one can deduct mathematically

$$M_i = \frac{M_t}{M_n}$$

where M_i is the average of indices which can be calculated from the said variables and M_t and M_n are the averages of the variables who appear resp. in the numerator and the denumerator of the index.

Let us substitute some particular values of the correlationcoefficient into the right side of the formula.

1. If $r = 0$ is substituted in equation 1 we get

$$v_i^2 = v_n^2 + v_t^2.$$

This simple relation may not be used, however to determine V_i from calculated values of V_t and V_n when the correlationcoefficient involved is approaching to zero. Because, as is shown in the above mentioned, unaltered formula of PEARSON, the correlationcoefficient, however small, is multiplied by $2\,V_n\,V_t$ to calculate V_i from the given V_t and V_n, which product will have a considerable and certainly not to be neglected value.

It is a pity that we have no small value of r at our disposal to illustrate this with numerical example.

At the same time this formula shows that, if we use two indices whereby the correlationcoefficients of the variables used in numerator and denumerator are in both cases 0, that index will vary most which has the largest $v_n^2 + v_t^2$.

2. For $r = 1$ the equation changes into $V_i = | V_t - V_n |$.

Here again applies that with a correlationcoefficient approaching 1, this reduced formula may not be used to calculate V_i from the given V_t and V_n.

This is shown as follows:

If we write the unaltered formula as $V_i^2 = V_n^2 + V_t^2 - 2\ V_n\ V_t\ r$ and compare this equation with the one we get for V_i^2 by squaring the reduced formula: $V_i^2 = V_n^2 + V_t^2 - 2\ V_n\ V_t$.

The difference lies with the last term of the right side.

In consequence, the value of the right term will be too low, i.o.w. we find a V_i much too low.

We can strikingly illustrate this with a numerical example:

Given: a calculated correlationcoefficient 0,983; $V_t = 7{,}90$; $V_n = 8{,}41$. From the formula where r is assumed to be 1 we get $V_i = 0{,}54$, while the calculated value, thus the correct V_i, amounts to 2,05.

Next to PEARSON, YULE in his wellknown book (9) derives a formula in which a connection between the magnitudes employed by us is also expressed. His derivation differs somewhat from PEARSON's (2) which is expressed in the final notation. YULE writes

$$r = \frac{V_t^2 + V_n^2 - \left(\frac{\sigma_i}{M_t/M_n}\right)^2}{2\ V_t\ V_n} \quad (2)$$

where σ_i is the standard deviation of the index.

CZUBER–BURKHARDT derive a formula and arrive at the same result as YULE (2). In the beginning of the derivation of formula (2) YULE neglects all powers higher than the second (PEARSON already neglected the second), and sofar the design of YULE's derivation is more accurate.

YULE does not find $M_i = \frac{M_t}{M_n}$ (3), but

$$M_i = \frac{M_t}{M_n}(1 - r\ V_t\ V_n + V_n^2) \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

which is more exact than (3).

In the course of his derivation however YULE neglects still more, which gives his final results no greater accuracy than PEARSON's formula. This is also proved by our results; the correlationcoefficients calculated from PEARSON's formula and from YULE's show no apparent difference (see table 2).

PEARSON's formula is preferable, the calculation being somewhat simpler. Furthermore it becomes clear from the preceding why YULE's formula contains $\left(\frac{\sigma_i}{M_t/M_n}\right)^2$ instead of the V_i^2 from PEARSON's formula. From PEARSON's assumption followed $M_i = \frac{M_t}{M_n}$ and as $V_i = \frac{\sigma_i}{M_i}$ we get $\frac{\sigma_i}{M_t/M_n} = \frac{\sigma_i}{M_i} = V_i$ accordingly.

YULE on the other hand finds $M_i \neq \frac{M_t}{M_n}$ (4) and he will keep $\frac{M_t}{M_n}$ as a final result.

Formula (4) which indicates the difference between the mean index M_i and the index of means $\frac{M_t}{M_n}$, has already been studied by some former anthropologists, not primarily in its mathematical aspect but as to the question which of the two is the best one.

BROCA in 1875 pleaded already for the index of means "qui est le véritable indice moyen" and because "le but des relevés craniométriques est de constituer un crâne moyen, caractérisé par l'état moyen de ses éléments craniométriques".

For less idealistic reasons, though holding more practical truth than those of BROCA, TOPINARD declares himself a defender of the index of means.

In SCHMIDT on the contrary, we find the means of indices prevalent, while the point of view of the two Frenchmen is denounced by MARTIN (5) as follows: "diese letztere von BROCA empfohlene Berechnungsmethode führt aber bei wenig homogenen Reihen zu ungenauen Resultaten und ist daher zu verwerfen".

MARTIN's statement as to the result becoming inaccurate if data used of a poor homogenity, indicates a role of the variationcoefficient.

Apparently MARTIN did not know YULE's above mentioned formula. As for the absolute differences existing between the means of indices calculated in both ways, these appeared to amount to 1 index-unit, for one out of 26 cases calculated by us, in the other cases always less; we may assign a half index-unit as the mean absolute difference.

The percentage-differences were less than 1 %; only in case the absolute magnitude of the variables composing the index is small, this percentage difference goes up to 2 %.

In most cases, the difference (index of means — mean of indices) appeared to the positive, less often negative, while evidently equivalence of both results can be considered an exception.

In nearly all cases the deviation does not exceed the limits of error of the mean index.

Reviewing what has been discussed in § 1, we see that the relation between the variability of index of two variables and the existing correlation between these two, cannot be represented by a simple orthogonal hyperbola, but is slightly more complicated, which is shown by the formulas of PEARSON and YULE. These formulas show little difference in practice, but the theoretical difference led to the problem of the mean index and the index of means. YULE also gave an accurate formula for this, which in practice suffices also splendidly.

In actual practice an important consequence of the discussed is, that aided by the formulas of PEARSON and YULE and of the mean index

formulas (resp. 1, 2 en 4) the correlationcoefficient between two variables can be calculated in a very simple manner, if their variationcoefficients and variationcoefficient of their index are known.

The results of the different formulas will be summed up in the next table 2.

TABLE 2.

	r BRAVAIS	r PEARSON	r YULE
1	0.983±0.008	0.969	0.969
2	0.855±0.027	0.848	0.845
3	0.771±0.045	0.781	0.782
4	0.770±0.038	0.767	0.768
5	0.798±0.034	0.797	0.798
6	0.804±0.039	0.807	0.806
7	0.745±0.048	0.760	0.761
8	0.824±0.033	0.842	0.844
9	0.835±0.024	0.809	0.810
10	0.710±0.047	0.703	0.704
11	0.719±0.050	0.694	0.693
12	0.592±0.068	0.579	0.575
13	0.683±0.046	0.667	0.665
14	0.645±0.046	0.607	0.607
15	0.764±0.033	0.758	0.755
16	0.516±0.063	0.513	0.512
17	0.576±0.063	0.579	0.574
18	0.628±0.053	0.626	0.622
19	0.668±0.044	0.685	0.683
20	0.418±0.070	0.389	0.382
21	0.536±0.057	0.465	0.461
22	0 225±0.076	0.232	0.232
23	0.519±0.058	0.560	0.554
24	0.244±0.084	0.127	0.103

The table shown above, indicates:

1. that the results obtained with PEARSON's and YULE's formulas are practically the same, PEARSON's formula being slightly simpler in its calculation is therefore preferable to YULE's.
2. that the results obtained by BRAVAIS and PEARSON (YULE) only differ occasionally and to such an extent that PEARSON's correlationcoefficient exceeds the limit of error of the correlationcoefficient of BRAVAIS.

   Concerning this latter statement we must consider that:

   a. the chance of mathematical-errors when using BRAVAIS' method is rather great.

   b. in the calculations we used numbers which were rounded off to 1 or 2 places of decimal. In our original aim, more places of decimal were not necessary.

From our point of view this is the most important explanation of the differences discussed above.

At the same time this is the reason why we did not calculate the correlationcoefficient according to the mean index (4), as in this case the influence of rounding off had to express itself much stronger. It should be mentioned that theoretically spoken formula 4 is the most accurate, and this will certainly become evident in practice, in view of the good results we obtained using the inaccurate formula of PEARSON.

We hope to show this later with other data.

§ 2. *The "Spurious Correlation".*

Let us try now, guided by the above discussed formulas, to obtain a conception of the nature of the so-called spurious correlation, which occurs at calculation of the correlationcoefficient between two indices with the same denumerator.

Suppose we want to know the correlationcoefficient between the indices A/C and B/C, making use of their variationcoefficient we now can apply PEARSON's formula, from which we get as correlationcoefficient:

$$r_{A/C;\,BC} = \frac{V_{A/C}^2 + V_{B/C}^2 - V_{A/C\,/\,BC}}{2\,V_{A/C}\,V_{B/C}}$$

With the aid of PEARSON's formula we can again change the variationcoefficients of the indices as they appear in the above-shown equation, for the variationcoefficients of the variables composing the index and their mutual correlationcoefficient, from which we derive:

$$r_{A/C;\,B/C} = \frac{V_C^2 - V_A\,V_C\,r_{A,C} - V_B\,V_C\,r_{B,C} + V_A\,V_B\,r_{A,B}}{\sqrt{(V_A^2 + V_C^2 - 2\,V_A\,V_C\,r_{A,C})\,(V_B^2 + V_C^2 - 2\,V_B\,V_C\,r_{B,C})}}$$

If we assume total absence of correlation between the variables A, B and C the formule changes into:

$$r_{A/C;\,B/C} = \frac{V_C^2}{\sqrt{(V_A^2 + V_C^2)\,(V_B^2 + V_C^2)}}$$

Now we can see, that even with total absence of any correlation between the variables concerned, the correlationcoefficient of the indices shows a not to be neglected magnitude.

If there is some correlation between the variables, then the correlationcoefficient between the indices will be different from the last-mentioned value of r; therefore to obtain a proper conception of the correlation between the indices, we must, according to PEARSON, substract a quantity from the correlationcoefficient, of which the magnitude is given by the magnitude of the correlationcoefficient between the indices in absence of any correlation between the mutual variables composing the indices.

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Anatomy. — *On spatial relations of the principal skull-measures.* By A. DE FROE. (Communicated by Prof. M. W. WOERDEMAN.)

(Communicated at the meeting of June 28, 1947.)

In craniometrics the shape of the skull is usually expressed in a quotient of two of the principal skull measures successively, which quotient, in order to simplify matters, is multiplied by 100, thus expressing one measure as a percentage of the other.

The short and accurate numerical expression of the form is of importance to the study of the cranium and its main proportions:

1. to the shape-analysis of the cranium in itself.
2. to the study of the variability of the shape of the cranium.
3. to the comparative morphology of the cranium.
4. to the research of the deviating skull-measurements, for instance those arising from the premature synostosis of the sutures.
5. to the study of the growth of the skull.
6. to the investigation of sex-differences.
7. to the research of the causalities determining the shape of the cranium: heredity, peristasis.

The expression of a form by a quotient of the principal measures, which quotient has obtained the suggestive title of "Index", is only then permitted:

1. when the choice of the measures is correct.
2. when the measures are highly and positively correlative.
3. when and if the sizes have a constant, structural relation in space, with regard to each-other.

The choice of the correct sizes will not be discussed here, also because the research of both other requirements may contribute to this choice.

As for the second requirement we refer to previous publications.

The mutual relations between these principal skull-measurements are definitely positive. There exists a higher degree of correlation in the cranium of the female than in that of the male.

In woman the correlation-figure is:

between skull-length and width	+ 0.460	± 0.085
between skull-length and height	+ 0.471	± 0.084
between skull-width and height	+ 0.425	± 0.089

In man the correlation-figure is:

between skull-length and width	+ 0.320	± 0.063
between skull-length and height	+ 0.228	± 0.068
between skull-width and height	+ 0.232	± 0.067

All values thus found are highly significant. A high correlative value however cannot be observed.

With regard to the third requirement it has to be considered that it would not be justified to demand a completely consistent, structural relationship, as we are not dealing with a product of handicraft but with a biological object, yet.

either 1. the fluctuations may not extensively influence the shape of the cranium;

or 2. if they do, we must accurately know their dispersion and trace the extent of their influence on our judgment;

or 3. we have to attempt to restrict these fluctuations by looking for different sizes.

These, also from a biological viewpoint, reasonable demands, remained hitherto alien to craniometrics.

The research brings about two main difficulties:

1. the *covariation* in shape and size of the skull, excluding the comparison of the skulls in itself. We can however exclude size by making diagrams and bringing those back to the same scale.
2. the orientation of the skull.

A natural, obvious, self-evident orientation of the skull or of the head cannot be indicated.

Working scientifically, we must limit ourselves to the question:

Is there a position to be found, which enlarges the comparability?

Of all positions we have been examining, that one sufficed best where the Frankforter Horizontal and the orbitalia are running concurrently. Nevertheless one could think of a better position.

Of 200 definitely male and 90 definitely female skulls, the mutual structural relation was examined, according to the following six data.

1. the Frankfurther Horizontal.
2. length of skull: Glabella-opisthocranion.
3. breadth of skull: euryon(d)-euryon(s).
4. height of skull: basion-bregma.
5. form of skull.
6. capacity of skull measured by the seedcorn-method.

To express the form of the skull in a manner, suiting this investigation, no use could be made of one single index, which only represents the skull in one single plane-section.

That is why length, breadth and height were expressed as a percentage of their sum, and according to their percentage-predomination groups were formed like this:

long, broad or high; long & broad; long & high; broad & high etc., etc.

The results of a similar grouping characterize the form of the skull very well, but are exclusively operative with the examined data.

We shall now confine ourselves to a rough reproduction of the definitely positive results of our investigation.

I. *The structural relations.*

1. apart from asymmetricals the position of the breadth with regard to the length varies in two ways:

A. the breadth may a) be situated above the length,
b) be just intersecting the length,
c) be situated under the length.

B. the breadth may be found in the course of the length in a more forward or a more rearward position.

A. Of 200 male skulls, 124 have the breadth situated above the length average 9.6 mm maximum 30 mm.

67 skulls had the breadth under the length, on an average of 5.1 mm and to a maximum of 23 mm.

9 skulls had the breadth intersecting the length.

In the female skull the relations are about the same.

B. The breadth lies perpendicular-projective to the length, apart from a few cases always nearer to the opisthocranion and at an average distance of 44.4 % of the length, varying from 35—51 %. This projectionpoint is always to be found behind the intersectionpoint of length and height. These relations correspond in the male and female skull.

2. The position of the height varies, with regard to that of the length in three ways:

A. The angle with which the height intersects the length varies, i.e. the height inclines more or less forward.

B. The point where height and length intersect, varies; this point may be situated more or less to the front.

C. The point where the length intersects the height, may be found to lie higher or lower in the course of the height.

A. The angle between skull-length and -height is found to be varying in male skulls from $66\frac{1}{2}$—$93\frac{1}{2}°$ (27°), its average is $83\frac{1}{2}°$.

The same average was found in female skulls.

The variation of this angle is greater than that of the angles between height & Frankfurter Horizontal and between length & Frankfurter Horizontal.

B. Height and length always intersect in front of the projectionpoint of the breadth.

Except a few cases the intersectionpoint in male skulls lies closer to the glabella, and at an average distance of 46 % of the length, varying from 42—51 %.

In female skulls this intersectionpoint lies distinctly more to the front, the average of 45 %, as well as the minimum and maximum 40 and 48 %.

In the female skull this point and the breadthpoint have a greater distance between them, than in the male skull.

C. The length intersects the height always closer to the basion than to the bregma.

In men the intersection-point lies at an average distance of 39.9 % from the height of the basion, varying from 27.5—48.3 %.

In women this average is 40.3 %, varying from 32.6—48.1 %.

II. *The connections.*

1. A connection of these data with the capacity of the skull generally fails. In male skulls the breadthpoint lies more to the front in larger skulls.

2. A connection with the form of the skull is generally present and often even evident, meaning that, in short, narrow, but high skulls (the female type):

   1. the breadth often lies above the length and also occasionally higher.
   2. the height is more inclining to the front.
   3. the intersectionpoint of length and height lies more in front.
   4. it is not so evident that the breadthpoint lies more backward.

3. The mutual connection is as follows:

   1. higher situated breadth correlates with more forward inclined height.
   2. higher situated breadth correlates with more backward-situated breadth. This corresponds to the correlation of the form.

4. When a great angle exists between the length and Frankfurter Horizontal:

   a. the breadth lies distinctly higher above the length.
   b. the height inclines more to the front we find this angle to be 1° higher on an average, in the female, than in the male.
   c. this angle evidently correlates with the angles between length and F.H. and between height and F.H.
   d. this angle correlates milaty with the form of the skull.
   e. this angle correlates with the capacity of the skull in women, a greater angle in larger skulls.

Conclusion.

We are negative in our answer to the question whether, as regards to the principal skull-measures, there might exist a reasonable, consistent, structural relation.

But not only an ample variation of the mutual positions exists, the variation of the positions itself shows a distinct influence upon the form of the skull.

This influence shows determinedly systematical, distinct connection.

1. Especially the height of the skull is involved.
2. Distinct sex-differences can be observed in it.
3. It bore the view that the orientation of the skull to the F.H. is not the same for each skull.

From this inquiry may finally be shown:

1. that the height may not be disregarded as a form-distinguishing measure.
2. that the problem of sex-dimorphism of the skull has not been solved by far.
3. that the results of the hitherto customary, craniometrical investigation, should most certainly be replenished, according to the directions indicated above.

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Physics. — *A calculation of the viscosity and the sedimentation velocity for solutions of large chain-molecules taking into account the hampered flow of the solvent through each chain molecule.* By H. C. BRINKMAN.

ERRATUM.

To our regret the above-mentioned calculation contains some numerical errors. The following corrections should be made.

Formula (17*a*) for the sedimentation constant should read:

$$f_1(\lambda R) = \frac{1 - \frac{T}{\lambda R}}{1 + \frac{3}{2\lambda^2 R^2} - \frac{3T}{2\lambda^3 R^3}} \quad \ldots\ldots \quad (17a)$$

Formula (18*a*) for the viscosity:

$$f_2(\lambda R) = \frac{1 - \frac{3C}{\lambda R} + \frac{3}{\lambda^2 R^2}}{1 + \frac{10}{\lambda^2 R^2} - \frac{30C}{\lambda^3 R^3} + \frac{30}{\lambda^4 R^4}} \quad \ldots\ldots \quad (18a)$$

These results agree with those obtained by DEBIJE (private communication to the author by prof. DEBIJE).

For the viscosity the correction has only a slight influence on the numerical results. In the interval $30 < N < 200$ the function $f_2(N)\sqrt{N}$ becomes larger by an approximately constant amount of 0,04. The exponent in the modified Staudinger law (24) becomes $\alpha = 0{,}71$.

In the same interval of N the function $\sqrt{N}/f_1(N)$ for the sedimentation should be multiplied approximately by a factor 0,7. In the applications this smaller value for the sedimentation results in a smaller value of the length A of a statistical chain-element. For nitrocellulose $A = 42$, for cellulose-acetate $A = 50$.

Chemistry. — *On the crystal structure of strychnine sulfate and selenate.* I. *Cell dimensions and space group.* By C. BOKHOVEN, J. C. SCHOONE and J. M. BIJVOET.

(Communicated at the meeting of September 27, 1947.)

In continuation of our previous work on the direct Fourier synthesis of crystals by comparison of the *X*-ray intensities of isomorphous compounds [1]), we intend to investigate the structure of strychnine.

After the chloride proved itself insuited because of twin formation, the isomorphous sulfate and selenate were chosen for the analysis in spite of their somewhat greater complexity $[(C_{21}H_{22}N_2O_2)_2 \cdot H_2SO_4\ 5\ aq]$.

This first account gives the cell-dimensions and space group as deduced from WEISSENBERG photographs:

	a	*b*	*c*	*β*	*n*	
Sulfate:	35.7 Å	7.53	7.84	107°20′	2	Spacegroup C_2
Selenate:	35.9	7.58	7.90	107°40′	2	

The *S* (*Se*) atom lying on the symmetry centrum of the [010] projection, the application of the Fourier method reduces itself for this projection to the well known simple case of the Phthalocyanine structure [2]).

A first rough synthesis gave a promising result. A detailed account will be published shortly.

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Astronomy. — *On the Temperature of Cometary Nuclei.* By M. G. J. MINNAERT.

(Communicated at the meeting of September 27, 1947.)

It is generally accepted, that comets are formed when some blocks of meteoritic matter come in the vicinity of the Sun and are heated by its radiation. The escaping gases are repelled by radiation pressure and form a tail, which scatters the solar light by optical resonance. Along different lines of approach it has been shown that the nuclei are probably composed of a considerable number of blocks, perhaps of the order of 100—200 meter [1], perhaps of the order of kilometers [2].

For a good understanding of the processes in comets it is important to ascertain the temperature in their nuclei. ZANSTRA, ORLOV and others have for simplicity pictured each meteoritic block as a sphere, which is in stationary condition at each moment, because it radiates an amount of energy equal to what it receives from the sun. Two extreme cases may be considered. (1). If the conductivity is considerable, the size of the nucleus small, the changes in the radiation slow, the rock will assume one and the same temperature throughout and we have: $T = \frac{276^\circ}{\sqrt{r}}$, r being the distance to the Sun in A.U. (2). If the conductivity is infinitely small: $T = \frac{392^\circ}{\sqrt{r}} (\cos \vartheta)^{1/4}$ in each surface element of which the normal is inclined under an angle ϑ to the rays of the Sun; in taking a mean value of $\cos \vartheta$, it should be remembered that $\cos \vartheta$ should be put equal to zero when $\vartheta > 90^\circ$.

Subsequently WURM showed [2] that the comet remains such a short time in the vicinity of the Sun, that a stationary condition is not reached; he computed, that a nucleus of 100 km would assume a temperature of 50° K only after 10 years, which of course is very much longer than the few weeks of a perihelion passage; and his conclusion was, that the nucleus will have actually a temperature of about 10° K only. WURM's argument definitely shows that the heating of the nucleus is to be considered as a dynamic process; but his conclusions would only be valid if the heat was instantaneously distributed through the whole of the nucleus. This manifestly is not true, the conductivity of rocks being very small indeed.

In order to get more exact results, we will compute the temperature distribution in the block at each depth and at each point of the orbit. Owing to the uncertainty considering the composition of the nucleus, we consider successively a stone meteorite and a nickel-iron meteorite. More-

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[1] B. VORONTSOV–VELYAMINOV, Ap. J. **104**, 226, 1947.

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over, we distinguish between the case of periodic comets with elliptic orbits and the case of parabolic or quasi-parabolic orbits.

1. *A stone meteorite in an elliptic orbit.*

The nucleus is continuously heated by a radiation which, though faint, is at least equal to its aphelion value. In the long run, this heat will penetrate the whole block. Therefore by applying the equilibrium calculation at aphelion distance, the following *minimum temperatures* may be found:

ENCKE	142° K
PONS–WINNECKE	122° K
TUTTLE I	90° K
PONS–FORBES	68° K
HALLEY	49° K

Let us consider HALLEY's comet as a representative case. The elements are: $T = 76^{y},02$, $e = 0{,}9673$, $a = 17{,}945$ A.U.; the greatest and the smallest radius vector are respectively 35,31 and 0,587 A.U. The nucleus is pictured as a stoney sphere with a diameter of the order of 1 km. We assume that its material has a specific weight 2,5. The heat conductivity of granite varies between 0,003 and 0,008 cal cm^{-1} sec^{-1} $degree^{-1}$; for acid rocks 0,004 seems a suitable value, for basic rocks 0,006; we will assume 0,005. As the conductivity of quartz glass increases with temperature, while that of ordinary quartz decreases, we will consider this coefficient as independant of T. The specific heat is about the same for quartz, quartz glass, ordinary glass and granite, and varies in the same way with temperature. We assume a smooth curve through the points:

$T =$	70°	120°	220°	330°	440°
$c =$	0,04	0,08	0,14	0,18	0,22

Because of this strong temperature-influence, the equation of heat conduction can not easily be solved by the classical methods. A numerical solution has also the advantage that the results are ready for use and that the tedious substitutions of actual numbers in the analytical formulae are avoided. We will find that the temperature variations are only important up to a depth of some tenths of meters; this being probably only a small fraction of the radius, we will treat the problem as a *plane problem* and apply the ordinary equation of conduction:

$$\frac{\partial T}{\partial t} = \frac{k}{c\varrho} \frac{\partial^2 T}{\partial x^2}. \quad \ldots\ldots\ldots \quad (1)$$

For the differentials we substitute finite differences. Let the nucleus be composed of plane strata, each 20 m thick, their representative central layers being located at depths of 10, 30, 50 ..... meter, and their tempe-

ratures being T_1, T_2, T_3, ... We investigate how the temperatures have changed after an interval of one year $= 3{,}15.10^7$ sec.

$$\triangle T_n = \frac{\triangle t}{(\triangle x)^2} \cdot \frac{k}{\varrho c} \cdot (T_{n+1} + T_{n-1} - 2T_n) = \frac{a}{c}(T_{n+1} + T_{n-1} - 2T_n), \quad (2)$$

where we have put

$$a = \frac{\triangle t}{(\triangle x)^2} \cdot \frac{k}{\varrho} = \frac{3{,}15.10^7}{4.10^6} \cdot \frac{0{,}005}{2{,}5} = 0{,}016.$$

It would be tedious if we had to follow the comet during many revolutions till stationary conditions have developed. A more practical method is to

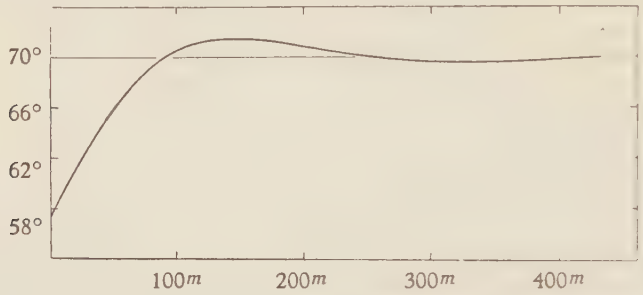


Fig. 1. Temperature distribution in o stone nucleus after one revolution of 76 years.

take as a start a rough estimate of the internal temperature which will be finally approached. This temperature will be somewhat higher than the minimum of 49°, say 70° K.

We assume that, at our start near aphelium, this temperature prevails in all the layers. The exterior layer absorbs solar radiation and radiates infrared radiation to space; let these amounts of energy be exchanged discontinuously at the beginning of each year. The solar radiation at the distance of the earth is known as the solar constant; from this we derive the radiation in our units at a distance r:

$$\frac{1{,}90 \,.\, 3{,}15 \,.\, 10^7}{60 \,.\, r^2 \,.\, 4} = \frac{250\ 000}{r^2} \text{ cal cm}^{-2} \text{ year}^{-1}.$$

The factor 4 has been introduced because the nucleus is probably in spinning motion, so that the radiation, falling on an effective cross section $\pi \varrho^2$, is actually distributed over an area $4 \pi \varrho^2$. The emission of the nucleus amounts to $1{,}37.10^{-12} \,.\, 3{,}15.10^7 \; T^4 = 4330 \left(\frac{T_0}{100}\right)^4$ cal cm^{-2} year^{-1},

T_0 being the temperature of the external radiating layer. The difference between the input and the output of radiation is the heat absorbed by the nucleus. We should take into account that such a rock has an appreciable albedo; however, by comparison with the planetoids, we estimate that this will be only of the order of 0,10, and may therefore be neglected.

We illustrate the method of computation by an example (table 1). On

TABLE I. — Temperature distribution in a stone meteorite for $t = 34^y$ and $t = 35^y$.

t	0	10	30	50	70	90	110	130	150	170	190	210 meter
34,0	67,10	65,97) 68,76)	64,41	63,76	64,40	65,44	66,51	67,44	68,20	68,78	69,20	69,49 m.
			+1,46	+0,51	+0,16	+0,01	−0,06	0,07	−0,07	−0,06	−0,05	−0,01 m.
35,0	70,70	68,76) 73,05)	65,87	64,27	64,56	65,45	66,45	67,37	68,13	68,72	69,15	69,45 m.
			+2,20	+0,74	+0,24	+0,04	−0,03	−0,06	−0,07	−0,06	−0,05	−0,04 m.

the first line we find the mean depth of each layer. On the second line, the temperatures as they are found 34 years after the aphelion. On the third line, the temperature changes during the next year, as computed from equation (2), for all the layers from 40 m on. The fourth line gives the temperatures for the epoch 35 years after the aphelion. The temperature of the first layer is supposed to increase discontinuously by $\triangle T_1$, at the epoch 34,0; if this increase is known, $\triangle T_2$ can also be found. $\triangle T_1$ is determined by the requirement, that the total gain of energy of the block is equal to the excess of the absorption over the emission. This computation will be given now in some detail.

In the interval considered, the mean distance of the comet is 11,5 A.U. and the irradiation 1890 cal. The emission of the nucleus depends on its surface temperature, which could be taken to be 65°,97. But a better approximation is obtained by extrapolating the temperature till the actual surface. A parabolic formula $T = a + bx + cx^2$ is fitted to the points T_1, T_2, T_3, corresponding with depths of 10, 30, 50 m; we find $T_0 = 1{,}875\ T_1 + 0{,}375\ T_3 - 1{,}25\ T_2 = 67°{,}1$. Moreover, judging from the increase between 33 and 34^y, we estimate that the temperature will rise over 1,9 ° during the next half year. The emission therefore should be computed for $T_0 = 69°{,}1$; it amounts to 990 cal.

So the block will have to absorb $I = 1890 - 990 = 900$ cal., which requires

$$I = \varrho \triangle x \left[c_1 \triangle T_1 + c_2 \triangle T_2 + \sum_3^\infty c_n \triangle T_n\right] =$$

$$= \varrho \triangle x \left[c_1 \triangle T_1 + c_2 (T_1' + T_3 - 2T_2) \frac{\triangle t}{(\triangle x)^2} \frac{k}{\varrho c^2} + \sum_3^\infty c_n \triangle T_n\right],$$

where $T_1' = T_1 + \triangle T_1$ is the temperature of the first layer, increased by $\triangle T_1$, at the beginning of the year. We solve the equation for $\triangle T_1$:

$$\left.\begin{aligned} \triangle T_1 &= \frac{I - \varrho \triangle x \left[a (T_1 + T_3 - 2T_2) + \Sigma c_n \triangle T_n\right]}{\varrho \triangle x (c_1 + a)} = \\ &= \frac{I - 5000 \left[0{,}016 (T_1 + T_3 - 2T_2) + \Sigma c_n \triangle T_n\right]}{5000\, c_1 + 80} \end{aligned}\right\} . \quad (3)$$

In many cases, the specific heat is equal to 0,04 at all depths except in the uppermost two strata; we then use the simpler formula:

$$\Delta T_1 = \frac{I - 200\,[0{,}40\,(T_1 + T_3 - 2T_2) + \sum_n^{\infty} \Delta Tn]}{5000\,c_1 + 80}.$$

At the epoch $t = 34_y$,

$$\Delta T_1 = \frac{900 - 200\,[0{,}40\,.\,0{,}91 + 0{,}25]}{200 + 80} = 2°{,}79 \text{ and } \Delta T_2 = 1°{,}46\,.$$

For $t = 35^y$ the surface temperature is found to be 70°,70. Apparently we should have computed the radiation for a mean temperature of $\frac{1}{2}$ (67°,1 + 70°,7) = 68°,9 instead of 69°. This difference will be neglected; in other cases it would be possible of course to repeat the computation and to find a better approximation.

Near the perihelion, the temperatures change so rapidly that a subdivision in smaller intervals is required; we simply divide Δt, I and a by an arbitrary factor. In this way, intervals of $\frac{1}{4}$ year and $\frac{1}{8}$ year were introduced.

The results of the whole computation are summarized in the diagrams 1, 2 and 3. Fig. 1 shows the temperature variation in the uppermost layers

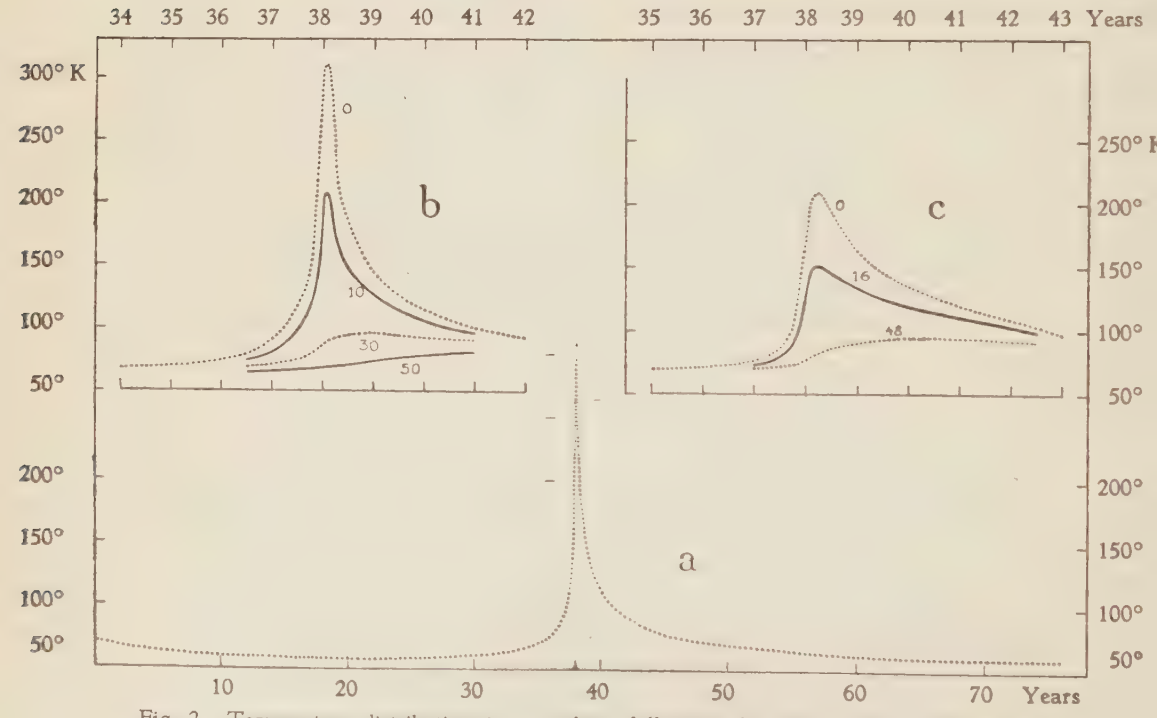


Fig. 2. Temperature distribution in a nucleus, following the orbit of HALLEY's comet. $T = 76$ years; perihelion distance = 0,59 A.U. *a*) Surface temperature of stone nucleus; *b*) temperature in deeper layers of a stone nucleus near perihelion; *c*) the same for an iron nucleus.

after one revolution. It appears that at the end the temperature distribution is not completely identical to that from which we started. In the outer layer the temperature is lower by about 10°; the difference decreases inward, and between 100 and 250 m the temperature is slightly higher than it was originally; this is the heat wave which originated near perihelion and which gradually reached greater depths. Altogether the heat content has decreased a little; apparently the temperature of 70° from which we started was somewhat too high, 60° would have been better. We will see later that such a difference has no influence at all on the surface temperature.

This surface temperature reaches much higher values than WURM had assumed (fig. 2a). Near the perihelion it rises to 305° K; of course this maximum value is subject to a considerable uncertainty, because the changes in radiation are so quick. The peak looks very sharp; but fig. 2b, where it is reproduced on an elarged scale, shows that during more than 8 months the surface temperature exceeds 200°, the temperature at 10 m exceeding 150°. The peak is asymmetric; the penetration of the heat wave has a velocity of the order of 10 m a year near the surface, 5 m a year at a depth of 100 meter. It is clear that a long time will elapse before a stationary condition is reached.

In fig. 3 we see how the radiation absorbed and the rediation emitted vary during a perihelion passage. The absorbed radiation is symmetric with

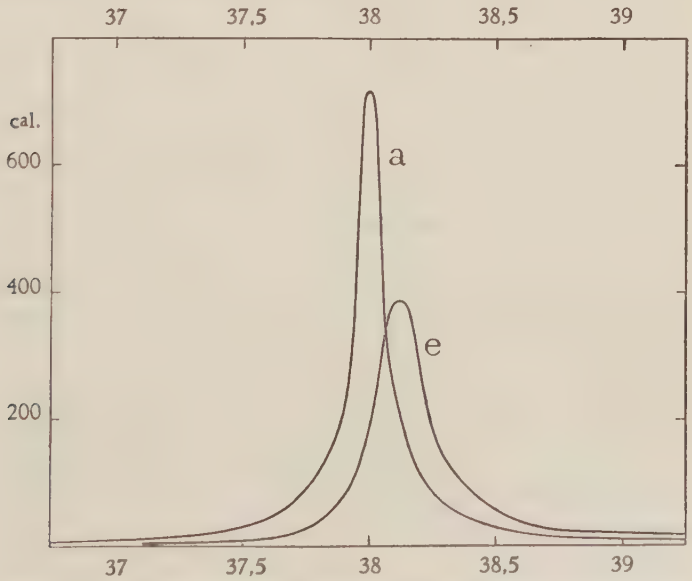


Fig. 3. Radiation absorbed (*a*) and emitted (*e*) during a perihelion passage by a stone nucleus. Unit: 1000 cal cm^{-2} $year^{-1}$.

respect to the perihelium epoch, the emitted radiation shows a lag of about $\frac{1}{8}$ year.

In order to investigate the effect of a more detailed subdivision of the layers on the numerical integration, the computation was repeated for the

years near the perihelion with layers only 10 meters thick. A maximum temperature of 310° was found, in good agreement with our first result.

2. *An iron-nickel meteorite in an elliptic orbit.*

In order to investigate the influence of the composition, we assume the same orbit as for the stone meteorite. The conductivity of pure iron would be of the order of 0,200, about 40 times higher than that of stone; but for allois it is considerably less: for nickel-steel with 30 % nickel, the coefficient is 0,030. We will adopt this last figure, but a better value could be determined by direct measurements on meteorites.

The specific heat of iron allois differs little from that of pure iron; this one we will assume, because its temperature dependence is better known.

$T = 60°$	70°	80°	100°	125°	150°	200°	250° K
$C =$ 0,025	0,032	0,039	0,050	0,065	0,078	0,095	0,103

The density is taken to be 8. The fraction $\frac{k}{c\varrho}$ is $\frac{0{,}030}{0{,}032.8} = 0{,}121$ at 70°, thus about 2,5 times as great as for stone.

In order to abbreviate the computation, we remark that the important temperature variations occur near the perihelion. We start from a rough estimate for the temperature distribution at the epoch 34^y, and apply our integration method to the years 34—44.

From the form of the differential equation, it is seen that the same relation holds, if $\frac{c}{k\varrho}$ is multiplied by n and the scale of depths by $\sqrt{n}$. Therefore the temperature distribution of the stone meteorite would apply to iron, if the strata were taken to be $20\sqrt{2{,}5} = 32$ m thick. This however would not satisfy the boundary conditions: the irradiation has been the same as for the stone meteorite, the difference in emission is negligible, so the heat content must be also the same. Now this is equal to $\Sigma c\varrho\ (T - 70)\ \triangle x = 0{,}032.8.3200.$ $(T - 70) = 820\ (T - 70)$ i.e. 3,9 times that of the stone meteorite. We therefore divide all temperature differences by 3,9, the relations still satisfying the differential equation. The estimated temperatures at 34^y deviate so little from 70° that an uncertainty in our approximations is immaterial for the further results.

The temperature peak is now computed as before, the results being summarized in fig. 2c. As a consequence of the greater heat capacity, all temperatures are lower than for stone. The surface temperature reaches now a maximum of 210°; during 8 months it exceeds 175°. Due to the greater conductivity, the temperature penetrates more rapidly in deeper layer; this is especially clear from a comparison between the layer at 50 m for stone and the layer at 48 m for iron. The asymmetry and the afteraction of the heating are very conspicuous, because the deeper layers play the role of a heat reservoir.

3. *A stone meteorite in a parabolic orbit.*

For a comparison with elliptic orbit, we select a parabola having the same perihelion distance as the ellipse of HALLEY's comet (0,587 A.U.).

At an infinite distance from the Sun, the nucleus will have a temperature near the absolute zero. It receives a very faint radiation but during a considerable time, because the motion of the comet is very slow.

We will assume that the energy absorbed until 40^y before the perihelion has communicated to the whole of the nucleus a temperature of 10° K. Doing this, we certainly will have rather exaggerated the difference between the elliptic and the parabolic case. We now compute the temperature distribution as in the parabolic orbit. At first the specific heat is very low and we have to consider very thick strata; 4 years before the perihelion the temperatures are already so high, that layers of normal thickness (20 m) can be introduced.

The results are surprising. Though we started from a much lower temperature, the surface temperature near the perihelion has nearly reached that of the elliptic orbit, and 1^y after the perihelium the difference has practically disappeared (fig. 4). This is understandable, because near the

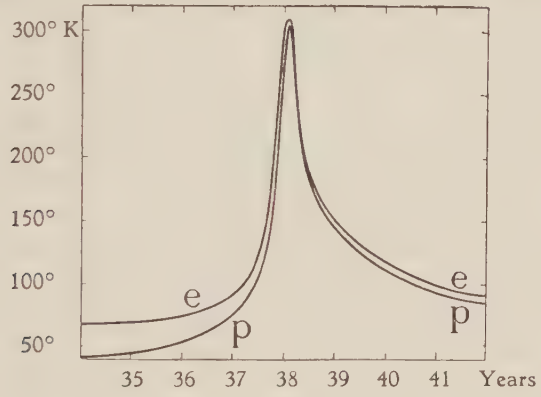


Fig. 4. Surface temperatures in an elliptic and in a parabolic orbit with the same perihelion distance.

Sun the parabolic motion is not very different from the elliptic motion; moreover, the emission is so sensitive to the temperature, that already small differences in T_0 are sufficient to compensate for differences in absorption. For the same reasons, a metallic meteorite would show the same surface temperatures near its perihelion in a parabolic as in an elliptic orbit.

Our result is also important, because it shows that the surface temperatures for an elliptic or parabolic orbit would have been found practically the same if we had assumed a different temperature at aphelion.

4. The desorption process by which the gases are liberated from the nucleus has been treated in a brilliant paper by LEVIN [3]). According to well-known laws, the quantity of gas liberated amounts to $n = n_0 \sqrt{\frac{kT}{2\pi m}} e^{-\frac{L}{RT}}$, L being the heat of adsorption. From provisional considerations on the

[3]) Russ. A. J. **21**, 48, 1943.

temperature of the nucleus, he derives that $T = \frac{350°}{\sqrt{r}}$ and writes:

$$n = n_0 \sqrt{\frac{kT_0}{2\pi m}}\, r^{-\frac{1}{2}}\, e^{-\frac{L\sqrt{r}}{350R}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The brightness of the comet, being due mainly to scattering of these gases, can be easily shown to be proportional to the number of gas molecules. By investigating the brightness of the comet at different distances from the Sun, the heat of adsorption is directly determined from the observations.

By applying this theory to ENCKE's comet, values are found of about 7000 cal./mol., in excellent general agreement with laboratory results; for HALLEY's comet a somewhat higher value of 8500 cal. mol. is derived.

It is now possible to improve these interesting results by making use of our computations. There is some doubt whether the gases are liberated only at the surface itself, or whether a layer of some depth, say the first 20 m layer, is involved. We compute for both cases the product $T\sqrt{r}$ about the perihelion:

t	r	T_0	$T_0\sqrt{r}$	T_1	$T_1\sqrt{r}$
37	5,12	95	214	84	191
37,50	2,87	131	222	110	186
38	0,59	298	229	203	154
38,50	2,87	175	296	141	239
39	5,12	130	294	117	204
		mean:	251		208

It is clear that the product $T\sqrt{r}$ should be taken rather equal to 200 or 250 than to 350. This gives an improved estimate for the adsorption heat, which proves to be 5000—6100 cal./mol. for HALLEY's comet. It would be 4000—5000 cal. for ENCKE's comet, if the same value of $T\sqrt{r}$ may be applied to an orbit so different in size.

5. The asymmetry in the temperature curve suggests, that in studying the brightness of comets as a function of the radius vector, we should carefully distinguish between observations before and after the perihelion. Unfortunately this has been often neglected; even the classic book of HOLETSCHEK does not make the difference. For ENCKE's comet, a provisional comparison is possible, and it is found that the brightness after the perihelion is rather less than before. This proves that the temperature is not the only controlling factor, but that there is also the effect of exhaustion, which is already apparent in the increased adsorption heat after the perihelion.

6. WURM required very low temperatures of the nucleus in order to explain the extraordinary sharp accessory rays which are observed in many comets. According to his calculations, any appreciable thermal motion should produce a blur.

Since however the surface temperature of the nucleus appears to be so much higher than he expected, we must look to another explanation for these rays. It looks very probable that their gases have considerably greater velocities than these in the tail itself; this would be in excellent agreement with their straight delineation. A factor 5 in the velocity would not seem impossible and would increase the allowed temperature limit by a factor 25.

An additional consideration in favour of the high nuclear temperatures is the observed liberation of gases itself. This could never take place at 10° K.

Summary.

The nucleus of a comet is pictured as a swarm of meteoritic blocks. For such a block the temperature is calculated by numerical methods as a function of the position in the orbit and for layers at different depths. We consider successively: 1. a stone meteorite in the elliptic orbit of HALLEY's comet; 2. an iron meteorite in the same orbit; 3. a stone meteorite in a parabolic orbit with the same perihelion distance. The maximum surface temperature near the perihelion proves to be 310° K.

Making use of this result, the adsorption heat of cometary gases is found to be about 5000 cal./mol. The sharp accessory rays observed in cometary tails cannot be attributed to low surface temperatures of the nucleus.

Chemistry. — *The interfacial tension of gum arabic-gelatine complex-caocervates and their equilibrium liquids.* By L. DE RUITER and H. G. BUNGENBERG DE JONG.

(Communicated at the meeting of September 27, 1947.)

1. *Introduction.*

Drops of complex caocervates, suspended in their equilibrium liquids, show motory an desintegration phenomena on application of a direct current field. [1])

It may be supposed that local changes of the interfacial tension coacervate equilibrium liquid will be of essential importance for the explanation of the motory phenomena. Here a serious lack of information exists as no interfacial tension coacervate/equilibrium liquid had ever been measured and thus the desired knowledge of the influence of variables on it in the special case of complex coacervates was not available.

We therefore have measured this interfacial tension (σ) for gum arabic (A)-gelatine (G) coacervates at various mixing proportions of the colloids at 40° C. and constant pH and the influence of some salt concentrations on it. The results of these measurements are given in the present article. Their application in the explanation of the phenomena mentioned above will be discussed in a following communication.

2. *Measuring methods.*

In some introductory experiments we found that the order of magnitude of the interfacial tensions discussed here was extraordinarily small:

1. With DU NOUY's ringmethod we could not measure any interfacial tension at all. Evidently the sensibility of this method was insufficient.

2. Determination of the height (h) to which the caocervate rises in narrow capillary tubes, brought into the phase boundary, gave h values ($\pm$ 0.1 mm) lying close to the margin of error (0.05 mm) of the kathetometer we used for these experiments. These values corresponded with $\sigma = \pm 2 \times 10^{-3}$ dynes/cm.

3. Determination of σ with the drop weight method (calculated from specific weights and microscopical dimensions of the just detaching coacervate drop) proved to be impossible, as the viscosity of the coacervate is relatively high : no well defined drops are formed, as every drop falling draws out a long thread of coacervate, which afterwards resolves into small separate droplets. Nevertheless some experiments with this method gave σ values in the above order of magnitude.

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[1]) H. G. BUNGENBERG DE JONG and E. G. HOSKAM. Proc. Ned. Akad. v. Wetensch., Amsterdam, **44**, 1099 (1941).

4. The "method of pendant drops" [2]) proved impracticable, as it was nearly impossible to form sufficiently small droplets (about 100 μ in diameter) of a constant volume. Only one determination was done in this way, which again came in the expected order of magnitude ($1{,}81 \times 10^{-3}$ dynes/cm at 20° C.) in a coacervate (50 % A-sol + 50 % G-sol) in which gelation had been prevented by means of addition of urea and resorcinol [3]).

Finally a method was developed based on the principle of the capillary rise method. Initially we applied this method in its simplest form: In a tube (fig. 1 A) containing coacervate (C) and equilibrium liquid (*E*) a second thin tube was brought, the end of which had been drawn out to a thin capillary. The top of this capillary tube was brought into the interface coacervate equilibrium liquid. An opening in the narrow tube below the surface of the equilibrium liquid allowed for free contact of the latter in both tubes in order to avoid hydrostatical pressure differences. The distance from the coacervate meniscus in the capillary tube to the surface of the surrounding coacervate was measured by means of a horizontal microscope (ocular-micrometer) placed just before the glaswindow of the thermostate (40° C.). Just behind the glaswindow a brass frame dipping into the thermostate allowed for the mounting of four tube combinations as described above and provided means to center the capillaries and to alter their depth of immersion in the interface.

The results obtained with this method were very unsatisfactory, for which fact several causes were responsible:

a. accurate determination of *h* is difficult, as the coacervate surface in the wide tube is seen as a broad diffuse dark band. Moreover the coacervate creeps up along the outside of the capillary tube. The zero level therefore should be read as far away from the capillary tube as is possible. This will increase the risk of making errors.

b. Alkali is secreted by the glass. This brings about changes in the properties of the coacervate and in the specific weights of the liquids in the capillary tube. As a result of the latter change the original capillary rise may diminish in a relatively short time to such an extent that a capillary depression seems to develop itself, though the meniscus of the interface still stands concave.

c. Very often the capillary tube is obstructed, as the funnal shaped narrowing of the narrow tube will conduct every sinking particle of dust into it.

The first and third of these 3 difficulties could be met by determining directly the difference between the heights to which the menisci rose in

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[2]) J. M. ANDREAS, E. A. HAUSER and W. B. TUCKER. J. Phys. Chem. **42**, 585 (1938).

[3]) H. G. BUNGENBERG DE JONG and E. G. HOSKAM, Proc. Ned. Akad. v. Wetensch., Amsterdam, **45**, 3 (1942).

2 short capillary tubes of different radii (fig. 1 B). The interfacial tension then may be found by means of the formula

$$\sigma = \tfrac{1}{2}(d_c - d_e)\, g\, (h_1 - h_2)\, \frac{r_1 \cdot r_2}{r_2 - r_1}$$

where $(d_c—d_e)$ stands for the difference of specific weights of coacervate and equilibrium liquid.

Because of the small values of h_1 and especially of h_2 the application of meniscus corrections could not be neglected.

In a first approximation this correction is equal to $1/3\ b$, when b is the "height" of the meniscus (as an accurate determination of b was not well practicable because of optical dificulties, we have refrained from more exact correction). As b is greater for wide capillary tubes than for narrow ones, the total correction $+ \frac{b_1 - b_2}{3}$ on the difference in capillary rise is negative.

In order to meet the second of the difficulties mentioned above the 2 capillary tubes used in this method should be very short. They were fixed by means of KHOTINSKY cement to the end of a glass rod which had been drawn out into a thin thread (fig. 1 B). It proved impracticable to do this

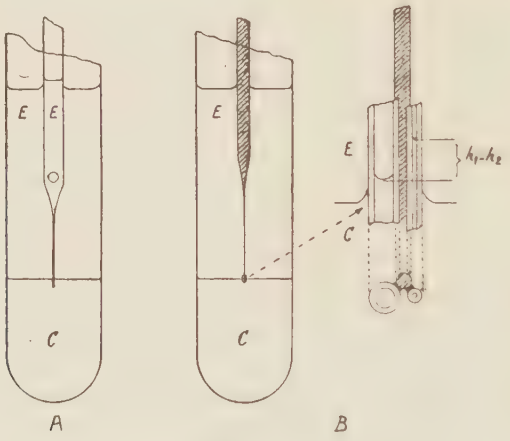


Fig. 1.

by means of paraffin. A very small droplet KHOTINSKY cement was brought on the end of the glass thread by dipping it into a molten piece of this substance. Then the 2 very small pieces (see under) of capillary tube, lying on a heated object glass were picked up by touching them with this small droplet. Usually they are all at once parallel to one another, or else can be brought easily into the right position which of course is of some importance in view of the exactness of the measurement.

It might be objected that soluble parts of KHOTINSKY cement might perhaps influence the properties of the coacervate. However, we never found evidence of such an influence in spite of the special attention we paid

to this question. Moreover the capillary tubes were always extracted in equilibrium liquid before the beginning of the experiment.

This bi-capillary method gave more or less satisfactory results only after the alkali production by the glass had been much reduced by working exclusively with Jena glass. As even this will probably produce some alkali we worked with the shortest capillary tubes we just could manipulate (about 2 mm) and extracted and washed them with equilibrium liquid before the beginning of every experiment.

We soon found that when we used the same capillary set more than once, we often obtained irregular results. Probably this is due to the fact that glass is not only dangerous on account of its alkali production, but that the silicate ions also have some influence on the coacervate that is left behind in the capillary tubes after the experiment. Initially we therefore tried to remove every remnant of coacervate from the capillary tubes by washing with concentrated salt solutions. As this did not satisfy, however, we used in the definitive experiments always a new set of capillary tubes for every single observation.

For the calibration of the capillary tubes we used an optical method given by KOHLRAUSCH [4]).

The radii of the capillary tubes were usually in the order of $\pm$ 180 μ for the wide one and $\pm$ 30 μ for the narrow one.

For the calculation of σ (see formula) data were wanted concerning the specific weights of the coacervate and equilibrium liquid studied. These were not yet available. We therefore determined these specific weights by means of small pycnometers (contents about 5 cc), which gave a sufficient accuracy for the present purpose.

In order to be able to work with systems of reproducible composition all definitive experiments were carried out on one set of stock sols. These contained 2 % colloid (*A*, resp. *G*) and were buffered with an acetic acid — sodium acetate mixture (pH $\pm$ 3.7) [5]).

For the preparation of the stock sols we started from:

60 g	*A* or *G* (air dry).
300 cc	0,1 n. sodium acetate.
150 cc	2 n. acetic acid.
2490 cc	dest. water.

When in the following pages we are speaking of a "33 % *A*" mixture, a system is meant consisting for 33 % of stock sol 2 % *A* and for 67 % of stock sol 2 % *G*.

After its preparation the gum arabic stock sol was filtered through an about 1 cm thick layer of filterpaper fibres, obtained by cooking pieces of

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[4]) F. KOHLRAUSCH, Lehrbuch der praktischen Physik, B. G. Teubner, Berlin, 13e Auflage (1921) p. 97.

[5]) H. G. BUNGENBERG DE JONG and E. G. HOSKAM, Proc. Ned. Akad. v. Wetensch., Amsterdam, **45**, 3 (1942).

filter paper. The gelatine stock sol was kept in the refrigerator in six separate flasks, of which only one at a time was molten in order to prevent repeated thermolytic effects. To both stock sols a drop of carbon tetrachloride was added against fungi.

For the investigation of the influence of the salt concentration on σ we added to 600 cc stock sol *A* 1,879 gr. KCl (i.e. about 42 m. aeq. p. l.). By mixing this solution with normal stock sol *A* we prepared such concentrations as were wanted for the experiments (7 and 10,5 m. aeq. KCl in 50 % *A* mixtures).

All glass used in the experiments was freed from fat by means of acetone. No other measures were taken to prevent errors caused by "capillary active" substances, as the stock sols themselves are already a source of impurities. Moreover there seems to be some justification for the expectation, that the interface we studied here will not be very much influenced by such substances, as it separates 2 media of only slightly different polarity.

The systems which we examined were prepared by mixing the stock sols in the desired proportion at 40° C., shaken in order to mix more thoroughly and kept over during one night at 40° C., during which time the coacervate sedimented. Next day coacervate and equilibrium liquid were separated by cooling below the gelating point (8 min. at 20° C.) and decanting, and separately heated to 40° C. again. Then the coacervate was brought into the tubes for σ-determination and once more gelated; some warm equilibrium liquid was then brought on the gelled coacervate and the tube immediately placed in the thermostate (40° C.) in which the observations were carried out. Before the experiment the capillary tubes were extracted in equilibrium liquid (40° C.) during 30 minutes, then washed with equilibrium liquid, and afterwards brought deeply into the coacervate, so that their inner surface was completely wetted with the latter (which took about 20 min.). Then they were half drawn out and fixed in that position until the menisci had fallen to their ultimate positions, which they usually had reached after 80 minutes. Then the capillary tubes were brought once more deeper into the coacervate. We then waited until the menisci had risen to their new positions again. We always followed the movements of the menisci as function of the time in order to detect possible irregularities. The definitive value of the height difference of the menisci in each case was found by taking the average of 4 separate measurings of this difference. The mean value of the 2 average height differences found after the falling and rising of the menisci was taken as the definitive value for the calculation of σ.

The method we used for the determination of the specific weights has an absolute error margin of about 0,0003 in the specific weights (i.e. a relative one of about 2 % in the differences of spec. w. between coac. and eq. liq.). The error margin of the σ-determination is about 15 % for the extreme mixing proportions (only about 7 % for the equivalent mixing proportion).

3. *Specific Weights.*

The results of these measurements have been summarized in table I. The values mentioned sub *A* have been obtained with the same mixtures which were used for σ-determination, those sub *B* from another set of stock sols of the same composition (each of the latter is the average of 2 or 3 separate determinations).

With d_{coac} and $d_{eq.liq.}$ is indicated the quotient of the masses of one cc coacervate (or equilibrium liquid) and one cc of destilled water at 40° C.

TABLE I.

Mixture No.	% A	m. aeq. p.l. KCl added	$d.\frac{40}{40}$ coac.	$d.\frac{40}{40}$ eq. liq.	$d_c - d_e$	average difference
A 1	33	0	1.0417	1.0052	0.0365	0.0366
2			1.0417	1.0050	0.0367	
3	40	0	1.0422	1.0044	0.0378	0.0378
4			1.0420	1.0042	0.0378	
5	50	0	1.0431	1.0038	0.0393	0.0393
6			1.0431	1.0039	0.0392	
7	60	0	1.0427	1.0047	0.0380	0.0381
8			1.0428	1.0046	0.0382	
9	67	0	1.0423	1.0057	0.0366	0.0366
10			1.0424	1.0059	0.0365	
11	50	7	1.0383	1.0054	0.0329	0.0326
12			1.0379	1.0057	0.0322	
13		10.5	1.0353	1.0065	0.0288	0.0288
14			1.0354	1.0066	0.0288	
B	0	0	—	1.0064		—
	33	0	1.0411	1.0052		0.0359
	40	0	1.0418	1.0044		0.0374
	50	0	1.0426	1.0039		0.0387
	60	0	1.0423	1.0047		0.0376
	67	0	1.0420	1.0057		0.0363
	100	0	—	1.0081		—

The agreement between the two series of different mixing proportions *A* and *B* is sufficient. The differences hardly surpass the error margin of the method. That the values sub *B* are systematically lower than sub *A* is no doubt due to the fact that we started from different stock sols.

Figure 2 gives the relation between mixing proportions and specific weights of coavervates and equilibrium liquids respectively.

The specific weight of the solmixtures if no coacervation had occurred is represented by a dotted nearly horizontal line, drawn between 1.0064 for the gelatin stock sol and 1.0081 for the gum arabic stock sol. From an earlier investigation it was known that the mixing proportion for optimal coacervation under the prevailing circumstances lies at 50 % *A* (see in fig. 5 the position of the reversal of charge point and the maximum of the

volume of the coacervate separated from 20 cc sol mixtures). Fig. 2 shows conform expectation a maximum for the specific weight of the coacervate and a minimum for the specific weight of the equilibrium liquid at the same

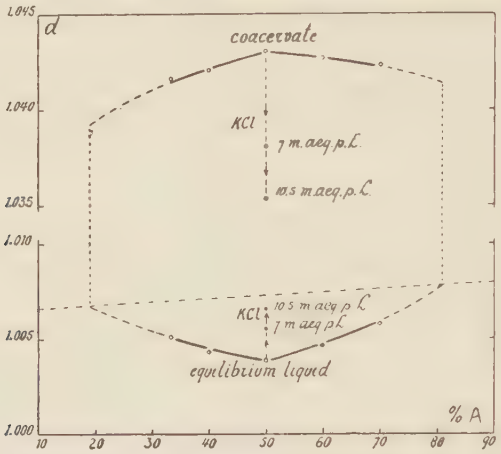


Fig. 2.

mixing proportion. Still more pronounced is the maximum at 50 % *A* for the difference in specific weight of coacervate and equilibrium liquid (see last column of table I).

The results show that the more one of the colloids is in excess the more the specific weights of coacervate and equilibrium liquid will come nearer to one another. By trying to extrapolate the specific weight curve of the equilibrium liquid to those values of the mixing proportion at which just no longer coacervation occurs (19 % *A* and 81 % *A*, see fig. 5) no improbable course is obtained (see fig. 2).

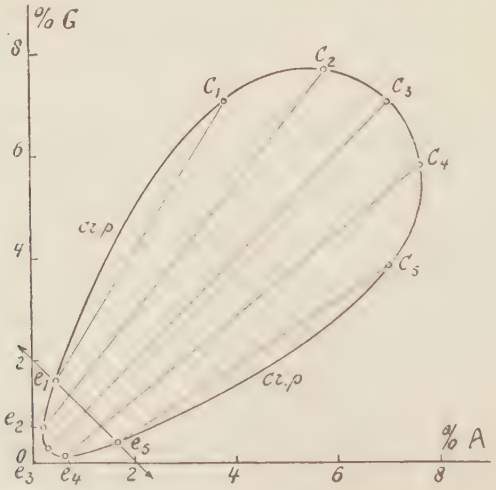


Fig. 3.

TABLE II.

Influence of mixing proportion.

33 % A		40 % A		50 % A		60 % A		67 % A	
Mixt no.	σ in dyne/cm $\times 10^3$	Mixt no.	σ in dyne/cm $\times 10^3$	Mixt no.	σ in dyne/cm $\times 10^3$	Mixt no.	σ in dyne/cm $\times 10^3$	Mixt no.	σ in dyne/cm $\times 10^3$
1	1.42	3	1.88	20	2.55	28	1.78	9	1.56
	1.45		2.14		2.39		2.25		1.65
2	1.49	4	1.93	21	2.51	29	2.21	10	1.60
	1.39		2.24		2.62		2.16	32	1.51
15	1.55	18	1.92	22	2.19	30	2.14		1.64
16	1.42		1.90		2.28		2.20	33	1.65
17	1.51	19	1.97	23	2.40	31	2.44		
			1.91		2.07				
				24	2.37				
					2.00				
				25	2.35				
					2.28				
				26	2.38				
					2.29				
				27	2.15				
					2.05				
1.46 ± 0.022		1.99 ± 0.046		2.31 ± 0.045		2.17 ± 0.075		1.60 ± 0.023	

TABLE III.

Influence of saltconcentration.

50 % A					
0 maeq KCl		7 maeq KCl		10.5 maeq KCl	
mixt no.	σ in dyne/cm $\times 10^3$	mixt no.	σ in dyne/cm $\times 10^3$	mixt no.	σ in dyne/cm $\times 10^3$
See Table II		11	0.75	13	0.34
		12	0.83		0.53
			0.78	14	0.39
		34	0.87		0.54
			0.85	36	0.41
		35	0.87		0.57
			0.98	37	0.51
					0.57
2.31 ± 0.045		0.85 ± 0.028		0.48 ± 0.031	

If this however is tried with the coacervate curve in fig. 2, it seems very unlikely that at 19 or 81 % *A* the specific weight should have become identical with the equilibrium liquid. It seems more probable that this coacervate curve takes a course as indicated in fig. 2. Such a course can even be foreseen, using a schematic phase diagram simplifying the coacerv-

ation as an unmixing in a ternary system $G + A +$ Water (see the schematic isotherm drawn in fig. 3).

On mixing 2 % A and 2 % G sols we enter and leave the area in which coacervation occurs in points so situated, that the two phases c_1 and e_1 and also c_5 and e_5, belonging to those points do not coincide. The equilibrium liquids (e_1 and e_5) have here the composition of the total sol mixture, the coacervates (c_1 and c_5), though infinitesimal small in volume, have compositions widely different from the equilibrium liquids.

The few data concerning the influence of the salt concentration on the specific weights indicate that with increasing KCl concentration the differences of composition between coacervate and equilibrium fluid decrease. (See fig. 2). This is just what might be expected, as an added indifferent salt increases the mutual solubility of coacervate and equilibrium liquid.

4. *Interfacial tension.*

In tables II and III the values found for σ in our experiments have been summarised.

At the bottom of tables II and III the mean value of σ for each series and its standard error are given.

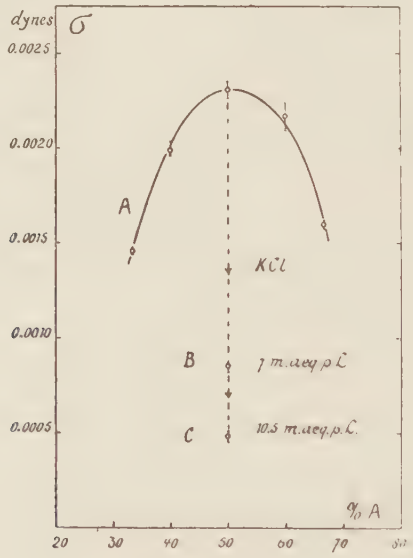


Fig. 4.

In fig. 4 the mean values for σ have been plotted against the percentage gum-arabic. (Moreover the mean values for σ in a 50 % A system to which 7 and 10,5 m. aeq. KCl resp. have been added have been indicated.)

To give some idea in how far these experiments can be reproduced

we may mention that with another set of stock sols we obtained (for 50 % *A*)

$$\sigma = (2{,}49 \pm 0.08) \times 10^{-3} \text{ dyne/cm} \quad (2{,}31 \pm 0.04 \times 10^{-3} \text{ dyne/cm in table II})$$

and with another set again

$$\sigma = (3{,}13 \pm 0.08) \times 10^{-3} \text{ dyne/cm.}$$

Latter value seems rather different from the other two. This may be due to the fact that in this series we did not keep the coacervate during one night at 40° C., but brought about sedimentation immediately by centrifugating during 30 min. at 38° C. and determined σ immediately afterwards. (Prevention of the slow hydrolysis of the gelatin component at 40° during some 16 hours.)

As always we again found in these experiments, especially in the determination of specific weights of the coacervate, that a very rigorous uniformity of treatment of the coacervate was a conditio sine qua non to obtain agreeing results.

As every complication in the experimental procedure involves a danger for uniformity of treatment, and this uniformity is highly desired as theoretically a maximum σ value was to be expected at a certain mixing proportion (see below), we preferred in the definitive experiments to bring about the separation into two clear liquid layers by spontaneous sedimentation during a fixed time and not by centrifugation.

5. *Discussion.*

a. *σ as function of the mixing proportion.*

Fig. 4 shows that in the isohydric mixing series (pH 3.7) the interfacial tension is not independent of the mixing proportion and that at a certain mixing proportion σ reaches a maximum value. From general considerations such a maximum curve of course can already be expected, as somewhere in the coacervation region the difference in composition of the two liquid layers will be maximal and this difference will become smaller in nearing the coacervation limits.

If we try to extrapolate both branches of the σ curve in fig. 4 (and 5) up to the coacervation limits (19 % and 81 % *A*) is seems probable that σ falls regularly to zero. We might however expect by a similar reasoning as given in 3a) that the interfacial tension will keep a finite value up to the coacervation limits. Evidently these finite values of σ are here very small.

b. *The postion of the σ maximum.*

Fig. 5 contains besides the σ curve from fig. 4 also results from an earlier investigation, in which the circumstances (2 % sols from the same colloid-preparations, same concentration of the acetate buffer) were exactly the same.

We may conclude that the interfacial tension reaches its maximum at

practically the same mixing proportion, at which the coacervate volume curve (coacervate volume separated from 20 cc sol mixtures) shows its maximum.

Of much importance is the fact that at this mixing proportion also the electrophoretic reversal of charge point is situated. Theoretically it should just be expected that an uncharged interface has a maximum interfacial tension, and that the acquisition of a positive (towards smaller % A) or of a negative (towards higher % A) capillary electric charge would lower the interfacial tension.

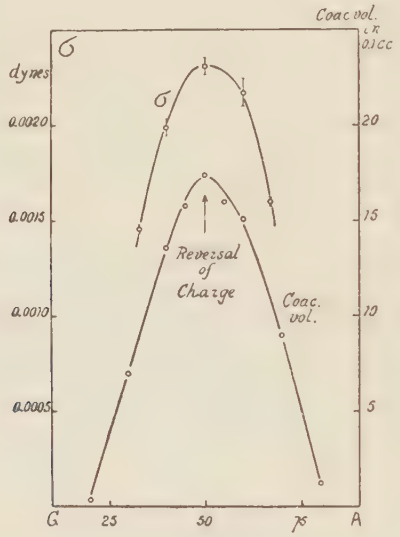


Fig. 5.

It seems possible that the above discussed (see a) strong decrease of σ up to only a very small definite value at the coacervation limits, is mainly due to this effect of the still increasing capillary electric charge of the interface.

c. *The influence of an added indifferent salt.*

Fig. 4 shows that 7 and 10.5 m. aeq. KCl diminishes considerably the interfacial tension of coacervate and equilibrium liquid. This result is in accordance with expectation as an indifferent salt increases the mutual solubility of the complex coacervate and the equilibrium liquid, thus diminishing considerably the original difference in composition of the two phases, which also expresses itself in a strong decrease of the original difference in specific weight of the two phases, see fig. 2.

d. *The order of magnitude of the interfacial tension.*

Most interesting is the very low order of magnitude of the interfacial tension, this even at its maximum value amounting to only a few thousandths of a dyne per cm. An argument for the correctness of this value

is to be found in the fact, that with several other less precise methods we mentioned above in 1, most of which were fully independent of the present method, we obtained values in the same order of magnitude.

Undoubtedly the interfacial tension we measured in the above experiments are lower than those to be expected if for the adjustment of the pH not a diluted buffer but only HCl had been used. The buffersalt present in our experiments (10 m. aeq. p. L. Na acetate) being of the same type (1—1) as KCl will have had a decreasing influence on the maximum of the σ curve of approximately 2.10^{-3} dynes p. cm.

By using HCl instead of a buffer, the coacervated system would still contain an indifferent salt, namely $CaCl_2$, originating from the counterions of both colloids (Ca from gum arabic and Cl from gelatin).

A rough estimation (assuming the decreasing influence of $CaCl_2$ to be twice as strong as of KCl) would lead for such ideal complex coacervates, which contain besides water only the oppositely charged macromolecular ions of both colloids, to a maximal value of the interfacial tension of approximately 8.10^{-3} dynes per cm, which still is surprisingly low.

e. *Interfacial tension and effective electrostatical attraction.*

Earlier investigations have sufficiently shown that the electrostatical atraction between the oppositely charged macromolecular ions of gelatin and gum arabic is the only factor which unites these colloids in the complex coacervate. The effective attraction depends on pH, mixing proportion and salts present, and gives an explanation of the occurrence at each given pH of an optimal mixing proportion, of the coincidence with a reversal of charge point, of the changes in composition of coacervate and equilibrium liquid and of the suppressing effect of added salts on complex coacervation. It seems logical to correlate the above found changes in interfacial tension also directly with the changes in effective electrostatical attraction.

The interfacial tension is thus no longer seen as an accidental property, which is modified by changes in composition of the two adjacent phases (as above in a and c) or by electrification of the interface (as above in b), but as the direct outcome of the electrostatical attraction of the oppositely charged macromolecular ions within the coacervate, which attractions in the interface have no longer a resultant zero.

Then at once the occurrence of a maximum σ value at the reversal of charge point (optimal attraction), the large fall of σ at both sides of this optimal mixing proportion (the effective attraction becomes small), and the strong reduction of σ by added salt (decrease of the effective attraction by the screening effect of both cation and anion of the added salt) becomes evident.

This conception may also explain qualitatively the low order of magnitude of the interfacial tension.

Compared with the surface tension of molten alkalihalides (e.g. NaCl/air $= \pm 100$ dynes per cm at 1000° C.), this interfacial tension must be very much lower because:

1. The "concentration" of the charges of both sign is relatively high in the molten salt, but relatively small in the coacervate (it contains ± 16 % colloids and 84 % water, from which the concentration of the "gelatin arabinate" expressed in aequivalent p. L. can be roughly calculated to be only 0.07 N).

2. The electrical attraction in the coacervate is diminished considerably (80 ×) because the macromolecular ions are inbedded in a medium (water) possessing a high dielectrical constant. This medium is the same as the equilibrium liquid (apart from some dissolved macromolecules in the latter), so that no contribution to the interfacial tension results from it.

Both factors would perhaps lead to even too small values for the interfacial tension of the coacervate, which may be due to the fact that we did not account for the very great difference in temperature.

Without doubt the two above points would not suffice for a quantitative theoretical treatment of the problem, and other factors (the attachment of the ionized group on flexible macromolecules) should also be taken into account.

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Biochemistry. — *Oleate systems containing potassium chloride in which the KCl concentration is still too low for coacervation.* I. *The viscosity-KCl curve.* By H. G. BUNGENBERG DE JONG and G. W. H. M. VAN ALPHEN.

(Communicated at the meeting of March 29, 1947.)

1. *Introduction. Technic of measurement.*

In previous research about coacervation of oleate sols by KCl, we have already mentioned that KCl concentrations still too low for coacervation, induce peculiar systems behaving as liquids of a high viscosity and showing elastic properties moreover [1]).

In this and the next communication, we discuss in principle their viscous behaviour, in the third communication follows a discussion of the elastic behaviour.

This investigation, being only of an orientating nature, does not aim in the first place, at measuring the characteristics already mentioned with a high degree of accuracy.

The purpose is rather to become acquainted with the variables which we have to deal with and especially to learn also something about the influence of non-electrolytes (in particular here the influence of primary normal alcoholes) exercising such an enormous great influence on oleate coarcervates, as shown in previous research [1]).

The expression $\frac{\eta_s - \eta_0}{\eta_0}$, in which we are interested in the first place at constant oleate concentrations, varies so enormously with increasing KCl concentrations that one has not enough of one and the same viscosimeter to record accurately the KCl curve but needs a whole set of them.

But in the circumstance that in these sols of high viscosity elastic properties are clearly present beyond a definite KCl concentration, the presented difficulties are not to overcome by increasing the accuracy of time measurement which may be obtained from the use of a set of mutually tested viscosimeters.

For our purposes we rather need to record with one viscosimeter the whole KCl curve, which necessitates the selection of a small value for the time of flow for water and KCl solutions resp.; at the same time we are limited to relatively small oleate concentrations, lest the times of flow become inpractically great in adding KCl.

But small oleate concentrations produce a relative time of flow of the sol

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[1]) H. G. BUNGENBERG DE JONG, H. L. BOOY and G. G. P. SAUBERT, Protoplasma **28**, 543 (1937).

with regard to water at a KCl conc = 0 being hardly the experiment fault greater than 1, so that the expression $\frac{\eta_s - \eta_0}{\eta_0}$ at KCl = 0 is not measurable accurately.

Thus in the next we just express the results in relative times of flow $\frac{t_s}{t_0}$ in which we will use as t_0 either the time of flow for water or that one of the KCl solution being present in the oleate system.

The viscosimeter (fig. 1) is especially constructed with the intention to perform serial work.

It may be easily pushed up and down as a whole (pulley and counter weight) in the vertical, along the rails of a heavy frame. It is filled by

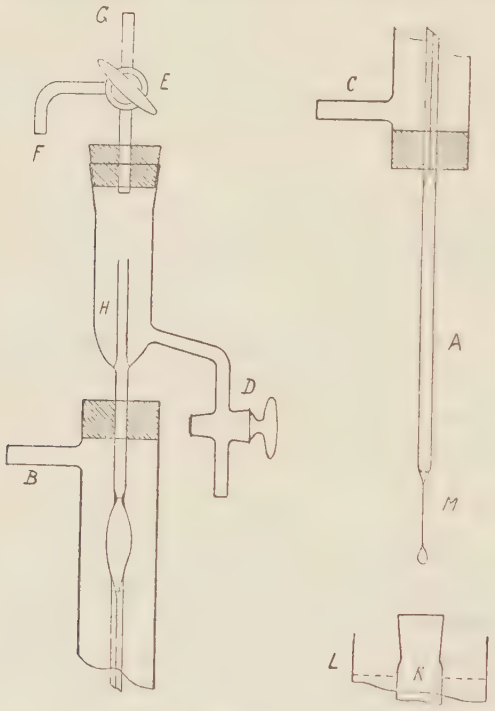


Fig. 1.

plunging of the lengthening-piece A into the system, which has to be measured and applying suction with closed tap D and correct position of the three-way tap E (F is connected with a tank filled with air of reduced pressure).

It is constructed in a glass-container with side tubes B and C through which thermostat water is pumped. The temperature (25.0° ± 0.05°) of this water may be controled by a thermometer in the container.

The systems which have to be measured are prepared at room temperature in a tube of Jenaglass (K), with ground tight fitting stopper and are shaked vigorously by hand (at least 1 minute).

After that, the tube is placed in a vessel (L) situated below the end of the viscosimeter through which is also running thermostat water. After 6 minutes the oleate system has the same temperature as the vessel.

The glass-stopper is removed and the viscosimeter pushed downwards; the end is now plunged deep in the oleate system and the mixture is sucked upwards as described above, whereby it flows over in H. In this way the rests of the previous contents are washed away.

When nearly everything has been sucked upwards, the threeway tap E is placed in such a position that a connection with the air outside is obtained via G, and the viscosimeter is emptying itself. Immediately there after one pushes the viscosimeter upwards and the liquid of the viscosimeter runs out freely in the air.

With a stop-watch one measures the time the liquid needs to pass the marks on the top and the lower end of the bulb.

The mixture is now sucked upwards again still a number of times (3 to 4) and measured, and the average time of flow is computed.

The width of the tube joined to the end of the capillary is chosen in such a manner that the share of friction existing herein may be disregard with respect to the friction in the capillary (length 24 cm, diameter 0.85 mm) and thus the temperature of the soap system passing this tube does not matter. To prevent the ascension of airbubbles, the end of the plunging tube had to be narrowed to $\pm$ 1 mm.

If the oleate system has a very viscous and gelatinous character after preparation and shaking the mixture is placed first in the thermostat of 25° to allow the airbubbles to ascend. Very small enclosed airbubbles do not influence the time of flow.

The viscosimeter is always used for the measurement of series mixtures, the order of measurement running in small steps from low to high values of the concerning variable (e.g. increasing KCl conc.). In this way a non-ideal removal of the rest of the preceding mixture can hardly influence the time of flow of the next mixture.

The free outlet into the air, though disputable from point of view of a theoretically correct technic of measurement, appeared to be of great value, as it was now possible to observe beyond a definite KCl conc., the formation of shorter or longer threads (M) instead of separate drops. Then the system becomes "thread-pulling".

2. *Viscosity-KCl-curve.*

In preliminary investigations about oleatecoacervates always Na-oleate of MERCK was used and coacervation was accomplished by adding a concentrated KCl solution.

The concentration of K-ions is large when coacervations starts with regard to the Na-ions originating from the Na-oleate solution, but it seemed more perfect to us if the system contained just K-ions.

Thus at first we made K-oleate stock solutions, by dissolving oleic acid purissimum of KAHLBAUM in a surplus of KOH.

The surplus of KOH was used to prevent "sensibilisation" by oleic acid hydrolytically liberated [2]).

With this oleate sols completely different viscosity-curves were obtained as similarly composed oleate sols from Na-oleate of MERCK.

Further investigation into this difference excluded immediately the small percentage of Na-ions in these latter sols as the cause.

Indeed, by the liberation of oleic acid from Na-oleate and preparation of K-oleate sols from this oleic acid in exactly the same manner as K-oleate sols were being prepared from oleic acid purissimum "KAHLBAUM", the same typical difference still existed (see 4).

Finally it was found that recrystallisation of Na-oleate prepared from oleic acid "KAHLBAUM" [3]) does profoundly alter the shape of the KCl-curve hithertoo found starting from oleic acid.

The recrystallised sample now shows the characteristic shape of Na-oleate (MERCK).

After these experiences we limited ourselves to Na-oleate MERCK, as oleic acid purissimum KAHLBAUM evidently contained a contamination, greatly influencing the shape of the KCl-curve.

As example of the characteristic viscosity KCl-curve we show fig. 2, in which the mixtures contain a final concentration of 0.454 % Na-oleate, always a constant quantity of 0.0454 N KOH [2]), and a variable KCl concentration.

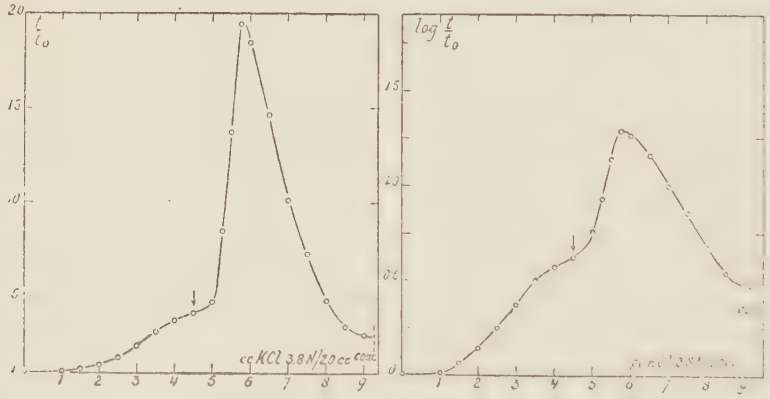


Fig. 2.

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[2]) Always a small constant KOH concentration is added. This serves to prevent complications resulting from the hydrolysis of the oleate. Free oleic acid actually acts as every non-electrolyte with long carbon chain very strong "condensing" on the oleate system. Thus we have to take care for a sufficient alcaline reaction to exclude this complication. Compare about this auto-sensibilisation, which was noticed for the first time in the study of oleate-coacervates, H. G. BUNGENBERG DE JONG, H. L. BOOY and G. G. P. SAUBERT, Protoplasma **29**, 536 (1938).

[3]) Oleic acid KAHLBAUM + the required quantity NaOH for neutralisation is dissolved by heating in alcohol and filtrated, after cooling the Na-oleate cristallises.

The mixtures were prepared from 5 cc of a 4 × concentrated Na-oleate sol [40 gr. Na-oleate + 2000 cc H_2O, to which is added after dissolving 200 cc KOH 2 N] + a cc KCl 3,8 N + (15—a) cc dest. water. See Table I.

TABLE I.

a (cc KCl 3.8 N per 20 cc)	*t* sec	*t* KCl solution	t/t_0 KCl	log. t/t_0	Observations about threads
0	9.0	8.9	1.01	0.00	pulls no threads
1	9.2	8.9	1.03	0.01	„
1.5	10.2	8.85	1.15	0.06	„
2	12.2	8.8	1.39	0.14	„
2.5	15.6	8.75	1.78	0.25	„
3	20.4	8.7	3.38	0.37	„
3.5	27.2	8.67	3.14	0.50	„
4	32.2	8.63	3.73	0.57	„
4.5	35.8	8.6	4.16	0.62	threads long and thin.
5	49.0	8.58	5.71	0.76	gradual transition to short and thick threads
5.25	72.8	8.56	8.50	0.93	
5.5	117.0	8.54	13.70	1.14	
5.75	165.6	8.52	19.44	1.29	
6	156.8	8.5	18.45	1.27	
6.5	124.	8.475	14.63	1.16	
7	85.4	8.45	10.11	1.00	
7.5	61.4	8.425	7.29	0.86	gradual transition to long and thin threads
8	40.2	8.4	4.79	0.68	
8.5	28.6	8.36	3.43	0.54	
9	25.2	8.33	3.03	0.48	
9.5	coacervated	—	—	—	

From the table it becomes clear (column 2) that the time of flow at KCl = 0 within the experimental faults, corresponds with that of water, but soon increases considerably, passes through a maximum and decreases strongly again before the KCl concentration is attained, where coacervation occurs.

We want to imply however these times of flow at those of similarly composed KCl solutions containing a constant quantity of 0.0454 N KOH. The times of flow of some of similar KCl mixtures were determined and the others of column 3 interpolated. Column 4 gives the relative times of flow with regard to the corresponding KCl solutions.

These values thus represent approximately [4]) the relative viscosity of the oleate systems showing an enormous influence of KCl.

While at KCl = 0, the relative viscosity within the experimental error is equal to water, it increases greatly and attains a value of more than 19 at the maximum of the curve.

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[4]) In fact the values t/t_0 are still troubled by an error, as the used test liquids do not possess the low σ of the soap systems. As this low σ is constant (see note 5), we have to deal here with a systematic error, but at the given dimensions of the viscosimeter (diameter measuring bulb and hydrostatic pressure) this error is practically to neglect.

The left ascending branch of the curve is not continuous but there appears to be an inflexion point. At this point or a somewhat higher KCl concentration (at a = 4.5) appears for the first time the phenomenon that a loosening drop from the point of the viscosimeter pulls a thread.

This thread is still here long and thin, but gets shorter and thicker at higher KCl concentrations, to get longer and thinner again at still higher concentrations.

When a drop is loosing from the bearing thread one may frequently observe that it was elastically stretched, it withdraws itself to the point of the viscosimeter.

Thus there must take place peculiar structure alterations in the oleate solution and we cannot leave the impression that KCl causes an appearance of elastic properties about which more in the third communication [5]).

To get a surveying diagram we rather choose in the following, the logarithmus of the relative time of flow as ordinate (column 5). In this manner the arised curve (fig. 2 B) shows a shape which is nearly the same as that of fig. 2 A. Here again we meet an inflexion point in the left ascending branch.

The fact, that the relative viscosity strongly decreases again (19 → 3) when approaching the coacervation limit (KCl concentration of $a = 9.25$) is highly important for the theory of the coacervation of oleates (association colloid).

In this respect there is an similarity with the behaviour of macromolecular sols.

Here a strong decrease in the relative viscosity does always occur when approaching the coacervation limit.

We may not attach to much value to this similarity however, as this strong decrease of the viscosity is missing at very small oleate concentrations, while nevertheless coacervation occurs.

Compare fig. 3 giving the results (log t) for different but for every curve constant oleate concentrations (the final concentration in % i.e. gram per 100 cc, are added to every curve), while the final concentration of KOH was always the same (0.0454 N).

It appears from fig. 3 that at increasing oleate concentrations, the KCl concentration whereat the threads are pulled for the first time gets smaller and further that the simple shape of the curve becomes more complicated (the curves at the two highest oleate concentrations were not further examined than indicated).

The viscosity maximum also moves to smaller KCl concentrations.

---

[5]) Measurements with the tensionmeter of DU NOUY have shown, that the surface tension of the oleate systems, though considerably lower than those of water (± 28 dynes) remains practically constant in the whole KCl concentration trajectory. The appearing phenomenon of pulling threads at the inflexion point of the viscosity KCl curve thus has no bearing on an alteration of surface tension, but must be due to alterations in the internal structure of the three dimensional oleate system.

On the other hand the coacervation limit does not, or hardly move with increasing KCl concentrations.

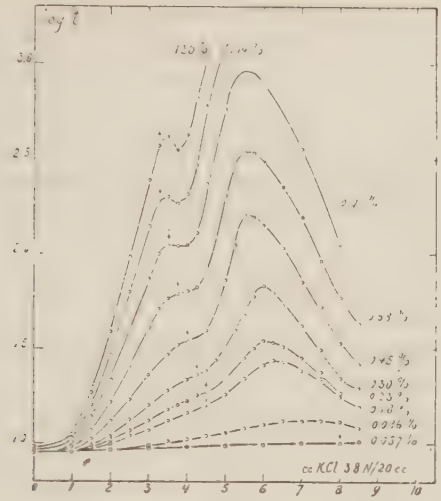


Fig. 3.

This limit was situated always very near to $a = 9$ cc KCl 3.8 N (per 20 cc), also for the highest used oleate concentrations.

3. *Influence of the temperature.*

Finally we observed, that the temperature also has a strong influence. One compares fig. 4, in which t_0 in the expression log t/t_0 presents the time of flow of the corresponding KCl—KOH solution at the indicated temperatures.

4. *Influence of auto-sensibilisation and of contaminations.*

In the note on page 852 we mentioned already that a small addition of a constant KOH concentration aimed at preventing the auto-sensibilisation. Different bottles of Na-oleate MERCK supply solutions in dest. water

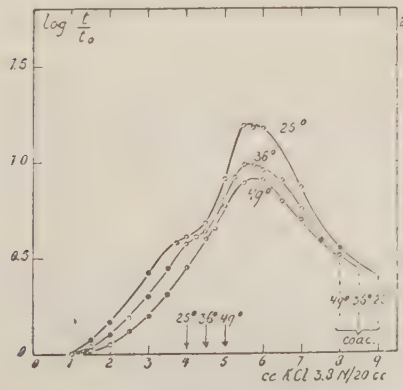


Fig. 4.

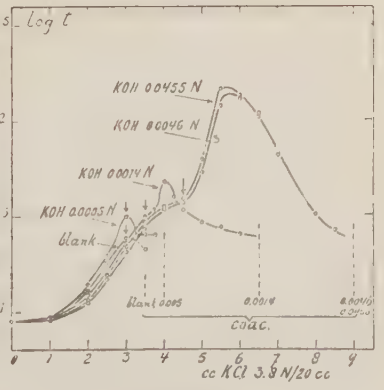


Fig. 5.

which may originate mutually very different viscosity KCl curves caused by their variable degree of auto-sensibilisation.

It may be recommended to open the bottles of Na-oleate powder just before use.

In opened situation the auto-sensibilisation may increase by absorbing CO_2 from the air.

Fig. 5 shows the viscosity KCl curves for a given oleate sample, which had been opened for a longer time and to the solutions of which no (blank), or increasing amounts of KOH were added (the final concentrations are added to the curves).

Apparently the oleate dissolved in dest. water produces strongly sensibilised oleate sols.

Here is nothing left of the characteristic appearence of the viscosity KCl curve in fig. 2 B.

The viscosity may still increase by adding KCl and the system may be just pulling threads (at 3 cc KCl 3.8 N in 20 cc final volume), but at a little higher concentration coacervation occurs already (dotted curve).

Already at small KOH concentrations the coacervation limit moves to higher KCl concentrations and gradually the shape of the curve develops itself into the typical one.

This is reached already at 0.0046 N, which curve hardly differs from that at 0.0455 N.

This latter KOH addition, which we always used previously and which will also steadily be maintained throughout the following communication, gives 10 times as large a security against the occurrence of auto-sensibilisation effects.

These effects we previously ascribed to non-ionised oleic acid, of which may be expected that it will act in principle in the same way as every other polar organic non-electrolyte with sufficient long carbon chain.

The series of curves of fig. 5 are indead very much alike to the curve-series such as we are to meet in the next communication (see fig. 1 and 2) about the influence of methyl-hexyl-keton or N. octanol resp.

In 2. we mentioned already that oleate sols prepared from oleic acid (KAHLBAUM) produce deviating viscosity KCl curves.

The difference with Na-oleate MERCK consisted of a much lower viscosity maximum, situated at much smaller KCl concentrations, a much lower coacervation limit, further the lack of an inflexion point in the left ascending curve branch i.e. phenomenons as occur in fig. 5 at a too small KOH concentration.

However sufficient KOH was present to exclude auto-sensibilisation. As after re-crystallisation of Na-oleate prepared from this oleic acid, a viscosity KCl curve was produced with all characteristic features of the Na-oleate MERCK, we must conclude that oleic acid KAHLBAUM contained a sensibilising non-electrolyte.

For the experiments in this and the next communication we always used therefore Na-oleate MERCK.

Finally it must be mentioned, that this preparation also cannot considered to be free from sensibilising contaminations, which latter must be always present in small but various amounts. Indeed, oleate sols prepared from different unopened bottles produce viscosity-KCl curves and coacervation limits, which do not entirely cover each other.

Thus it will be advisable to use one and the same oleate stock solution in examining the influence of one variable.

Summary.

1. The viscosity of non sensibilised oleate sols (0.3 % and higher) increases through KCl to a maximum and afterwards decreases considerably again before the concentration has been reached where coacervation occurs.

2. The left ascending curve branch contains an inflexion point.

Here the system starts to show "thread-pulling".

This property at first strongly increases, but only to decrease strongly again when approaching the coacervation limit.

3. The typical shape of the viscosity-KCl-curve becomes less pronounced with increasing temperature. It disappears at extremely low oleate concentrations although in both instances coacervation occurs in a normal manner.

4. The influence of auto-sensibilisation and of contaminations, in the oleate system on the shape of the viscosity-KCl-curve has been discussed.

Physics. — *Recovery and recrystallization viewed as processes of dissolution and movement of dislocations.* (Concluding part.) By W. G. BURGERS. (Laboratorium voor Physische Scheikunde der Technische Hoogeschool, Delft.) (Communicated by Prof. J. M. BURGERS.)

(Communicated at the meeting of April 26, 1947.)

III, 6. *Occurrence of "stimulated" crystals.*

We shall conclude this paper by discussing one other remarkable pecularity deduced from the analysis of the forms of crystals in recrystallized aluminium plate. This refers to the presence of such "pointed" crystals as shown at *B* in fig. 18. From the analysis by SANDEE (62) and MAY (63) it follows that such crystals did not start to grow from their nucleus until another crystal (the "surrounding" crystal *A*), the growth of which

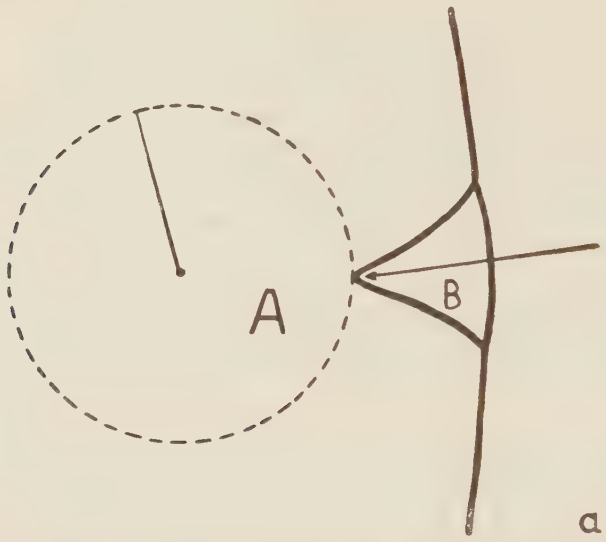


Fig. 19a. Formation of a "stimulated" crystal: Crystal *B* did not start to grow from its nuclear spot (indicated by the arrow) before crystal *A* had attained a size indicated by the circle with radius *r*.

had started previously, had come into contact with it in a way as described in fig. 19. In a discussion of this phenomenon (70) we have spoken in this connection of "stimulation" of growth of crystal *B* by crystal *A*. In order that the stimulated crystal obtains a measurable size, its rate of growth must be larger than that of the stimulating crystal. It was also found that both stimulating and stimulated crystals had a mutual orientation as found in spinel twins, that is they have a (111) plane and a (110) direction in common. As this plane and direction are those of preferential gliding in aluminium, it was suggested that the action of stimulation might be caused by something like a "discharge" of the stresses in the nucleus, which might take place by movement of dislocations along the common glide plane, at the moment that contact was established.

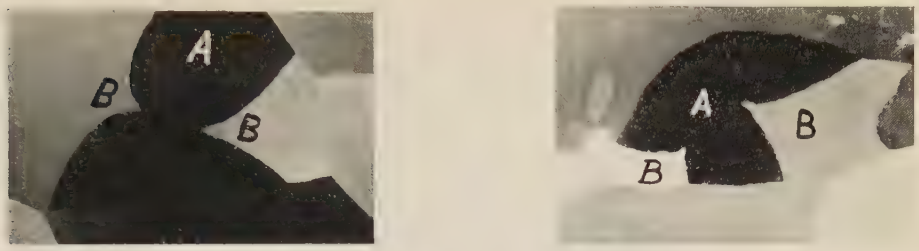


Fig. 18. Examples of stimulated crystals in recrystallized aluminium plate: the pointed crystals *B* have been "stimulated", in the sense as described in fig. 19*a*, by crystal *A*. Natural size.

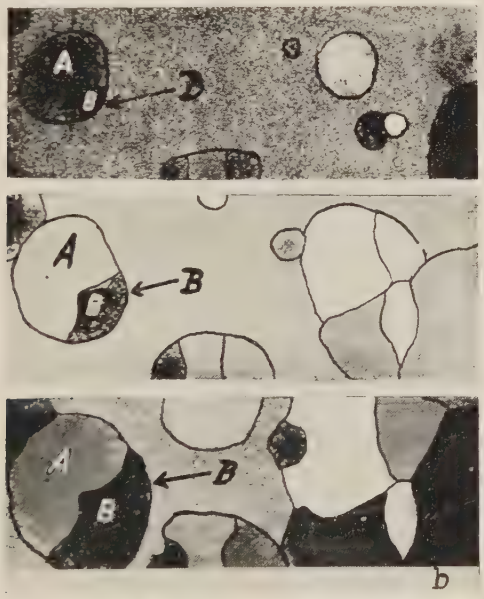


Fig. 19*b*. Formation of a "pointed" crystal in aluminium (cf. Fig. 19*a*). Natural size.

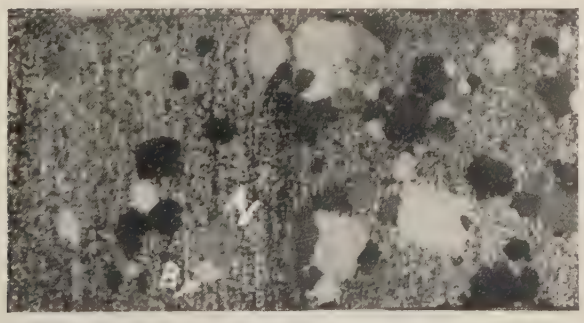


Fig. 21. Occurrence of "pointed" ("stimulated") crystals in the recrystallization experiments of ANDERSON and MEHL (10): it is highly probable that the pointed crystallite *B* (white on the reproduction) is "stimulated" by the growing crystal *A* (grey on the reproduction) (After ANDERSON and MEHL (10)). Magnif. about 7 ×.

The occurrence of stimulated crystals, which seems very common in recrystallized aluminium and which gives rise to most remarkable orientation relationsships, is still a subject of closer investigation [33]). Here, however, we mention this remarkable effect only in connection with some recent results, obtained by ANDERSON and MEHL (10) in their investigation of the recrystallization of thin aluminium sheet. From the increase of the

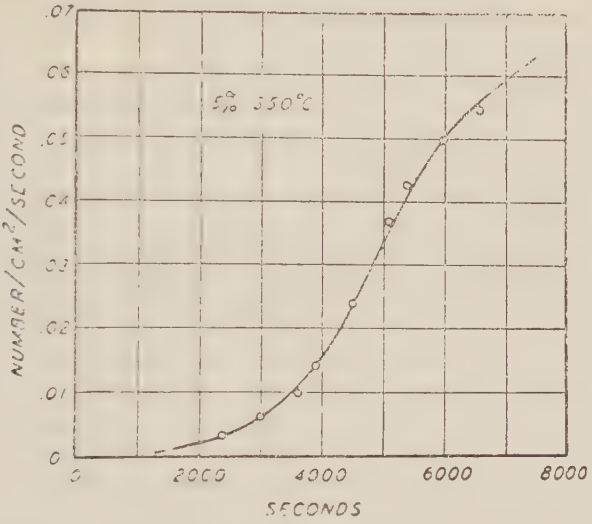


Fig. 20. Rate of nucleation N (number of grains formed per cm² of the unrecrystallized matrix per second) as a function of time of heating in isothermal recrystallization of thin aluminium sheet (350° C, 5.1 % elongation). *N is seen to increase in the course of the recrystallization process* (after ANDERSON and MEHL (10)).

recrystallized fraction and of the number of new crystallites as a function of time of heating these authors deduced that, whereas the rate of growth of the new crystallites remained constant with time (in agreement with for example fig. 12), the rate of nucleation (that is the number of new grains formed per cm² of the unrecrystallized matrix per second) *increased* with time, at least in finegrained material [34]). This is shown in fig. 20. With regard to this result ANDERSON and MEHL remark that the observed curve appears autocatalytic in nature and that such an auto-acceleration would appear to argue for an action at a distance; that is, that the occurrence and growth of one nucleus should transmit through the unrecrystallized matrix an accelerating stimulus to nucleation. Viewed in the light of the "stimulated" crystals discussed above, the actual occurrence of such an accelerating stimulus follows from the presence of "pointed" crystals of the type shown in fig. 18. That actually such crystals appear also in the recrystallized test-pieces of ANDERSON and MEHL is evident

[33]) We are inclined to think that several other phenomena, observed while studying recrystallization, for example that of "secondary recrystallization" ("exaggerated graingrowth") and the formation of twins, is closely related to this stimulating action.

[34]) In coarse-grained material a *decrease* was observed.

from fig. 21, which is taken from their paper: the small crystallite *B* is most probable an example of a stimulated crystal.

I am indebted to my brother, Prof. J. M. BURGERS at Delft, for helpful discussions.

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Delft, March 1947. *Technische Hoogeschool*

Mathematics. — *Interprétation topologique d'un théorème de* M. MAHLER *concernant les pseudo-évaluations.* By F. LOONSTRA. (Communicated by Prof. L. E. J. BROUWER.)

(Communicated at the meeting of June 28, 1947.)

M. MAHLER appelle les pseudo-évaluations d'un anneau commutatif R ayant un élément unité, $w_1(a)$, $w_2(a)$, ..., $w_n(a)$ indépendantes [1]), si l'on peut faire correspondre à tout ensemble de n nombres $a^{(1)}, a^{(2)}, \ldots, a^{(n)}$ de R une suite $a_1, a_2, \ldots, a_n, \ldots$ d'éléments de R, possédant les propriétés suivantes:

$$\lim_{m\to\infty} w_h(a^{(h)} - a_m) = 0 \quad (h = 1, 2, \ldots, n).$$

En vertu de cette définition il démontrait: Étant données des pseudo-évaluations indépendantes d'un anneau R: $w_1(a)$, $w_2(a)$, ..., $w_n(a)$ et désignant par w leur somme directe, on peut considérer l'extension complète R'_w comme une somme directe des anneaux complets R'_{w_i} $(i = 1, 2, \ldots, n)$. Nous démontrerons le théorème suivant, exprimant le rapport topologique entre les espaces complets R'_w et R'_{w_i} $(i = 1, 2, \ldots, n)$:

Soient $w_1, w_2, \ldots, w_n$ des pseudo-évaluations indépendantes d'un anneau commutatif (ayant un élément unité), $w = w_1 + w_2 + \ldots + w_n$ leur somme directe, R'_w, R'_{w_i} $(i = 1, 2, \ldots, n)$ les extensions complètes, l'espace R'_w est topologiquement homéomorphe au produit topologique des espaces $R'_{w_1}, R'_{w_2}, \ldots, R'_{w_n}$.

Démonstration. En vertu de la pseudo-évaluation w, R'_w possède un système de voisinages satisfaisant aux axiomes d'un espace topologique. L'espace produit $R'_{w_1} \times R'_{w_2} \times \ldots \times R'_{w_n}$, étant l'ensemble des éléments $(A_1, A_2, \ldots, A_n)$, où A_i est un élément quelconque de R'_{w_i}, est également un espace topologique; on appelle voisinage de $(A_1, A_2, \ldots, A_n)$ le produit des voisinages $U(A_i)$ de A_i $(i = 1, 2, \ldots, n)$.

Soit A un élément de R'_w; il s'ensuit qu'il existe une w-suite fondamentale $\{a_1, a_2, \ldots\}$ d'éléments de R, tel que $A = \lim_{m\to\infty} a_m(w)$. La suite $\{a_i\}$ est aussi une w_i-suite fondamentale $(i = 1, 2, \ldots, n)$; alors il y a un élément $A_1 \subset R'_{w_1}$, $A_2 \subset R'_{w_2}, \ldots, A_n \subset R'_{w_n}$, de sorte qu'on a en même temps $A_i = \lim_{m\to\infty} a_m(w_i)$ $(i = 1, 2, \ldots, n)$.

L'application $A \to (A_1, A_2, \ldots, A_n)$ est univoque. Inversement il est

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[1]) K. MAHLER, Ueber Pseudobewertungen I, Acta Math^a 66, 79—119.

possible de déterminer univoquement un élément $A \subset R'_w$ correspondant à un élément $(A_1, A_2, \ldots, A_n)$ quelconque de l'espace produit. Cette possibilité nous sera fournie en vertu d'un théorème de M. MAHLER [1]): Soient $w_1, w_2, \ldots, w_n$ des pseudo-évaluations indépendantes de R et puis $a_1^{(h)}, a_2^{(h)}, \ldots$ $(h = 1, 2, \ldots, n)$ n suites infinies de R, de sorte que pour $h = 1, 2, \ldots, n$ la h^{me} suite est une w_h-suite fondamentale, de façon que $A_h = \lim_{m \to \infty} a_m^{(h)}(w_h)$ $(h = 1, 2, \ldots, n)$, il existe une suite infini $a_1, a_2, \ldots$ de R, étant en même temps une w_i-suite fondamentale pour $i = 1, 2, \ldots, n$ et de façon qu'en outre $A_h = \lim_{m \to \infty} a_m(w_h)$ $(h = 1, 2, \ldots, n)$. Cette suite est aussi une w-suite fondamentale, ainsi elle admet une limite A en R'_w. L'application $(A_1, A_2, \ldots, A_n) \to A$ est également univoque. Alors, il existe une application biunivoque des éléments A de R'_w sur les éléments $(A_1, A_2, \ldots, A_n)$ de $R'_{w_1} \times R'_{w_2} \times \ldots \times R'_{w_n}$. En outre cette application sera bicontinu. Pour cela nous démontrerons que la suite, correspondant à une suite quelconque d'éléments $A_1, A_2, \ldots$ de R_w avec $\lim_{i \to \infty} A_i = A(w)$, tend dans l'espace produit vers l'élément correspondant de A; c'est à dire: appelons les composants de A_k: $(A_k^{(1)}, A_k^{(2)}, \ldots, A_k^{n})$ et de A: $(A^{(1)}, A^{(2)}, \ldots, A^{(n)})$, nous aurons à démontrer que la suite $A_1^{(k)}, A_2^{(k)}, \ldots$ convergera en R'_{w_k} vers $A^{(k)}$ $(k = 1, 2, \ldots, n)$; en effet, pour qu'une suite d'éléments de l'espace produit soit convergente, il faut et il suffit que les suites des composants soient convergentes.

Désignant la suite fondamentale $\{a, a, \ldots\}$ des éléments égaux par $\{a\}$, nous déterminons pour chaque élément A_m de la suite convergeant vers A une suite fondamentale $\{a_1^{(m)}\}, \{a_2^{(m)}\}, \{a_3^{(m)}\}, \ldots$ de façon que $w(A_m - \{a_\nu^{(m)}\}) \leqq \frac{1}{\nu}$ $(m = 1, 2, 3, \ldots; \nu = 1, 2, 3, \ldots)$. On a

$$\begin{array}{l}
\{a_1^{(1)}\}, \{a_2^{(1)}\}, \{a_3^{(1)}\}, \ldots \to A_1 \\
\{a_1^{(2)}\}, \{a_2^{(2)}\}, \{a_3^{(2)}\}, \ldots \to A_2 \\
\quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\
\{a_1^{(m)}\}, \{a_2^{(m)}\}, \{a_3^{(m)}\}, \ldots \to A_m \\
\qquad\qquad\qquad\qquad\qquad\qquad \downarrow \\
\qquad\qquad\qquad\qquad\qquad\qquad A
\end{array}$$

Chaque suite horizontale est une w-suite fondamentale, alors elle est aussi une w_h-suite fondamentale $(h = 1, 2, \ldots, n)$, c'est à dire

$$\lim_{i \to \infty} a_i^{(m)} = A_m^{(h)}(w_h) \quad (h = 1, 2, \ldots, n;\ m = 1, 2, 3, \ldots).$$

Comme $\lim_{i \to \infty} A_i = A(w)$, on peut faire correspondre à tout nombre ε positif arbitrairement petit un nombre positif entier $q = q(\varepsilon)$, tel que $w(A_m - A_{m'}) \leqq \varepsilon$, $m \geqq q(\varepsilon)$, $m' \geqq q(\varepsilon)$. Supposons $r = r(\varepsilon) = \max\left(\frac{3}{\varepsilon}, q\left(\frac{\varepsilon}{3}\right)\right)$,

nous aurons

$$w(A_m - A_{m'}) \leqq \frac{\varepsilon}{3} \text{ pour } m, m' \geqq r(\varepsilon),$$

$$w(A_m - \{a_\mu^{(m)}\}) \leqq \frac{\varepsilon}{3} \text{ pour } m, \mu \geqq r(\varepsilon),$$

$$w(A_{m'} - \{a_\nu^{(m')}\}) \leqq \frac{\varepsilon}{3} \text{ pour } m', \nu \geqq r(\varepsilon),$$

alors $w(\{a_\mu^{(m)}\} - \{a_\nu^{(m')}\}) = w(a_\mu^{(m)} - a_\nu^{(m')}) \leqq \varepsilon$ pour $m, m', \mu, \nu \geqq r(\varepsilon)$ et puis $w(a_m^{(m)} - a_{m'}^{(m')}) \leqq \varepsilon$ pour $m, m' \geqq r(\varepsilon)$.

La suite $\{a_m^{(m)}\}$ $(m = 1, 2, 3, \ldots)$ est une w-suite fondamentale et en outre R'_w la suite $\{a_m^{(m)}\}$ tend vers l'élément A: en effet, à tout nombre δ, on peut faire correspondre un $r(\delta)$, tel que pour $\varrho \geqq r\left(\frac{\delta}{2}\right)$

$$w(A - A_\varrho) \leqq \frac{\delta}{2} \quad \text{et} \quad w(A_\varrho - \{a_\varrho^{(\varrho)}\}) \leqq \frac{\delta}{2},$$

c'est à dire

$$w(A - \{a_\varrho^{(\varrho)}\}) \leqq 2 \cdot \frac{\delta}{2} = \delta;$$

alors $\lim_{m \to \infty} a_m^{(m)} = A(w)$; en outre ce résultat implique que $\lim_{m \to \infty} a_m^{(m)} = A^{(h)}(w_h)$ $(h = 1, 2, \ldots, n)$. Maintenant nous pourrons démontrer tout de suite que dans l'espace R'_{w_h} la suite $A_1^{(h)}, A_2^{(h)}, \ldots$ tend vers $A^{(h)}$. Pour cela nous rappelons que

$$\lim_{i \to \infty} \{a_i^{(m)}\} = A_m(w), \quad \lim_{i \to \infty} \{a_i^{(m)}\} = A_m(w_h) \quad (m = 1, 2, \ldots;\ h = 1, 2, \ldots, n).$$

A tout $\alpha_h > 0$ on peut faire correspondre un $\sigma_h = \sigma_h\left(\frac{\alpha_h}{2}\right)$, de façon que pour $m > \sigma_h$

$$w_h(A^{(h)} - \{a_m^{(m)}\}) \leqq \frac{\alpha_h}{2} \quad \text{et} \quad w_h(A_m^{(h)} - \{a_m^{(m)}\}) \leqq \frac{\alpha_h}{2}.$$

Soit $\max_{h=1,2,\ldots,n} \alpha_h = \alpha$, $\max_{h=1,2,\ldots,n} \sigma_h = \sigma$, alors on a:

$$\begin{aligned} w_h(A^{(h)} - A_m^{(h)}) &= w_h((A^{(h)} - \{a_m^{(m)}\}) - (A_m^{(h)} - \{a_m^{(m)}\})) \\ &\leqq w_h(A^{(h)} - \{a_m^{(m)}\}) + w_h(A_m^{(h)} - \{a_m^{(m)}\}) \leqq \alpha \\ &\qquad \text{pour } m \geqq \sigma \quad h = 1, 2, \ldots, n, \end{aligned}$$

c'est à dire $\lim_{m \to \infty} A_m^{(h)} = A^{(h)}$ $(h = 1, 2, \ldots, n)$. (1)

Pour que $\lim_{m \to \infty} (A_m^{(1)}, A_m^{(2)}, \ldots, A_m^{(n)}) = (A^{(1)}, A^{(2)}, \ldots, A^{(n)})$, il faut et il

suffit que $\lim_{m\to\infty} A_m^{(h)} = A^{(h)}$ $(h = 1, 2, \ldots, n)$, alors en vertu de (1) on a:

$$\lim_{m\to\infty} (A_m^{(1)}, A_m^{(2)}, \ldots, A_m^{(n)}) = (A^{(1)}, A^{(2)}, \ldots, A^{(n)}).$$

Supposons maintenant $(A_m^{(1)}, A_m^{(2)}, \ldots, A_m^{(n)})$ $(m = 1, 2, \ldots)$ une suite convergeant dans l'espace produit vers $(A^{(1)}, A^{(2)}, \ldots, A^{(n)})$, nous démontrerons que la suite correspondante A_m converge vers l'élément correspondant A de $(A^{(1)}, \ldots, A^{(n)})$. Pour cela nous déterminons pour chaque élément $(A_m^{(1)}, A_m^{(2)}, \ldots, A_m^{(n)})$ de l'espace produit une suite fondamentale $(\{a_1^{(m)}\}, \{a_2^{(m)}\}, \ldots, \{a_k^{(m)}\}, \ldots)$, $a_k^{(m)} \subset R$, de sorte que cette suite tend pour w_1 vers $A_m^{(1)}$, pour w_2 vers $A_m^{(2)}, \ldots$, pour w_n vers A_m, alors pour w vers A_m, tandis que $w_h(A_m^{(h)} - \{a_\nu^{(m)}\}) \leqq \frac{1}{\nu}$ pour $h = 1, 2, \ldots, n$. Maintenant on peut faire correspondre à tout $\varepsilon > 0$ un $q_h = q_h(\varepsilon)$ de sorte que $w_h(A_m^{(h)} - A_{m'}^{(h)}) \leqq \varepsilon$ pour $m, m' \geqq q_h(\varepsilon)$, $h = 1, 2, \ldots, n$.

Désignant $r = r(\varepsilon) = \max\left(\frac{3}{\varepsilon}, q_h\left(\frac{\varepsilon}{3}\right)\right)$ pour tous h, on a

$$w_h(A_m^{(h)} - A_{m'}^{(h)}) \leqq \frac{\varepsilon}{3} \quad \text{pour } m, m' \geqq r(\varepsilon) \text{ et } h = 1, 2, \ldots, n,$$

$$w_h(A_m^{(h)} - \{a_\mu^{(m)}\}) \leqq \frac{\varepsilon}{3} \quad \text{pour } m, \mu \geqq r(\varepsilon) \text{ et } h = 1, 2, \ldots, n,$$

$$w_h(A_{m'}^{(h)} - \{a_\nu^{(m')}\}) \leqq \frac{\varepsilon}{3} \quad \text{pour } m', \nu \geqq r(\varepsilon) \text{ et } h = 1, 2, \ldots, n,$$

alors

$$w_h(\{a_\mu^{(m)}\} - \{a_\nu^{(m')}\}) = w_h(a_\mu^{(m)} - a_\nu^{(m')}) \leqq \varepsilon \text{ pour } m, m', \mu, \nu \geqq r(\varepsilon)$$
$$h = 1, 2, \ldots, n.$$

Nous obtenons

$$\lim_{m\to\infty} a_m^{(m)} = A^{(1)}(w_1), \lim_{m\to\infty} a_m^{(m)} = A^{(2)}(w_2), \ldots, \lim_{m\to\infty} a_m^{(m)} = A^{(n)}(w_n),$$

ainsi que dans la partie de la démonstration mentionné ci-dessus. Comme la suite $\{a_m^{(m)}\}$ est une w_i-suite fondamentale $(i = 1, 2, \ldots, n)$, elle est aussi une w-suite fondamentale et elle converge vers A, l'élément correspondant de $(A^{(1)}, A^{(2)}, \ldots, A^{(n)})$, c'est à dire $\lim \{a_m^{(m)}\} = A(w)$.

En vertu de la relation

$$w(A - A_m) \leqq w(A_m - \{a_m^{(m)}\}) + w(A - \{a_m^{(m)}\})$$

et du fait que pour un m suffisamment grand les termes droits seront volontairement petits, on conclut $\lim_{m\to\infty} A_m = A(w)$. Alors nous avons démontré la bicontinuité de la correspondance biunivoque entre R'_w et $R'_{w_1} \times R'_{w_2} \times \ldots \times R'_{w_n}$; la correspondance est donc topologique.

L'anneau R_g des nombres g-adiques.

Les nombres g-adiques, introduits par M. HENSEL [2]), se rapprochent intimement de ce qui précède.

Soit R le corps des nombres rationnels, g un entier rationnel positif, on appelle nombre g-adique toute de la forme:

$$a = a_m g^m + a_{m+1} g^{m+1} + \ldots + a_{m+k} g^{m+k} + \ldots$$

(dans laquelle a_m, $a_{m+1}, \ldots$ représentent des nombres de la suite $0, 1, 2, \ldots, g-1$; pour $a \neq 0$, on a $a_m \neq 0$).

L'ensemble des nombres g-adiques est un anneau R_g, contenant le corps des nombres rationnels. Supposons $p, q, \ldots, r$ les différents nombres premiers de g, tout nombre g-adique, a, peut être désigné univoquement par

$$a = a_{(p)} + a_{(q)} + \ldots + a_{(r)}$$

comme une somme des nombres g-adiques $a_{(p)}, a_{(q)}, \ldots, a_{(r)}$, où $a_{(p)}, a_{(q)}, \ldots, a_{(r)}$ sont déterminés par

$$\left.\begin{array}{l} a_{(p)} = a\,(p)\,,\ a_{(p)} = 0\,(q)\,, \ldots, a_{(p)} = 0\,(r), \\ a_{(q)} = 0\,(p)\,,\ a_{(q)} = a\,(q)\,, \ldots, a_{(q)} = 0\,(r), \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{(r)} = 0\,(p)\,,\ a_{(r)} = 0\,(q)\,, \ldots, a_{(r)} = a\,(r), \end{array}\right\} \qquad (2)$$

(où par exemple $a_{(p)} = a\,(p)$ veut dire que a, écrit comme nombre p-adique est précisément $a_{(p)}$). Si au contraire $a_{(p)}, a_{(q)}, \ldots, a_{(r)}$ sont respectivement des nombres p-adiques, q-adiques, ..., r-adiques, il n'y a qu'un seul nombre g-adique de R_g, a, satisfaisant aux conditions (2). Nous écrivons simplement $a = (a_{(p)}, a_{(q)}, \ldots, a_{(r)})$. HENSEL a démontré que R_g est la somme directe de $R_p, R_q, \ldots, R_r$ (dans laquelle R_p représente le corps des nombres p-adiques, etc.). En outre HENSEL définit pour tout nombre g-adique, a, une évaluation $w(a) = (\alpha_{(p)}, \alpha_{(q)}, \ldots, \alpha_{(r)})$, où $\alpha_{(p)}$ est l'évaluation p-adique conventionnelle de a etc., tandis que $w(0) = (0, 0, \ldots, 0)$ et $w(a) \leqq w(b)$, si $\alpha_{(p)} \leqq \beta_{(p)}, \alpha_{(q)} \leqq \beta_{(q)}, \ldots, \alpha_{(r)} \leqq \beta_{(r)}$; maintenant on remarque aussitôt la relation étroite avec les pseudo-évaluations indépendantes. Soient $\varepsilon_p, \varepsilon_q, \ldots, \varepsilon_r$ des nombres arbitrairement positifs, on appelle l'ensemble des nombres g-adiques x avec $w(a - x) \leqq (\varepsilon_p, \varepsilon_q, \ldots, \varepsilon_r)$ le voisinage U_a de $a \subset R_g$. Evidemment le système des voisinages appartenant à l'évaluation de HENSEL équivaut au systèmes des voisinages, définis par $w = w_p + w_q + \ldots + w_r$.

En outre $w_p, w_q, \ldots, w_r$ sont des pseudo-évaluations indépendantes, de sorte qu'on a: L'anneau R_g des nombres g-adiques peut être considéré comme le produit topologique des corps $R_p, R_q, \ldots, R_r$. En résumant les propriétés de R_g, nous avons:

1. R_g est un espace métrique, donc normal;

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[2]) K. HENSEL, Zahlentheorie, Berlin 1913.

2. R_p possède un base dénombrable, parce que l'ensemble dénombrable des des nombres p-adiques rationnels est dense en R_p; de même R_q, R_r possèdent une base dénombrable. L'espace produit R_g possède de même une base dénombrable.

3. En vertu de l'évaluation non-archimédienne de $R_p, R_q, \ldots, R_r$, ces espaces sont nuldimensionnels et comme l'espace produit d'un nombre dénombrable d'espaces nuldimensionnels est aussi nuldimensionel, R_g est également nuldimensionnel;

4. Comme $R_p, R_q, \ldots, R_r$ sont denses en soi, R_g est dense en soi aussi;

5. R_g est complet (parce que $R_p, R_q, \ldots, R_r$ sont complets);

6. R_g est localement compact (parce que $R_p, R_q, \ldots, R_r$ sont localement compacts).

En vertu de 1. en 2. R_g est un sous-ensemble de l'espace de HILBERT homéomorphe; en vertu de 1., 2. et 3. R_g est homéomorphe à un sous-ensemble du Discontinu de CANTOR. En ajoutant un seul point, l'espace localement compact R_g sera un espace nuldimensionnel, compact, dense en soi, donc homéomorphe au Discontinu même. On appelle voisinage de ce point ajouté tout ensemble de nombres g-adiques $x = (x_{(p)}, x_{(q)}, \ldots, x_{(r)})$, où $w_p(x_{(p)}) > k_p, w_q(x_{(q)}) > k_q, \ldots$, où l'on a pris $k_p, k_q, \ldots$ dans l'ensemble des évaluations de $R_p, R_q, \ldots R_g$ est homémorphe au Discontinu de CANTOR jusqu'à un point près.

Rectification.

M. FREUDENTHAL m'a fait observer que le théorème 3.8, énoncé dans ma communication: "La structure des pseudo-évaluations d'un anneau élémentaire", Proceedings Amsterdam XLIX, N°. 8, 1946, n'est pas valable en général. Toutes les pseudo-évaluations d'un anneau R, contenant un nombre fini d'éléments, sont d'un type

$$w_{\mathfrak{r}}(a) = \begin{cases} 0 \text{ pour } a \equiv 0 \,(\mathfrak{r}) \\ 1 \text{ pour } a \not\equiv 0 \,(\mathfrak{r}) \end{cases}$$

dans lequel $\mathfrak{r}$ présente un idéal quelconque de R.

Il y a une correspondence biunivoque entre les idéaux $\mathfrak{r}$ et les pseudo-évaluations $w_{\mathfrak{r}}$ de R, de sorte que, si $\mathfrak{r}_1 \longleftrightarrow w_{\mathfrak{r}_1}$, $\mathfrak{r}_2 \longleftrightarrow w_{\mathfrak{r}_2}$, on a $\mathfrak{r}_1 \cup \mathfrak{r}_2 \longleftrightarrow w_{\mathfrak{r}_1} \cap w_{\mathfrak{r}_2}$, $\mathfrak{r}_1 \cap \mathfrak{r}_2 \longleftrightarrow w_{\mathfrak{r}_1} \cup w_{\mathfrak{r}_2}$. Alors les propriétés de structure des pseudo-évaluations de R coïncident avec celles des idéaux de R. Les idéaux d'un anneau forment une structure modulaire, alors les pseudo-évaluations d'un anneau, contenant un nombre fini d'éléments, constituent de même une structure modulaire.

Mathematics. — *A Note on Irreducible Star Bodies.* By C. A. ROGERS. (University College, London.) (Communicated by Prof. J. A. SCHOUTEN.)

(Communicated at the meeting of September 27, 1947.)

In a recent paper MAHLER [1]) has developed the general theory of lattice points in n-dimensional star bodies. In one section of the second part of his paper (334—343) MAHLER developes a theory of irreducible star bodies. In this note I indicate a modified treatment of this portion of MAHLER's work. In particular I obtain a necessary and sufficient condition for a star body to be irreducible.

MAHLER's notation and terminology will be used; but instead of working with MAHLER's concept of a "free" lattice we introduce the concept of reducible and irreducible points. A point P of a star body S will be called reducible if there is a star body $T \prec S$ not containing P, i.e. a star body T, contained in S but not containing P, for which

$$\Delta(T) = \Delta(S).$$

A point of S which is not reducible will be called irreducible.

Theorem 1. *A star body S of the finite type is irreducible if and only if every point on its boundary is irreducible.*

Proof. (*a*) If S is irreducible then it is impossible to find a star body $T \prec S$. So every point of S is irreducible, and in particular every point on the boundary of S is irreducible.

(*b*) If S is reducible then there is a star body $T \prec S$. Every point in S but not in T is thus reducible. Thus we have only to show that there is a point, on the boundary of S, which is not in T; i.e. we have only to prove the following lemma.

Lemma. *Let T be a star body contained in but different from a star body S of the finite type. Then there is a point on the boundary of S which is not in T.*

Proof. Suppose, if possible, that every point on the boundary of S belongs to T. As T is contained in but different from S it follows that there is a point P in the interior of S which is not in T. Now the set of points outside T and interior to S is open and contains P. So we can choose a number $\varepsilon > 0$, so small that the sphere

$$|X - P| \leqq \varepsilon \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

is outside T but is interior to S.

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[1]) K. MAHLER, Proc. Royal Soc., A **187**, 151—187 (1946) and Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam., **49**, 331—343, 444—454, 525—532, 622—631 (1946).

Suppose X is any point in the sphere (1); it is interior to S but outside T. Suppose that for some positive λ the point λX is outside S. Then as X is in the interior of the star body S there is μ with $1 < \mu < \lambda$ such that μX is on the boundary of S. But as X is not in the star body T, the point μX is not in T, contrary to our original supposition. Thus for every positive λ the point λX is in S and so S contains the sphere

$$|X - \lambda P| \leqq \varepsilon \lambda. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

If Λ is any lattice, λ can be chosen so large that there is a point of Λ, other than the origin O, in the sphere (2). Thus no lattice Λ is admissable for S and S is a star body of the infinite type. This contradiction proves the lemma.

Theorem 2. *If S is a star body of the finite type then the set Σ of irreducible points of S is closed* [2]).

Proof. Suppose, if possible, that P is a limit point of the set Σ, not in Σ. Then P is a reducible point of S and there is a star body $T < S$ not containing P. Then as the limit point P of Σ is outside the closed set T, there is a point Q of Σ outside T. But such a point Q of S outside T is reducible. This contradiction proves that P is in Σ and that Σ is closed.

Theorem 3. *A point P on the boundary of a star body S of the finite type is irreducible if and only if, for every $\varepsilon > 0$, there exists a lattice Λ with*

$$d(\Lambda) < \Delta(S),$$

such that the only points of Λ in the interior of S are points O, Q and $-Q$ with

$$|P - Q| < \varepsilon.$$

Proof. (*a*) Suppose that P is an irreducible point of the boundary of S and ε is a positive number. Let S be the body given by $F(X) \leq 1$, where, as usual, we suppose that $F(X)$ is non-negative, bounded and continuous and, for all real t,

$$F(tX) = |t| F(X).$$

Write

$$\sigma = \varepsilon/(1 + |P|), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$G(X) = \begin{cases} 0, \text{ when } F(X) = 0, \\ F(X)\left[1 + \sigma - \min\left\{\sigma, \left|\frac{X}{F(X)} - \frac{P}{F(P)}\right|, \left|\frac{X}{F(X)} + \frac{P}{F(P)}\right|\right\}\right] \text{ otherwise,} \end{cases} \quad (4)$$

Since $F(tX) = |t| F(X)$, we have $G(tX) = |t| G(X)$, and $G(X)$ is a

[2]) If S is bounded it is possible to prove that Σ is the closure of an open set containing the origin.

non-negative bounded continuous function of X. Thus the body T given by

$$G(X) \leqslant 1$$

is a star body. Further as $G(X) \geq F(X)$ and $G(P) > F(P) = 1$, the star body T is contained in S but does not contain P. As P is an irreducible point of S this implies that

$$\Delta(T) < \Delta(S).$$

So there is a lattice Λ which is admissible for T and which has

$$d(\Lambda) < \Delta(S).$$

Suppose Q is any point of Λ other than O, in the interior of S. Then

$$F(Q) < 1 \leqslant G(Q).$$

But, by (4),

$$G(Q) \leqslant F(Q)\{1+\sigma\},$$

and

$$G(Q) = F(Q)$$

unless, for one of the signs,

$$\left| \frac{Q}{F(Q)} \pm \frac{P}{F(P)} \right| < \sigma.$$

Thus

$$\{1+\sigma\}^{-1} \leqslant F(Q) < 1, \quad \ldots\ldots\ldots \quad (5)$$

and (since $F(P) = 1$), for one of the signs

$$|Q \pm P F(Q)| < \sigma F(Q).$$

Consequently, for one of the signs,

$$\begin{aligned} |Q \pm P| &\leqslant |Q \pm P F(Q)| + |P| \cdot |1 - F(Q)| \\ &< \sigma\{1 + |P|\} = \varepsilon, \end{aligned}$$

using (5) and (3).

We may without loss of generality assume that ε is so small that the sphere

$$|X| \leqslant 2\varepsilon$$

is within the star body T. Then, if Q' were any point of Λ other than O or Q or $-Q$ in the interior of S, we should have either

$$|Q - Q'| < 2\varepsilon \text{ or } |Q + Q'| < 2\varepsilon$$

and Λ would not be admissible for T. This establishes the existence of a lattice with the required properties.

(*b*) Suppose that for some point P on the boundary of S, for every $\varepsilon > 0$, there exists a lattice Λ, with

$$d(\Lambda) < \Delta(S),$$

such that the only points of Λ in the interior of S are points O, Q and $-Q$ with

$$|P - Q| < \varepsilon.$$

Suppose, if possible, that P is reducible. Then there exists a star body $T < S$ not containing P. Then as P is outside the closed symmetrical set T, we can choose $\varepsilon > 0$ so small that all points X, satisfying either

$$|X - P| < \varepsilon \text{ or } |X + P| < \varepsilon,$$

lie outside T. We can also choose a lattice Λ, with

$$d(\Lambda) < \Delta(S) = \Delta(T), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

such that the only points of Λ in the interior of S are points O, Q and $-Q$ with

$$|P - Q| < \varepsilon.$$

As this means that Q and $-Q$ are outside T, the lattice Λ is admissible for T and so

$$d(\Lambda) \geqq \Delta(T),$$

contrary to (6). This proves that P is irreducible and completes the proof of the theorem.

Theorem 4. *If P is an irreducible point on the boundary of a star body S of the finite type, then P is a point of at least one of the critical lattices of S.*

Proof. As P is an irreducible point on the boundary of S, it follows from Theorem 3, that we can choose an infinite sequence of lattices $\Lambda_1, \Lambda_2, \Lambda_3, \ldots$, with

$$d(\Lambda_r) < \Delta(S), \quad r = 1, 2, 3, \ldots,$$

and such that the only points of Λ_r in the interior of S are points O, P_r and $-P_r$, with

$$|P_r - P| < 1/r.$$

We have:

(*a*): $$\overline{\lim_{r \to \infty}}\, d(\Lambda_r) \leqq \Delta(S);$$

(*b*): For every r, the lattice Λ_r is admissible for the body given by

$$F(X) \leqq F(P_r);$$

(*c*): The points $P_1, P_2, P_3, \ldots$ tend to the limit point P on the boundary of S.

The proof can now be completed by the method used by MAHLER in proving his Theorem *B*. Briefly his argument shows that there is a subsequence of the sequence $\Lambda_1, \Lambda_2, \Lambda_3, \ldots$, which converges to a lattice Λ. The point P is a point of Λ. The lattice Λ is admissible for S. Further

$$\Delta(S) \leqslant d(\Lambda) \leqslant \overline{\lim_{r\to\infty}}\, d(\Lambda_r) \leqslant \Delta(S),$$

so that Λ is a critical lattice.

MAHLER's Theorem *C* is an immediate consequence of Theorems 1 and 4. It is also clear from Theorem 3 and MAHLER's definition *B* (reading $\pm P_k^*$ for $\pm P_k$ in condition (*c*)) that every point of the boundary of S which is a lattice point of a free lattice is irreducible in our sense. Using this, MAHLER's Theorem *D* follows at once from Theorems 1 and 2.

I am grateful to Professor DAVENPORT and Dr. MAHLER for reading the manuscript and making a number of useful suggestions.

University College, London.

Mathematics. — *Some examples concerning the relations between homology and homotopy groups.* By HSIEN-CHUNG WANG. (Communicated by Prof. L. E. J. BROUWER.)

(Communicated at the meeting of September 27, 1947.)

1. The relations between homology and homotopy groups were firstly discussed by H. HOPF. To improve and generalize HOPF's results, ECKMANN, EILENBERG, FREUDENTHAL and MACLANE have made extensive researches. The investigation is still going on. Two of the main known results can be stated as follows:

A. $H^2(M, G)$ is determined by the groups $\pi_1(M)$, $\pi_2(M)$ and a certain homotopic invariant $\mathfrak{k}^3$ [4].

B. For a space M with

$$\pi_2(M) = \pi_3(M) = \ldots = \pi_{n-1}(M) = 0 \quad . \quad . \quad . \quad . \quad (1)$$

the factor group $H^n(M)/\Sigma^n(M)$ is determined by the fundamental group $\pi_1(M)$ [3, 5].

Here, as throughout this note, M denotes an arcwise connected space, $\pi_n(M)$ the n^{th} homotopy group, $H^n(M)$ the n^{th} Betti group, $H^n(M, G)$ the n^{th} homology group with coefficient group G, and Σ^n the spherical subgroup of $H^n(M)$.

It is the aim of this note to show that all the homotopy groups together with the dimension of the space are not sufficient to determine H^n $(n = 2, 3, \ldots)$, nor the factor group H^{n+1}/Σ^{n+1} even when the space is an analytic closed manifold having the property (1). The proof consists of very simple arguments. However, this tells us that the above two results of EILENBERG and MACLANE cannot be further improved and the relation between homology and homotopy groups is quite loose.

2. Let S^k and P^k denote the k-sphere and the k-dimensional real projective space respectively. The homotopy groups of spheres have been well discussed. In what follows, we need the following results [2]:

$$\pi_r(S^k) = 0, \quad \pi_k(S^k) \approx I, \quad (r < k,\ k > 0). \quad . \quad . \quad . \quad . \quad (2)$$

where I denotes the free cyclic group. The k-sphere is a covering space of P^k with two leaves. It follows that [6] [1])

$$\pi_1(P_k) \approx I_2, \quad \pi_r(P^k) \approx \pi_r(S^k), \quad (k > 1,\ r > 1). \quad . \quad . \quad . \quad (3)$$

Let us consider the topological products

$$M_1 = S^n \times P^{n+2}, \quad M_2 = P^n \times S^{n+2}, \quad (n > 1). \quad . \quad . \quad . \quad (4)$$

both of which are analytic closed manifolds of $2n + 2$ dimensions. The

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[1]) Here, as well as throughout this note, I_2 denotes a cyclic group of order two.

homotopy group of a topological product is ismorphic to the direct sum of the homotopy groups of all the factors [7, p. 289]. Therefore,

$$\pi_1(M_1) \approx \pi_1(M_2) \approx I_2, \quad \pi_r(M_1) \approx \pi_r(S^n) + \pi_r(P^{n+2})$$
$$\pi_r(M_2) \approx \pi_r(S^{n+2}) + \pi_r(P^n), \quad (r > 1)$$

which, on account of (3), imply that

$$\pi_r(M_1) \approx \pi_r(M_2), \quad (r = 1, 2, 3, \ldots). \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Furthermore, we have from (2)

$$\pi_2(M_i) \approx \pi_3(M_i) \approx \ldots \approx \pi_{n-1}(M_i) = 0, \quad (i = 1, 2). \quad . \quad . \quad (6)$$

3. Let H^n denote the n^{th} Betti group, i.e. homology group with integer coefficients. We know that

$$\left.\begin{aligned} H^0(S^k) \approx H^k(S^k) \approx H^0(P^k) \approx I, \quad H^r(S^k) = 0, \quad (r < k) \\ H^{2r+1}(P^k) \approx I_2, \quad (0 < 2r+1 < k) \\ H^{2r}(P^k) = 0, \quad (0 < 2r < k) \\ H^k(P^k) \approx \begin{cases} 0, & \text{if } k \text{ is even,} \\ I, & \text{if } k \text{ is odd.} \end{cases} \end{aligned}\right\} . \quad (7)$$

It follows then from the Theorem of KÜNNETH [1, p. 308] that

$$H^n(M_1) \approx I + I_2, \quad H^n(M_2) \approx I$$

when n is odd, and

$$H^n(M_1) \approx I, \quad H^n(M_2) = 0$$

when n is even. In both cases, $H^n(M_1)$ and $H^n(M_2)$ are not isomorphic.

Now we shall show that the factor groups $H^{n+1}(M_1)/\Sigma^{n+1}(M_1)$ and $H^{n+1}(M_2)/\Sigma^{n+1}(M_2)$ are also non-isomorphic where Σ^{n+1} denotes the spherical subgroup of H^{n+1}. For this purpose, let us first consider $\Sigma^{n+1}(M_1)$. By definition it is the image of the group $\pi_{n+1}(M_1)$ under the natural homomorphism

$$\nu_{n+1}: \pi_{n+1}(M_1) \to H^{n+1}(M_1).$$

Suppose z to be an element of $\pi_{n+1}(M_1)$ with base point b represented by the map $f(S^{n+1}) \subset M_1$. Since $M_1 = S^n \times P^{n+2}$, we can write the point b and the map f in the following form

$$b = b_1 \times b_2, \quad b_1 \in S^n, \quad b_2 \in P^{n+2},$$
$$f(p) = f_1(p) \times f_2(p), \quad p \in S^{n+1}, \quad f_1(p) \in S^n, \quad f_2(p) \in P^{n+2}$$

where f_1 and f_2 are maps of S^{n+1} into S^n and P^{n+2} respectively. From (2) and (3) we have

$$\pi_{n+1}(P^{n+2}) \approx \pi_{n+1}(S^{n+2}) = 0.$$

Therefore the map $f_2(S^{n+1}) \subset P^{n+2}$ is null homotopic so that f is homotopic to the map g defined by

$$g(p) = f_1(p) \times b_2, \quad p \in S^{n+1}.$$

However, $g(S^{n+1}) \subset S^n \times b_2$ is at most n-dimensional. Considered as singular cycle, g and then f must be null homologous. In other words, under the natural homomorphism ν_{n+1}, z is mapped into the zero of $H^{n+1}(M_1)$. Since z is arbitrary, we have

$$\Sigma^{n+1}(M_1) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

From the Theorem of KÜNNETH, it can be easily seen that

$$H^{n+1}(M_1) \approx I_2, \quad H^{n+1}(M_2) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

when n is odd, and

$$H^{n+1}(M_1) \approx I_2 + I_2, \quad H^{n+1}(M_2) = 0 . \quad . \quad . \quad . \quad . \quad (10)$$

when n is even. Formulae (8), (9) and (10) tell us at once that

$$H^{n+1}(M_1) / \Sigma^{n+1}(M_1) \neq 0, \quad H^{n+1}(M_2) / \Sigma^{n+1}(M_2) = 0.$$

Hence $H^{n+1}(M_1)/\Sigma^{n+1}(M_1)$ and $H^{n+1}(M_2)/\Sigma^{n+1}(M_2)$ are not isomorphic.

Summing up the above results, we have the following:

To each n $(n > 1)$ we can construct two closed manifolds M_1 and M_2 of $2n + 2$ dimensions having the properties (5) *and* (6). *However,*

$$H^n(M_1) \approx H^n(M_2), \quad H^{n+1}(M_1)/\Sigma^{n+1}(M_1) \not\approx H^{n+1}(M_2)/\Sigma^{n+1}(M_2).$$

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Mathematics. — *Topological characterization of all subsets of the real number system.* By J. DE GROOT. (Communicated by Prof. L. E. J. BROUWER.)

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1. Let M be an arbitrary separable metric space (i.e., a normal space with countable base) and N a subset of M; *under which conditions will N be homeomorphic with a subset of the R_1 (the ordered system of real numbers)?*

The answer will be given in 2., Theorem II.

For *compact* subsets N and a *plane* set M this problem is not difficult and has been solved long ago. To this end a *M.K.-set* (MOORE–KLINE) is defined as a subset K of M, which satisfies the following conditions:

1°. K is compact,

2°. the (quasi)components [1]) of K are points of (closed) simple arcs,

3°. no interior point of a simple arc A is limit-point of the set $K—A$: the arcs are "free".

Now according to MOORE–KLINE [1] (comp. also [2]) a M.K.-set is a subset of a simple arc (belonging to M). In particular K is homeomorphic with a subset of R_1. Generalizations of these problems are treated by ZIPPIN [3].

If, conversely, K has been proved to be homeomorphic with a subset of R_1, then this does not show, however, that it is possible to pass a simple arc, lying in M, through K. On the other hand, the above-mentioned theorem of MOORE–KLINE has been proved only for special spaces M (for instance, M is a plane). Therefore our first step will be to prove that the M.K.-spaces, considered as subsets of an arbitrary separable metric space M, are topologically identical with the compact subsets of R_1.

The proof follows a method which is totally different from the one used for the mentioned MOORE–KLINE problem.

Theorem I. *The M.K.-subsets K of a separable metric space M are topologically identical with the compact subsets of R_1.*

Proof. It is evident that a compact subset of R_1 is a M.K.-space. Conversely, we shall have to prove that a M.K.-set K is topologically equivalent with a subset of R_1. According to 3°. K contains only countably many simple arcs A_1, A_2, A_3, ... ad inf. The endpoints of A_i we call a_i^1 and a_i^2.

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[1]) In compact separable metric spaces the components are identical with the quasi-components.

The diameters α_i of A_i are converging to zero; this may easily be seen by applying a theorem of ZORETTI, which says that (for a compact set) the limes superior of a sequence of connected sets is connected if the limes inferior is not vacuous. Let the A_i be counted in such a way that $\alpha_{i+1} \leqq \alpha_i$ $(i = 1, 2, \ldots)$. We consider the pointset K', which is formed by removing all A_i out of K except the endpoints a_i^1 and a_i^2. K is a 0-dimensional compactum. Now we define a system of coverings of K'. The first covering consists of K' itself: $K' = U_1$. Then we divide K' into a finite number of disjoint closed subsets $U_{11}, U_{12}, \ldots, U_{1j_1}$, such that

1^0. a_1^1 and a_1^2 do not belong to the same U_{1j},

2^0. a_i^1 and a_i^2 always belong to the same U_{1j} for fixed $i > 1$,

3^0. the diameter of every U_{1j} is less than $\alpha_2 + {}^1/_2$.

From the mentioned properties it follows that this covering is possible.

Now every U_{1j} $(j = 1, 2, \ldots, j_1)$ is subdivided analogically, i.e. a covering of U_{1j} is determined in disjoint closed subsets $U_{1j1}, U_{1j2}, \ldots, U_{1jk_j}$ such that

1^0. $a_{m_j}^1$ and $a_{m_j}^2$ do not belong to the same U_{1jl}; here m_j is the smallest index for which $a_{m_j}^1$ and $a_{m_j}^2$ belong to U_{1j}. Should U_{1j} contain a_i^1 (respectively a_i^2), then this point is not in the same U_{1jk} with $a_{m_j}^1$ or $a_{m_j}^2$. Should no pair of endpoints belong to U_{1j}, then this condition (and condition 2^0. as well) is satisfied in a trivial way.

2^0. If a U_{1jl} contains one endpoint a_i^1, then it also contains a_i^2 for every $i > m_j$.

3^0. The diameter of every U_{1jk} is less than $\alpha_{m_{j+1}} + {}^1/_3$ (should U_{1j} contain no endpoints then we put $\alpha_{m_{j+1}} = 0$).

In this manner the coverings

$$\{U_1\}\ ,\ \{U_{1r_1}\}\ ,\ \{U_{1r_1r_2}\}, \ldots$$

of K' are formed. It is clear how by means of induction the covering $\{U_{1r_1r_2\ldots r_{n+1}}\}$ is defined. In doing this we take care that a $U_{1a_1a_2\ldots a_n}$ is divided into finitely many disjoint closed sets $U_{1a_1a_2\ldots a_n a_{n+1}}$ $(a_{n+1} = 1, 2, \ldots, s)$, such that a_p^1, a_p^2 (where p is the smallest index for which a_p^1 and a_p^2 belong to $U_{1a_1a_2\ldots a_n}$) and a possibly appearing point a_q^1 (resp. a_q^2) (where a_q^2 (resp. a_q^1) does not belong to $U_{1a_1\ldots a_n}$) are lying in three separate sets $U_{1a_1a_2\ldots a_n a_{n+1}}$, such that further a_t^1 and a_t^2 $(t > p)$ are in the same $U_{1a_1a_2\ldots a_n a_{n+1}}$, while, lastly, the diameter of the sets $U_{1a_1a_2\ldots a_n a_{n+1}}$ is less than $\alpha_{p+1} + \dfrac{1}{n+2}$ $\left(\text{eventually } \dfrac{1}{n+2}\right)$. Such a division is apparently possible.

Now we are going to map K' topologically on a subset of the

segment [01] of the real axis. Therefore we let K' correspond with $[01] = I_1$. Consider the second covering $\{U_{1r_1}\}$ $(r_1 = 1, 2, \ldots, j_1)$.

We consider j_1 subsequent closed intervals $I_{11}, I_{12}, \ldots, I_{1j_1}$, lying disjoint on [01], each of which has a length $< 1/2$. We map

$$U_{1j} \longleftrightarrow I_{1j}.$$

Let U_{11} contain a point a_1^1 and U_{12} a point a_1^2; then we map a_1^1 on the right endpoint of I_{11} and a_1^2 on the left endpoint of I_{12}.

Now we consider, whithin I_{1j}, k_j disjoint closed intervals $I_{1j1} \ldots I_{1jk_j}$, each of a length $< 1/3$, where and endpoint, which is already an image, is contained (as an endpoint) in an interval I_{1jl}.

Let U_{111} contain point a_1^1, U_{112} point $a_{m_1}^1$, and U_{113} point $a_{m_1}^2$, then we let the sets U_{11s} and I_{11s} $(s = 1, 2, \ldots k_1)$ correspond one-to-one, in such a manner that the interval I_{11t}, which contains the image of a_1^1, corresponds with U_{111}, while U_{112} and U_{113} correspond with two immediately subsequent intervals I_{11s_1} and I_{11s_2}, where, moreover, $a_{m_1}^1$ is mapped on the right endpoint of I_{11s_1} and $a_{m_1}^2$ on the left endpoint of I_{11s_2}. Analogically the sets U_{12s} and I_{12s} are mapped on each other, where the points a_1^2, $a_{m_2}^1$ and $a_{m_2}^2$ appear. In general the sets U_{1jk} $(k = 1, 2, \ldots, k_j$; j fixed) are corresponding one-to-one with the intervals I_{1jk}, in such a way that the two U-sets containing $a_{m_j}^1$ and $a_{m_j}^2$ correspond with immediately subsequent I-intervals, where $a_{m_j}^1$ and $a_{m_j}^2$ are mapped on the two middle endpoints of the I-intervals. By means of induction the system of intervals $\{I_{1r_1r_2 \ldots r_n}\}$ of a length $< 1/n$ is generally defined in analogous way and corresponding one-to-one with the sets of $\{U_{1r_1r_2 \ldots r_n}\}$. It is essential that the already mapped endpoints a_i^1 and a_i^2, which appear successively while the construction is made, all belong to disjoint U-sets and therefore that the images belong to disjoint I-intervals, while moreover there is no I-interval or part of an I-interval between the images of a_i^1 and a_i^2. Every point of K' is defined as an intersection of exactly one sequence

$$U_1 \supset U_{1a_1} \supset U_{1a_1a_2} \supset \ldots \ldots \quad (1)$$

while, conversely, every such sequence (1) defines exactly one point of the compact set K'.

The corresponding I-sequence

$$I_1 \supset I_{1a_1} \supset I_{1a_1a_2} \supset \ldots \ldots \quad (2)$$

determines exactly one point of the interval [01] as intersection. The set of all those points obviously is homeomorphic with K'. If we add to this set all the open intervals between the images of a_i^1 and a_i^2 then the thus formed subset of [01] is homeomorphic with K, since every A_i may be mapped topologically on the corresponding interval of [01] connecting

the images of a_i^1 and a_i^2), which mapping together with the mapping of K' gives the required topological mapping of K.

2. Our proper purpose is the characterization of arbitrary pointsets of the R_1.

To this end we define a *generalized M.K.-space* as a separable metric space K, which satisfies the conditions:

1^0. the quasicomponents of K are points or open, half open or closed simple arcs (i.e., homeomorphic with the sets $\{x\}$ defined by $0 < x < 1$, or $0 \leqq x < 1$, or $0 \leqq x \leqq 1$);

2^0. the arcs are free; no interior point of a simple arc A is limitpoint of the set $K—A$.

A topological space is, as is known, *semicompact* if every point has arbitrarily small neighbourhoods with compact boundaries. Now our contention is that the semicompact generalized M.K.-spaces K are homeomorphic with the subsets of R_1.

Remark. The condition of semicompactness cannot be omitted, as will be shown by the following examples.

Example 1. Let P be a subset of the plane containing the following points (x, y):

(o, o)

$\left(x, \frac{1}{n}\right)$, where $(0 \leqq x \leqq 1)$, $(n = 1, 2, \ldots)$.

P is a generalized M.K.-space, but not semicompact (in point $(0, 0)$).

P is not homeomorphic with a subset of R_1, as may easily be seen.

The condition of semicompactness cannot be omitted even in the case that all quasicomponents consist of one point, as is shown by spaces constructed (to another purpose, however) by SIERPIŃSKI [8] and MAZURKIEWICZ [9]:

Example 2. For every natural number n there exist totally disconnected (i.e., the quasicomponents consist of one point) n-dimensional separable spaces.

These spaces are obviously not homeomorphic with a subset of R_1.

The theorem becomes untrue if (in 1^0.) components are considered instead of quasicomponents, as is shown by

Example 3. Let S be a subset of the plane containing the following points (x, y):

$\left(x, \frac{1}{n}\right)$ where $(0 \leqq x \leqq 1)$, $(n = 1, 2, \ldots)$,

$(x', 0)$ where $(0 \leqq x' \leqq 1$; x' rational).

S is a semicompact space satisfying the modified condition 1^0. (components are considered instead of quasicomponents) and condition 2^0. Yet apparently S is not homeomorphic with a subset of R_1.

Theorem II. *The semicompact generalized M.K.-spaces are — topologically spoken — identical with the subsets of the* R_1 [2]).

2.1. **Preliminary remarks.**

The proof may be given by a — be it far-reaching — generalization of the proof of Theorem I. This is, however, a very long and complicated way.

Therefore we shall give a proof, which implies a direct use of Theorem I: we shall embed the given semicompact generalized M.K.-space in a M.K.-space; from the homeomorphism of this last space with a subset of R_1 it will follow immediately that the former space is also homeomorphic with a subset of R_1.

For our proof we shall use a number of different theorems and notions which — in order to keep the proof brief — will be mentioned here.

(α) Every semicompact separable metric space M may be compactified to a compact separable metric space $\overline{M}$ by an 0-dimensional set $\overline{M}—M$ (where M is everywhere dense in $\overline{M}$); conversely, every subset M of a compactum $\overline{M}$, for which $\overline{M}—M$ is 0-dimensional, is semicompact (see [5], p. 45).

If a space M is the sum of a collection of disjoint closed sets A:

$$M = \Sigma A \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

a new space $^*M$ — *the decomposition space* of M corresponding with the decomposition (1) — is formed by identifying (in the well-known way) each set A to one point $^*a$. — The decomposition (space) is called *upper semi-continuous* [3]) (comp. for instance WHYBURN [7], p. 122 a.f.), provided that for every neighbourhood $U = U(A/M)$ of an arbitrary A in M may be found a neighbourhood $V = V(A/M)$ such that any A-set intersecting V is totally contained in U:

$$A \cdot V \neq 0 \rightarrow A \subset U.$$

Now

(β) Every upper semi-continuous decomposition space $^*M$, corresponding with a decomposition (1) of a separable metric space M in disjoint compact sets A, is a separable metric space (comp. for instance WHYBURN [7], p. 123, 124).

We also use another partition-space of M, the so-called *space of quasi-components* $Q(M)$ of M, corresponding with that partition (1) of M for which the sets A are exactly the quasicomponents of M.

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[2]) This theorem was mentioned in [4] without proof. In that paper was, in defining semicompact generalized M. K.-spaces, erroneously spoken of components instead of quasicomponents.

[3]) The upper semi-continuous decomposition of MOORE is identical with the "continuous decomposition" (stetige Zerlegung) of ALEXANDROFF (comp. [10], p. 66—67).

The neighbourhoods in $Q(M)$ of a point $^*a$ are defined as follows: consider the quasicomponent A of M, corresponding with point $^*a$ in $Q(M)$; take an arbitrary set S, both open and closed, with $M \supset S \supset A$; the set of points $^*a$ corresponding with all quasicomponents $A \subset S$ is a neighbourhood of $^*a$ in $Q(M)$. Therefore $Q(M)$ is always 0-dimensional.

(γ) There be given a separable metric space M and the partition of M in its quasicomponents: $M = \Sigma A$. If for every neighbourhood $U = U(A/M)$ of an arbitrary A in M may be found a set S, both open and closed, such that

$$A \subset S \subset U, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

then the decomposition space $^*M$, corresponding with $M = \Sigma A$, is upper semi-continuous and $^*M$ and $Q(M)$ are identical.

Further we apply a simple but useful theorem (first used by LEFSCHETZ):

(δ) If f is a continuous mapping of a separable metric space M on a separable metric space $f(M)$, then the subset

$$\sum_{m \subset M} m \times f(m)$$

of the topological product $M \times f(M)$ is homeomorphic with M.

2.2. **Proof of Theorem II.**

It is evident that every subset of R_1 is a semicompact generalized M.K.-space. Conversely, let K be a semicompact generalized M.K.-space. — Every open arc O of K is mapped topologically within an arbitrary closed arc $O' \subset O$. Every half-open arc H of K is mapped topologically on a proper subset H' of H such that the endpoint belonging to H is mapped on itself. Now K is homeomorphic with

$$K' = K - \Sigma O - \Sigma H + \Sigma O' + \Sigma H'.$$

The above defined mapping of ΣO on $\Sigma O'$ and ΣH on $\Sigma H'$ and the identical mapping of the remaining points of K apparently give a topological mapping of K on K'.

Now apparently $K' \subset K$. Consider

$$M = K' + \Sigma \overline{O} + \Sigma \overline{H} \text{ (closures with regard to } K).$$

M is formed out of K' by adding all endpoints of simple arcs (as far as they were not already contained in K').

M satisfies conditions 2^0. and 3^0. of 1. Moreover M is semicompact, for M is semicompact in the points of K' as well as K, while M is, apparently, also semicompact in the added set of endpoints $M—K'$. It is not necessary, however, that M is compact.

Consider the decomposition of M in its quasicomponents (points or closed simple arcs): $M = \Sigma A$. We shall prove that for every neighbourhood $U = U(A/M)$ of an arbitrary A there may be found a both open and closed set S such that $A \subset S \subset U$.

Within U is a neighbourhood V with compact boundary such that $A \subset V \subset U$; if the quasicomponent A consists of one point this follows immediately from the semicompactness of M; if A is a simple closed arc, and therefore compact, there may be found a finite number of neighbourhoods O_i (open in M) of points of A, each of which has a compact boundary, such that ΣO_i covers the set A. Now the boundary of the neighbourhood $V = \Sigma O_i$ of A is apparently compact.

For a fixed but arbitrary point p lying in the boundary $R(V)$ of V and for A may be found a division of M into disjoint closed sets M_1 and M_2, such that

$$M = M_1 + M_2 \qquad p \in M_1 \quad , \quad A \in M_2.$$

This is possible because p and A belong to different quasicomponents. M_1 is obviously open and a neighbourhood of p not containing A. For every $p \in R(V)$ may be found such a M_1 and M_2. Since V is compact, V is covered by a finite number of such sets M_1. Then the intersection I of the corresponding M_2-sets is a both open and closed set, just like the intersection $I \cdot U = S$.

Now, however, $A \subset S \subset U$; q.e.d.

Therefore we may apply (γ), from which follows that ${}^*M = Q(M)$ and therefore upper semi-continuous. Since the sets A are compact we may apply (β), from which it is evident that ${}^*M$ is separable and metric. Since $Q(M)$ is always 0-dimensional, ${}^*M$ is a 0-dimensional separable metric *space* [4].

According to (α) the semicompact set M may be compactified to a compactum $\overline{M}$ by an 0-dimensional set $\overline{M} - M$. According to a well-known theorem of SIERPIŃSKI the 0-dimensional set ${}^*M$ is homeomorphic with a subset of the discontinuum D of CANTOR, which we also denote by ${}^*M$, such that $\overline{{}^*M}$ (closure in D) is a compact 0-dimensional set.

The mapping $A \to {}^*a$ of M on ${}^*M$ is an apparently continuous mapping f of M on ${}^*M = f(M)$.

Consider in the topological product $\overline{M} \times \overline{{}^*M}$ the subset

$$M' = \sum_{m \subset M} m \times f(m).$$

M' is homeomorphic with M according to (δ).

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[4]) *Remark.* From this part of the proof it follows that we have proved, in general, the following theorem:

Theorem. The space of quasicomponents of a semicompact separable metric space of which the quasicomponents are compact, is an 0-dimensional metric space.

In this theorem we meet two conditions: the semicompactness of the space in question and the compactness of the quasicomponents. If one of these conditions is omitted the theorem becomes untrue. Example 2 gives an instance of a separable metric space with quasicomponents consisting of one point (therefore certainly compact), the space of quasicomponents of which is not a separable metric space, according to [6], p. 134. Semicompact separable metric spaces of which the space of quasicomponents is not a separable metric space, may also be constructed.

We shall prove that $\overline{M'}$ (closure in $\overline{M} \times {}^*\overline{M}$) is a M.K.-space. Then we shall have attained our object, for the original set K is, as follows from what precedes, homeomorphic with a subset of $\overline{M'}$. By applying Theorem I on the M.K.-space $\overline{M'}$, K becomes homeomorphic with a subset of R_1; q.e.d.

$\overline{M'}$ is obviously a M.K.-space, if we can prove that a quasicomponent of $\overline{M'}$ is exactly a quasicomponent of M' or a point of $\overline{M'}-M'$ [5]), for in that case the conditions 1^0. and 2^0. from 1. will be satisfied, and also, since M' is everywhere dense in $\overline{M'}$, condition 3^0.

The projection, i.e., the mapping g

$$m \times {}^*\overline{M} \to m \qquad \text{for every } m \in \overline{M}$$

of $\overline{M} \times {}^*\overline{M}$ on $\overline{M}$ is obviously continuous. g is even topological on M' and its image $g(M') = M$. — Further $g(\overline{M'}) = \overline{M}$.

A point $p \in \overline{M'}-M'$ has an image $g(p)$ belonging to $\overline{M}-M$, for suppose $g(p)$ should belong to M; consider a sequence of points $m_i' \in M'$, converging to p; then $g(m_i')$ converges to $g(p)$ of M; since g is topological on M', $m_i' \to m'$, where m' is that point of $g^{-1}(p)$ which belongs to M'. This gives a contradiction with $m_i' \to p$.

Therefore

$$g(\overline{M'}-M') = \overline{M}-M.$$

Therefore $\overline{M'}-M'$ totally belongs to the topological product of the 0-dimensional sets $\overline{M}-M$ and ${}^*\overline{M}$; this product is, however, 0-dimensional, according to a well-known theorem, and therefore $\overline{M'}-M'$ is also 0-dimensional.

Consider, on the other hand, the projection, i.e., the continuous mapping h

$$\overline{M'} \times {}^*a \to {}^*\overline{M} \qquad \text{for every } {}^*a \in {}^*\overline{M}$$

of $\overline{M'} \times {}^*\overline{M}$ on ${}^*\overline{M}$. Apparently $h(\overline{M'}) = {}^*\overline{M}$, h being continuous! Since $\overline{M'}$ is compact the quasicomponents of $\overline{M'}$ are identical with its components. By a continuous mapping a connected set is mapped on a connected set. A component C of $\overline{M'}$ is mapped by h, since ${}^*\overline{M}$ is 0-dimensional, on exactly one point ${}^*m$ of ${}^*\overline{M}$. The set C therefore belongs to the intersection

$$h^{-1}({}^*m) \cdot \overline{M'} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

If this intersection (3) contains a point of M', then it contains exactly one quasicomponent of M', since different quasicomponents of M (and also of the homeomorphic set M') are mapped (by f) on different points

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[5]) This is by no means the case in $\overline{M}$; $\overline{M}$ may even be connected. For this reason it is necessary to apply the method of topological products.

of $^*M$. Besides this possible quasicomponent Q of M', (3) may only contain points of $\overline{M'}—M'$, therefore an 0-dimensional pointset N. The intersection (3) therefore is the sum of a *compact* quasicomponent Q (point or simple closed arc) and an 0-*dimensional* set N (where Q or N may be vacuous).

A component of $Q + N$, however, is apparently either Q or a point of N; therefore it follows from what precedes that every component C of $\overline{M'}$ is either a quasicomponent of M' or one point of $\overline{M'}—M'$, which is what we had to prove.

2.3. **Remark.** From Theorem II it is particularly evident that in a semicompact generalized M.K.-space every point has an order 2, 1 or 0. Naturally we might have conditioned this property instead of the semicompactness; but then our theorem would have been less general.

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Mathematics. — *On the existence and uniqueness of the solution of the fundamental equation in the theory of metallic conduction.* By L. J. F. BROER. (Communicated by J. D. VAN DER WAALS.)

(Communicated at the meeting of September 27, 1947.)

1. *The fundamental equation.* According to the standard theories of electrons in metals [1]) the state of the electrons in the conduction band can be described by the discrete distribution function f_k. The argument k refers to the L^3 different values of the reduced wave vector $\vec{k}$, where L is the number of atoms per unit length. It is assumed that f_k can change only by the quantum jumps caused by the interaction with the lattice and by the acceleration due to the applied electrical field, not by mutual interaction of the electrons. When we denote the total number of electron transitions from the state k to the state k' per unit time by $I_{kk'}$, the stationary states of the system are found from the fundamental LORENTZ–BLOCH equation:

$$\left(\frac{\partial f_k}{\partial t}\right)_{\text{field}} - \sum_{k'} I_{kk'} + \sum_{k'} I_{k'k} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The set (1) consists of L^3 equations in the L^3 unknowns f_k. However, if we suppose that the field is not so strong as to cause ionisation, the set (1) is interdependent, as summation of all equations yields $0 = 0$. If we now add the condition:

$$\sum_k f_k = N \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where $N < L^3$ is the number of electrons, we can expect that (1) and (2) together allow an unique solution under suitable conditions. The object of this note is to supply these conditions, which apparentely never have been stated in literature. (This problem constitutes the so called third fundamental problem of the theory of metallic conduction [2]).

For our purposes we can assume that the electric field F and the temperature of the lattice are constant throughout the metal. In this case it is shown in the current treatments that:

$$\left(\frac{\partial f_k}{\partial t}\right)_{\text{field}} = -\frac{eF}{h}\frac{\partial f_k}{\partial k_x}$$

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[1]) See e.g. A. SOMMERFELD and H. BETHE, Handbuch der Physik, Vol. 24, part 2, (Berlin 1930).

[2]) F. SAUTER, Ann. der Phys. 42, 110 (1942).

when the electric field has been taken in the x-direction. We may regard $\frac{\partial f_k}{\partial k_x}$ as an abbreviation for the difference:

$$\frac{L}{2}\{f(k_{x_{p+1}}, k_{y_q}, k_{z_r}) - f(k_{x_{p-1}}, k_{y_q}, k_{z_q})\} = \frac{\partial f(k_{x_p}, k_{y_q}, k_{z_r})}{\partial k_x}$$

We note here that $\sum_k \frac{\partial f_k}{\partial k_x} = 0$ as the states $(k_{x_o}, k_{y_q}, k_{z_r})$ and $(k_{x_L}, k_{y_q}, k_{z_r})$ have to be identified according to the concept of reduced wave vector.

When the transition probability per unit time from an occupied state k to an empty state k' is denoted by $P_{kk'}$ we have, according to the FERMI–DIRAC statistics,

$$I_{kk'} = P_{kk'} f_k (1 - f_{k'}). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The fundamental equation can therefore now be written in the form:

$$\frac{eF}{h}\frac{\partial f_k}{\partial k_x} = -\sum_{k'} P_{kk'} f_k (1 - f_{k'}) + \sum_{k'} P_{k'k} f_{k'} (1 - f_k). \quad . \quad . \quad . \quad (4)$$

The problem is now to find conditions for $P_{kk'}$ guaranteeing the existence of one and only one solution of (4) together with (2) which satisfies the condition

$$0 < f_k < 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

imposed by the FERMI–DIRAC statistics. It will be seen that these conditions are such that the question whether and why they are realised in practice needs no special investigation.

2. *Properties of the* $P_{kk'}$. From the definition of $P_{kk'}$ we can infer the two following properties:

A. $P_{kk} \geq 0$ (clear)

B. For every *closed* polygon in k-space the relation:

$$P_{kk'} \cdot P_{k'k''} \ldots . P_{k^{(n)}k} = P_{kk^{(n)}} \ldots . P_{k''k'} \cdot P_{k'k}$$

must be valid. This is a consequence of the fact that the $P_{kk'}$ are calculated from a wave equation invariant for inversion of time. The physical meaning of this relation is that, when only one electron is present, the probabilities of going round this polygon in each direction are equal.

Moreover we will assume, as always safely can be done:

C. The system is ergodic, that is, for every pair of states kk' there exists at least one product:

$$P_{kk''} \cdot P_{k''k'''} \ldots . P_{k^{(n)}k'}$$

which is different from zero. This means that every state k is accessible from every other state k'.

D. The $P_{kk'}$ are analytic functions of F.

We shall show that these four conditions are sufficient for our purpose.

We proceed to prove two useful theorems on systems of numbers $A_{kk'}$ which obey the conditions A, B and C. First we mention that, from B and C [3]), it can be inferred that, whenever $A_{kk'}$ is different from zero, the same applies to $A_{k'k}$. For, according to C, k is accessible from k'. If $A_{k'k}$ would be zero, there is therefore a product $A_{k'k''}...A_{k_{(n)}k}$ different from zero. As $A_{kk'}$ was not zero we conclude from B that $A_{k'k}\,A_{kk_{(n)}}\,\ldots\,A_{k''k'}$ and therefore $A_{k'k}$, is not zero. Now we can state:

Theorem I. When $A_{kk'}$ satisfy A, B and C the equations:

$$-\sum_{k'} A_{kk'}\,x_k+\sum_{k'} A_{k'k}\,x_{k'}=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

have, apart from a constant factor, one and only one solution. This solution has the same sign everywhere

Proof. We select an arbitrary positive number x_l^0. Now there is, by B and C, at least one pair $A_{ll'}$, and $A_{l'l}$ both different from zero. Then $x_{l'}^0$, is defined as $x_l^0\,A_{ll'}/A_{l'l}$.

Next there is an $A_{l'l''}>0$. Then $x_{l''}^0$ is defined as $x_{l'}^0.A_{l'l''}/A_{l''l'}$. When $A_{ll''}>0$ then, by B, we have $x_{l''}^0=x_l^0.\,A_{ll''}/A_{l''l}$. This process is continued till $x_{l_{(n)}}^0$ is defined for all states accessible from l, that is, by C, for all states. The discrete function x_k^0 clearly satisfies:

$$A_{kk'}\,x_k^0=A_{k'k}\,x_{k'}^0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

therefore x_k^0 is a solution of (6).

Now x_k^0 is, by A, non-negative. Moreover, as $A_{kk'}$ and $A_{k'k}$ vanish neither or both and we started from a positive x_l^0, it follows from a repeated application of (7) that all x_l^0 are positive and proportional to x_k^0.

Finally the solution x_k^0 is unique. For, let there exist an essentially different solution x_k^1 (which then would not satisfy (7)). Then there is a minimal value of the ratio x_k^0/x_k^1. Let 1 be one of the states for which this minimum is assumed. Then there is a linear combination

$$y_k=\lambda\,x_k^0+\mu x_k'$$

so that y_k is non-negative and $y_l=0$. y_k has to satisfy (6). As by C there must be some $A_{ll'}$ different from zero we see by A and (6) that the corresponding $y_{l'}$ must be zero. This argument can, by C, be continued till every y_k is seen to be zero. This completes the proof.

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[3]) For this purpose C could be replaced by the weaker condition that, when k is accessible from k', k' is accessible from k, which means that the states k can be divided into groups of mutual accessible states.

Theorem II. When $A_{kk'}$ satisfy A, B and C the determinant:

$$T = \begin{vmatrix} 1 & 1 & 1 & \dots \\ A_{12} & -\sum_{k'} A_{2k'} & A_{32} & \dots \\ A_{13} & A_{23} & -\sum_{k'} A_{3k'} & \dots \\ \vdots & \vdots & \vdots & \end{vmatrix}$$

is different from zero.

P r o o f. We consider the linear homogeneous equations corresponding to T. If the first of these is omitted a set aequivalent to (6) is left. This set therefore has one and only one solution, which is positive throughout. This positive solution cannot satisfy the first equation $\sum_k x_k = 0$. the complete set therefore has only the zero solution. The determinant T thus must be different from zero.

3. *The field-free case.* When $F = 0$ we have to look for a solution of the equations:

$$0 = -\sum_{k'} P_{kk'} f_k (1 - f_{k'}) + \sum_{k'} P_{k'k} f_{k'} (1 - f_k) \quad . \quad . \quad . \quad . \quad (8)$$

which satisfies (2) and (5). By theorem I there exists one positive solution g_k of the equations:

$$0 = -\sum_{k'} P_{kk'} z_k + \sum_{k'} P_{k'k} z_{k'}.$$

We now put $f_k = \frac{\lambda g_k}{1 + \lambda g_k}$. Then f_k satisfies (8) and (5). (This can be seen by substituting $\lambda g_k = \frac{f_k}{1 - f_k}$ in (7)). Furthermore $\sum_k f_k$ is a monotoneous increasing function of λ, so there is only one value λ^0 for which f_k^0 satisfies (2).

From (7) (in the form $P_{kk'} g_k = P_{k'k} g_{k'}$) and (3) we obtain the relation:

$$I_{kk'}^0 = P_{kk'} f_k^0 (1 - f_{k'}^0) = P_{k'k} f_{k'}^0 (1 - f_k^0) = I_{k'k}^0. \quad . \quad . \quad . \quad (9)$$

This equation, which expresses the so-called principle of detailed balance, indicates that there is no net transport along any traject in k-space in equilibrium.

f_k^0 is the only solution derivable from g_k. To complete the proof of uniqueness we still have to show that there are no solutions of (8) for which (2) and (5), but not (9) are valid. To this purpose we first remark that, from (3) it is seen that, whenever $P_{kk'}$ satisfies B, this same property applies to $I_{kk'}$. Furthermore, by (5) all $I_{kk'}$ are positive. Therefore there exists no closed polygon in k-space so that the net transport $I_{kk'} - I_{k'k}$ has the same direction along each of its edges. Now, if there would be a

traject kk' for which $I_{kk'} \neq I_{k'k}$ we see from (1) (with $\frac{\partial f}{\partial t} = 0$) that there must be at last one traject $k'k''$ in which the net transport is in the same direction as in kk'. We can continue in this way till we have constructed a closed polygon consisting of trajects with net transport in the same direction as the number of states is finite. By this contradiction we see that $I_{k'k} = 0$, therefore f_k^0 is the only solution satisfying (2) and (5).

Usually (9) is given either without commentary or by founding it on ONSAGER's theorem of reversibility [4]). The uniqueness of the solution is then dealt with by referring to some proof of the ergodic theorem [5]). We do not think this procedure to be quite exact as both theorems mentioned have been proved for closed systems, attaining equilibrium by mutual interaction of their components. Though we do not doubt that an extension to our case can easily be made, we thought it of interest to present the direct elementary proof given above as both theorems are rather complicated and the proof of the first one often is considered to contain a weak chain [6]).

4. *Solution of the fundamental equation.* With the aid of the results of the two preceeding sections it is now easy to deal with equation (4). We have to solve f_k from L^3 equations of the form:

$$R_k(f_{k'};\ F) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

(viz. (2) and (L^3—1) of the equations (4)). The functions R_k are algebraic in f_k and analytic in F. We know from section 3 that there is one solution f_k^0 satisfying (5) for $F = 0$. We now compute the functional determinant $J = \frac{\partial(R_k)}{\partial(f_{k'})}$ for $f_k = f_k^0$ $F = 0$. We find, using (2), (4) and (9):

$$J = \begin{vmatrix} 1 & 1 & 1 & \cdots \\ P_{12}\frac{(1-f_2^0)}{(1-f_1^0)} & -\sum_{k'} P_{k'2}\frac{(1-f_2^0)}{(1-f_{k'}^0)} & P_{32}\frac{(1-f_1^0)}{(1-f_3^0)} & \cdots \\ P_{13}\frac{(1-f_3^0)}{(1-f_1^0)} & P_{23}\frac{(1-f_3^0)}{(1-f_2^0)} & -\sum_{k'} P_{k'3}\frac{(1-f_3^0)}{(1-f_{k'}^0)} & \cdots \\ \vdots & \vdots & \vdots & \end{vmatrix}$$

As $P_{kk'}(1-f_{k'}^0)/(1-f_k^0)$ satisfies the conditions A, B and C the determinant J is seen, on account of (5), to be of the type T in theorem II,

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[4]) L. ONSAGER, Phys. Rev., **37**, 405 (1931); **38**, 2265 (1931).

[5]) E.g. those of: J. NEUMANN, Z. Phys. **57**, 30 (1929); or: W. PAULI and M. FIERZ, Z. Phys. **106**, 572 (1937).

[6]) See e.g. S. R. DE GROOT, Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam (1945); H. B. G. CASIMIR, Rev. Mod. Phys., **17**, 343 (1945).

therefore $J \neq 0$. From a well known theorem of analysis we can conclude now that there will be one solution of (10), analytic in F within some finit range, which reduces to f_k^0 for $F = 0$.

In order to validate the customary procedure of finding this solution as a power series in F we substitute:

$$f_k = \sum_{\alpha=0}^{\infty} f_k^{(\alpha)} F^\alpha$$

and

$$P_{kk'} = \sum_{\alpha=0}^{\infty} P_{kk'}^{(\alpha)} F^\alpha$$

in (2) and (4) and put the coefficients of F equal to zero separately. We obtain for the terms with F:

$$\left.\begin{aligned} 0 &= \sum_k f_k^1 \\ \frac{e}{h}\frac{\partial f_k^0}{\partial k_x} + \sum_{k'}\{P_{kk'}^1 f_k^0 (1-f_{k'}^0) - P_{k'k}^1 f_{k'}^0 (1-f_k^0)\} &= -\sum Q_{kk'} f_k^1 + \sum Q_{k'k} f_{k'}^1 \end{aligned}\right\}, \quad . \quad . \quad . \quad (11.1)$$

for the terms with F^2:

$$\left.\begin{aligned} 0 &= \sum_k f_k^2 \\ \frac{e}{h}\frac{\partial f_k^1}{\partial k_x} &+ \sum_{k'}\{P_{kk'}^2 f_k^0 (1-f_{k'}^0) - P_{k'k}^2 f_{k'}^0 (1-f_k^0)\} \\ &+ \sum_{k'}[P_{kk'}^1\{f_k^0 f_{k'}^1 + f_k^1 (1-f_{k'}^0)\} - P_{k'k}^1\{f_k^0 f_k^1 + f_{k'}^1 (1-f_k^0)\}] \\ &+ \sum_{k'}(P_{kk'}^0 - P_{k'k}^0) f_k^1 f_{k'}^1 \\ &= -\sum_{k'} Q_{kk'} f_k^2 + \sum_{k'} Q_{k'k} f_{k'}^2 \end{aligned}\right\}, \quad (11.2)$$

etc. where:

$$Q_{kk'} = P_{kk'}^0 \frac{(1-f_{k'}^0)}{(1-f_k^0)}.$$

It will be seen that the sum of all the first members of each set equations (11. n) vanishes. Furthermore, by (5), $Q_{kk'}$ satisfy the conditions A, B and C. We can conclude therefore from theorem II that the sets of equations (11. n) can be solved subsequently and will have only one solution. In fact, the determinant of their right-hand sides is exactly J.

Finally we have to consider condition (5). We proved that f_k^0 satisfies (5). If, upon increasing F steadily, f_k would attain the value 0 or 1 for some wave vector $\vec{k}$ it would clearly have an extremal value here. But, inspection of (4) shows that, whenever f_k would be 0 or 1, then $\frac{\partial f_k}{\partial k_x} > 0$

or < 0 respectively. Therefore our solution f_k is in accordance with (5) for all values of F for which it does exist.

The solution obtained in this way breaks down at the boundary of convergence of the series. Moreover in strong fields equation (3) is not adequate as it does not describe ionisation or passage between bands by the tunnel effect. Nevertheless we have proved that the usual treatment of this equation is valid up to some finite value of the field strength.

Mathematics. — *The congruence order of the elliptic plane.* By J. HAANTJES and J. SEIDEL. (Communicated by Prof. W. VAN DER WOUDE.)

(Communicated at the meeting of September 27, 1947.)

A *metric space* is a set of abstract elements called points such that to each pair of points p, q there is attached a non-negative real number pq, called the distance of p and q, satisfying the conditions

1. $pq = 0$ implies $p = q$
2. $pq + qr \geq pr$ (triangle inequality)

Two metric spaces M and M' are said to be *congruent* if there exists a mapping of each into the other preserving the distances of the points.

A metric space M is called *imbeddable* in a metric space S if M is congruent with a subset of S.

In order to find a metric characterization of euclidean spaces MENGER showed in the "Zweite Untersuchung" [1]) that if every set of $n + 3$ points of a metric space M is congruent with $n + 3$ points of a euclidean n-dimensional space R_n, then M is congruent with a subset of R_n. The number $n + 3$ is a metric invariant of the R_n and is called the congruence order of the R_n. The notion congruence order is then defined as follows:

A space S' has *congruence order* k provided that any metric space M is congruent with a subset of S' whenever each set of k points of M is congruent with k points of S'.

It has been proved that the congruence order of the n-dimensional hyperbolical space and of the n-dimensional spherical space is also $n - 3$ [2]). It is however easily seen that the congruence order of the elliptic plane I_2 is at least seven [3]). For an I_2 of total curvature r^2 contains equilateral sextuples [3]) [4]) with edge $d = r \arccos \frac{1}{5}\sqrt{5}$. Therefore, a metric space M with more than 6 points, for which the distance of every pair of different points equals d, has the property that every subset of 6 points is congruent with 6 points of I_2, whereas M is not congruent with a subset of I_2. This example shows that the congruence order of I_2 is at least 7. It is the purpose of this paper to show that this congruence order is 7. For the sake of brevity we have to confine ourselves to a general outline of the proof. The full proof will be published shortly in the thesis of J. SEIDEL.

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[1]) K. MENGER, Untersuchungen über allgemeine Metrik, Math. Annalen **100**, 113—141 (1928).

[2]) L. M. BLUMENTHAL, The geometry of a class of semimetric spaces. Tôhoku Math. Journal **43**, 205—214 (1937).

[3]) L. M. BLUMENTHAL, Metric characterization of elliptic space. Trans. Amer. Math. Soc. **59**, 381—400 (1946).

[4]) J. HAANTJES, Equilateral point-sets in elliptic two- and three-dimensional spaces. Nieuw Arch. v. Wiskunde XXII.

The methods used in order to find the congruence order of euclidean and hyperbolic spaces are wholly unsuitable for elliptic spaces. One of the main reasons is that the congruence of two subsets of an elliptic space I_2 does not imply *superposability* (that is the existence of a congruent transformation of the space on itself which carries one set into the other). A simple example is furnished by two triples a, b, c and a', b', c' with $ab = bc = ac = a'b' = b'c' = a'c' = \frac{1}{3}\pi r$, where the points a, b and c lie on a straight line and a', b' and c' form the vertices of a proper triangle.

As a first result the following theorem is obtained:

Theorem 1. *A metric space M consisting of exactly 8 points is imbeddable in I_2 whenever each seventuple is.*

In order to prove this theorem the following cases are distinguished:

A. M contains four points, which are congruent with four points of I_2 forming the vertices and the orthocenter of a proper triangle of which no side equals $\frac{1}{2}\pi r$. Let it be the points 1, 2, 3, 4. This property is denoted by O (1234).

B. M contains four points which are congruent with four points on a straight line in I_2. This property is denoted by L (1234).

C. M contains five points congruent with five points of I_2 three of which (1, 2, 3) are linear, whereas the line joining the two other points (4, 5) is perpendicular to the line (123). Moreover no four points of this fivetuple have the property O. This property is denoted by V (123,45).

D. M contains no point sets with the property O, L or V.

The point sets with one of the properties O, L and V play an important part in the proof because it can be shown that two congruent sets of this kind in I_2 are at the same time superposable.

In the case A [O(1234)] we consider the point sets in I_2 which are congruent with the sets 1234567, 1234568, 1234578, 1234678. The corresponding points in I_2 are denoted by 1234567, 12345′6′8, 12345″7′8′, 12346″7″8″, which means that the elliptic representation of the first four points is the same in each set. This may be supposed because according to the above remark any two congruent representations of these four points are superposable. Now the triple 5, 5′, 5″ may consist of only one point (the points are identical) or it may contain two different points or three different points. If $5 \neq 5'$ the perpendicular bisectors of the segment 55′ must contain the points 1, 2, 3 and 4 because the points 5 and 5′ have the same distances to the points 1234. If 5, 5′ and 5″ are three different points the triple is wholly determined by the set 1234, the points of which being the four circumcenters of the triangle 55′ 5″. The same can be said of the triples 66′ 6″, 77′ 7″ and 88′ 8″. This leads to several cases which are treated seperately. In either possible case it can be shown that from each triple one point can be chosen in such a way that these points together with the points 1234 are congruent with the eight points of M. Thus M is imbeddable in I_2.

The cases mentioned under B and C are dealt with in much the same way. The treatment is somewhat simpler because as is easily seen at least two of the points 55′ 5″ (66′ 6″ etc.) must coincide.

If the metric space M contains no point sets with the property O, L or V (the case mentioned under D) it is proved that there exist three points (1, 2, 3) such that in the congruent representations of the sets 1234567, 1234568, 1234578 the representations of the points 1, 2, 3 are superposable. Then the corresponding points may be denoted by 1234567, 1234′5′6′8, 1234″5″7′8′. Again the points 5, 5′, 5″ etc. may be different or equal. This leads to several cases, which are all treated separately. Again it turns out that in either case eight points in I_2 can be found congruent with the metric space M. Therefore, M is imbeddable in I_2.

Then the proof that the congruence order is seven is completed by showing:

Theorem 2. *A metric space M is imbeddable in I_2 whenever each eighttuple is.*

This theorem may also be stated as follows:

The congruence order of the I_2 is ≤ 8.

The proof of this theorem is similar to that of the first theorem. From theorem 1 and 2 it is seen that the congruence order is ≤ 7, whereas an example shows, as we have seen, that it is ≥ 7. So the congruence order of the elliptic plane is seven.

Mathematics. — *On the zeros of composition-polynomials.* By N. G. DE BRUIJN and T. A. SPRINGER. (Communicated by Prof. W. VAN DER WOUDE.)

(Communicated at the meeting of September 27, 1947.)

1. Introduction.

Recently [1]) we proved some inequalities, expressing that the zeros of the derivative of a polynomial lie, in the mean, closer to a given line or a given point in the complex plane than the zeros of the polynomial itself. One may expect that similar inequalities are valid if, instead of the derivative, a polynomial derived in a more general way from the given one is considered. Here we shall prove inequalities of this type for polynomials obtained by composition from two given polynomials.

This composition is defined in the following way [2]): if

$$A(z) \equiv \binom{n}{0} a_0 + \binom{n}{1} a_1 z + \binom{n}{2} a_2 z^2 + \ldots + \binom{n}{n} a_n z^n$$

and

$$B(z) \equiv \binom{n}{0} b_0 + \binom{n}{1} b_1 z + \binom{n}{2} b_2 z^2 + \ldots + \binom{n}{n} b_n z^n$$

are the two given polynomials, then the composition-polynomial is

$$AB(z) \equiv \binom{n}{0} a_0 b_0 + \binom{n}{1} a_1 b_1 z + \binom{n}{2} a_2 b_2 z^2 + \ldots + \binom{n}{n} a_n b_n z^n.$$

If $B(z) \equiv nz(1+z)^{n-1}$ we have $AB(z) \equiv zA'(z)$. For this special choice of $B(z)$ most of the theorems proved in this paper give rise to results already proved in I and II.

Throughout this paper the zeros of $A(z)$, $B(z)$ and $AB(z)$ will be denoted by $\alpha_1, \ldots, \alpha_n$; $\beta_1, \ldots, \beta_n$ and $\gamma_1, \ldots, \gamma_n$, respectively. Furthermore we put

$$\{A, B\} = \binom{n}{0} a_0 b_n - \binom{n}{1} a_1 b_{n-1} + \ldots + (-1)^n \binom{n}{n} a_n b_0.$$

We shall often use the following well-known theorem of J. H. GRACE

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[1]) Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, **49**, 1037—1044 (1946) = Indagationes Mathematicae 8, 635—643; Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, **50**, 458—464 (1947) = Indagationes Mathematicae **9**, 264—270. These papers are referred to as I and II.

[2]) This way of composition was introduced by G. SZEGÖ, Math. Zeitschrift, **13**, 28—55 (1922).

("Grace's Apolarity Theorem") [3], which is rather important in problems concerning the geometric properties of the roots of polynomials:

Theorem 1: *If the two polynomials $A(z)$ and $B(z)$ satisfy $\{A, B\} = 0$, then any circle-domain [4] containing all zeros of $A(z)$ contains at least one zero of $B(z)$.*

From this theorem we deduce our fundamental

Lemma: *Let $A(z)$ and $B(z)$ be two polynomials and let D_1 and D_2 be the two closed domains determined by a circle (or a straight line) C. The polynomial $A^*(z)$ is derived from $A(z)$ by replacing the zeros of $A(z)$ which lie in D_1 by their images with respect to C, and is normed such that $|A^*(z)| = |A(z)|$ on C [5]; $B^{**}(z)$ is obtained from $B(z)$ in a similar way, but now the zeros in D_2 are replaced by their images with respect to C. Then we have*

$$|\{A, B\}| \leqq |\{A^*, B^{**}\}|. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Proof: We may suppose that $A(z)$ and $B(z)$ are $\neq 0$ on C: the general case then follows by an argument of continuity.

Since no zeros of $A^*(z)$ lie in D_1, and since $|A(z)| = |A^*(z)|$ if z lies on C, it follows by application of the maximum modulus theorem, that $|A(z)| \leq |A^*(z)|$ in D_1. In the same way it can be shown that $|B(z)| \leq B^{**}(z)|$ in D_2. Hence the polynomials $A^*(z) - \lambda A(z)$ and $B^{**}(z) - \lambda B(z)$ have no zeros in D_1 and D_2 respectively, if λ is any number satisfying $|\lambda| < 1$. By theorem 1 we now have $\{A^* - \lambda A, B^{**} - \lambda B\} \neq 0$ for $|\lambda| < 1$. Thus the polynomial

$$\lambda^2\{A, B\} - \lambda(\{A, B^{**}\} + \{A^*, B\}) + \{A^*, B^{**}\}$$

has no zeros in $|\lambda| < 1$, and it follows that $|\{A, B\}| \leq |\{A^*, B^{**}\}|$.

In section 2 inequalities are derived for the distances of the zeros to a line. The following theorem shows the character of the results obtained in that section:

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[3] J. H. GRACE, Proc. Cambridge Phil. Soc. **11**, 352—357 (1900—'02). — G. SZEGÖ, loc. cit. [2]). — G. PÓLYA and G. SZEGÖ, Aufgaben und Lehrsätze aus der Analysis, II, p. 64, Aufgabe 145.

[4] A circle-domain is a closed part of the plane whose boundary is a circle or a straight line.

[5] So if $A(z) \equiv A(z-\varrho_1)\ldots(z-\varrho_n)$, $\varrho_1, \ldots, \varrho_k$ in D_2, $\varrho_{k+1}, \ldots, \varrho_n$ in D_1 but not on C, and if σ lies on C then

$$A^*(z) \equiv \varepsilon A \prod_{i=1}^{k}(z-\varrho_i) \prod_{i=k+1}^{n}(z-\varrho_i^*) \cdot \frac{\sigma-\varrho_i}{\sigma-\varrho_i^*},$$

where ϱ_i^* represents the image of ϱ_i with respect to C, and $|\varepsilon| = 1$. The true value of ε is irrelevant.

Theorem 2: *If all β_i are ≤ 0, then we have*

$$\sum_{\nu=1}^{n} |Im\, \gamma_\nu| \leqq \frac{b_{n-1}}{b_n} \sum_{\nu=1}^{n} |Im\, \alpha_\nu|. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

This is a special case of the more general theorem 3.

Theorems 4 and 5, which are derived from theorem 3, may also be of some interest.

In section 3 we are dealing with the moduli of the zeros. There the main result is

Theorem 9: *With the same notations as before we have, if p and k are real numbers, $k \geq 1$, $\frac{1}{k} + \frac{1}{k'} = 1$,*

$$\sum_{\nu=1}^{n} |\gamma_\nu|^p \leqq \left(\sum_{\nu=1}^{n} |\alpha_\nu|^{pk}\right)^{\frac{1}{k}} \cdot \left(\sum_{\nu=i}^{n} |\beta_\nu|^{pk'}\right)^{\frac{1}{k'}} \quad . \quad . \quad . \quad . \quad . \quad (3)$$

This inequality is deduced from a more general one, which contains an arbitrary convex function (Theorem 8).

2. We prove the following theorem on the distances of the zeros of a composition-polynomial to a straight line:

Theorem 3: *If $\psi(z, l)$ denotes the distance of the point z in the complex plane to a line l we have*

$$a) \quad \sum_{\nu=1}^{n} \psi(\gamma_\nu, l) \leqq \frac{b_{n-1}}{b_n} \sum_{\nu=1}^{n} \psi(\alpha_\nu, l),$$

if the line l contains the point $z = 0$ and all zeros β_i satisfy $\beta_i \leq 0$ $(i = 1, 2, \ldots, n)$,

and

$$b) \quad \sum_{\nu=1}^{n} \{\psi(\gamma_\nu, l) - \psi(0, l)\} \leqq \frac{b_{n-1}}{b_n} \sum_{\nu=1}^{n} \{\psi(\alpha_\nu, l) - \psi(0, l)\},$$

if l is arbitrary and all β_i satisfy $-1 \leq \beta_i \leq 0$ $(i = 1, 2, \ldots, n)$

Proof: Let H_1 and H_2 be the closed half-planes determined by the line l. If l does not contain the point $z = 0$ we suppose that this point lies in H_2. We introduce the auxiliary function $A^*(z)$ obtained from $A(z)$ by replacing the zeros of $A(z)$ which lie in H_1 by their images with respect to l, and which is normed such that $|A^*(z)| = |A(z)|$ on l.

Let y be a number on l. The polynomial

$$P(z) \equiv \binom{n}{0} b_0 z^n - \binom{n}{1} b_1 z^{n-1} y + \ldots + (-1)^n \binom{n}{n} b_n y^n$$

(considered as a polynomial in z) has the zeros $-\frac{y}{\beta_1}, \ldots, -\frac{y}{\beta_n}$. These numbers belong to H_1 or to l. This follows from the assumptions under a)

as well as from those under b). By application of the lemma [6]) to $A(z)$ and $P(z)$ we now obtain

$$\left|\binom{n}{0} a_0 b_0 + \binom{n}{1} a_1 b_1 y + \ldots + \binom{n}{n} a_n b_n y^n\right| \leqq$$

$$\leqq \left|\binom{n}{0} a_0{}^* b_0 + \binom{n}{1} a_1{}^* b_1 y + \ldots + \binom{n}{n} a_n{}^* b_n y^n\right|,$$

if y lies on l (the coefficient of z^k in $A^*(z)$ being denoted by $\binom{n}{k} a_k^*$).

Evidently $|a_n{}^*| = |a_n|$, and it follows, $\gamma_\nu{}^*$ $(\nu = 1, 2, \ldots, n)$ being the zeros of $A^*B(z)$, that

$$\sum_{\nu=1}^{n} \log|y-\gamma_\nu| \leqq \sum_{\nu=1}^{n} \log|y-\gamma_\nu^*|,$$

if y lies on l.

Now if y is a variable point on l we may write $y = xe^{i\varphi} + a$, where x is a real variable and φ and a are fixed real numbers. Thus

$$\sum_{\nu=1}^{n} \int_{-A}^{A} \log|xe^{i\varphi} + a - \gamma_\nu|\, dx \leqq \sum_{\nu=1}^{n} \int_{-A}^{A} \log|xe^{i\varphi} + a - \gamma_\nu^*|\, dx. \qquad (4)$$

By application of the formula

$$\int_{-A}^{A} \log|x-\alpha|\, dx = 2(A \log A - A) + \pi|Im\,\alpha| + O\left(\frac{1}{A}\right)$$

(cf. II, formula (9)), we obtain from (4) by making $A \to \infty$

$$\sum_{\nu=1}^{n} \psi(\gamma_\nu, l) \leqq \sum_{\nu=1}^{n} \psi(\gamma_\nu^*, l).$$

It is easily deduced from GRACE's theorem [7]) that the $\gamma_\nu{}^*$ all lie in H_2; hence

$$\Sigma\, \psi(\gamma_\nu^*, l) = n\, \psi\left(\frac{1}{n} \Sigma \gamma_\nu^*, l\right)$$

and

$$\Sigma\, \psi(\gamma_\nu, l) \leqq n\, \psi\left(-\frac{a_{n-1}^* b_{n-1}}{a_n^* b_n}, l\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

The numbers

$$-\frac{a_{n-1}^*}{a_n^*} = \frac{1}{n} \Sigma a_\nu^* \quad \text{and} \quad -\frac{a_{n-1}^* b_{n-1}}{a_n^* b_n} = \frac{1}{n} \Sigma \gamma_\nu^*$$

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[6]) Since the zeros of $P(z)$ lie in H_1, we have $P^{**}(z) \equiv \varepsilon P(z)$ $(|\varepsilon| = 1)$.

[7]) Cf. SZEGÖ, loc. cit. [2]), Satz 2; PÓLYA–SZEGÖ II, p. 65, Aufg. 151.

both lie in H_2. Since b_{n-1}/b_n is positive, it is easily verified that

$$\psi\left(-\frac{a_{n-1}^* b_{n-1}}{a_n^* b_n}, l\right) - \psi(0, l) = \frac{b_{n-1}}{b_n}\left\{\psi\left(-\frac{a_{n-1}^*}{a_n^*}, l\right) - \psi(0, l)\right\}. \quad (6)$$

Finally using

$$\Sigma\,\psi(a_\nu, l) = \Sigma\,\psi(a_\nu^*, l) = n\,\psi\left(-\frac{a_{n-1}^*}{a_n^*}, l\right),$$

we obtain from (5) and (6)

$$\Sigma\{\psi(\gamma_\nu, l) - \psi(0, l)\} \leqq \frac{b_{n-1}}{b_n}\,\Sigma\{\psi(a_\nu, l) - \psi(0, l)\}.$$

This proves our theorem, for if l contains $z = 0$, we have $\psi(0, l) = 0$.

As to the validity of theorem 3 for other functions $\psi(z)$ we make the following remarks. *b*) is also true if $\psi(z, l)$ is replaced by a constant function of z, but *a*) is not. Both *a*) and *b*) remain true if $\psi(z, l)$ is replaced by an arbitrary homogeneous linear function of $Re\,z$ and $Im\,z$. It follows by an argument used in II that statement *b*) of theorem 3 holds true if $\psi(z, l)$ is replaced by a function that can be obtained by superposition of functions of the types $|Im\,(\alpha z + \beta)|$ and $Im\,(\alpha z + \beta)$. In II the class of these functions was called C*. Examples of such functions are $|Im\,z|^\lambda$ and $|z|^\lambda$ $(\lambda \geq 1)$. Thus we obtain

Theorem 4: *If $\psi(z)$ is a function of the class C* and $-1 \leq \beta_i \leq 0$ $(i = 1, 2, \ldots, n)$, we have*

$$\sum_{\nu=1}^{n}\{\psi(\gamma_\nu) - \psi(0)\} \leqq \frac{b_{n-1}}{b_n}\sum_{\nu=1}^{n}\{\psi(a_\nu) - \psi(0)\}.$$

By an integration-process, as carried out in the proof of theorem 4 in II, we can deduce from case *a*) of theorem 3 the following

Theorem 5: *If $\beta_i \leq 0$ $(i = 1, 2, \ldots, n)$, we have*

$$\left|\sum_{\nu=1}^{n}\gamma_\nu\right| \leqq \frac{b_{n-1}}{b_n}\sum_{\nu=1}^{n}|a_\nu|.$$

This also follows from Theorem 4 by a simple transformation.

Finally we remark that it is easy to infer theorems of the following type:

Theorem 6 [8]**:** *Let the polynomial*

$$f(z) \equiv a_n z^n + \ldots + a_1 z + a_0$$

have the zeros $a_1, \ldots, a_n$ and let $P(y)$ be a polynomial all of whose zeros are real and ≤ 0. Let $\gamma_1, \ldots, \gamma_n$ be the zeros of

$$a_n P(n) z^n + \ldots + a_1 P(1) z + a_0 P(0).$$

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[8]) In I (theorem 4) this result was proved under the assumption that all a_i are real.

Then

$$\sum_{\nu=1}^{n} |Im\, \beta_\nu| \leqq \frac{P(n-1)}{P(n)} \sum_{\nu=1}^{n} |Im\, \alpha_\nu| .$$

This follows from theorem 3 (case a)) and from the fact that

$$\binom{n}{n} P(n)\, z^n + \ldots + \binom{n}{1} P(1)\, z + \binom{n}{0} P(0)$$

has real, non-positive zeros only [9]).

3. In the proof of theorem 3 we integrated along a straight line. In this section we prove, by integration along a circle, inequalities of another type.

We use a well-known formula of JENSEN: if

$$f(z) \equiv c_n z^n + c_{n-1} z^{n-1} + \ldots + c_1 z + c_0$$

has the zeros $\xi_1, \ldots, \xi_n$, then we have

$$\frac{1}{2\pi} \int_0^{2\pi} \log \left| \frac{f(re^{i\vartheta})}{c_0} \right| d\vartheta = \sum_{|\xi_\nu| \leq r} \log \frac{r}{|\xi_\nu|} .$$

The sum is taken over those values of ν, which satisfy the condition indicated under the summation sign. We can also write it in the form

$$\sum_{|\xi_\nu| \leq r} \log \frac{r}{|\xi_\nu|} = \sum_{\nu=1}^{n} \log \left\{ \operatorname{Max} \left(\frac{r}{|\xi_\nu|}, 1 \right) \right\} .$$

Theorem 7: *With the same notation as in the introduction we have*

$$\prod_{|\gamma_\nu| \leq r_1 r_2} \frac{r_1 r_2}{|\gamma_\nu|} \leqq \prod_{|\alpha_\nu| \leq r_1} \frac{r_1}{|\alpha_\nu|} \cdot \prod_{|\beta_\nu| \leq r_2} \frac{r_2}{|\beta_\nu|} \quad (r_1 > 0,\ r_2 > 0) \quad . \quad . \quad (7)$$

Proof. Without loss of generality we may take $r_1 = r_2 = 1$.

We will use the notation A^*, A^{**}, explained in the introduction, taking for D_1 the domain $|z| \leq 1$.

The polynomials $A^*(z)$ and $B^*(z)$ having no roots in $|z| < 1$, if follows from GRACE's theorem [10]) that their composition $A^* B^*(z)$ is also $\neq 0$ for $|z| < 1$. Furthermore

$$|A\, B(y)| \leqq |A^*\, B^*(y)| \qquad (|y| = 1) \quad . \quad . \quad . \quad . \quad . \quad (8)$$

To prove this, take

$$A_1(z) = A(y\, z), \quad B_1(z) = z^n\, B\left(-\frac{1}{z} \right) .$$

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[9]) Cf. G. PÓLYA and G. SZEGÖ, Aufgaben und Lehrsätze aus der Analysis, II, p. 47, Aufg. 67.

[10]) Cf. SZEGÖ, loc. cit. [2]), Satz 3·1; PÓLYA–SZEGÖ II, p. 65, Aufg. 152.

Then we have

$$A_1^*(z) = A^*(y\,z), \quad B_1^{**}(z) = z^n\,B^*\left(-\frac{1}{z}\right)$$

and

$$\{A_1, B_1\} = A\,B\,(y), \quad \{A_1^*, B_1^{**}\} = A^*\,B^*\,(y)$$

Now (8) follows from (1).

Since A^*, B^* and A^*B^* are $\neq 0$ for $|z|<1$, JENSEN's formula gives

$$\frac{1}{2\pi}\int_0^{2\pi} \log\left|\frac{A^*(e^{i\varphi})}{A^*(0)}\right| d\varphi +$$

$$+\frac{1}{2\pi}\int_0^{2\pi} \log\left|\frac{B^*(e^{i\varphi})}{B^*(0)}\right| d\varphi = 0 = \frac{1}{2\pi}\int_0^{2\pi} \log\left|\frac{A^*\,B^*(e^{i\varphi})}{A^*\,B^*(0)}\right| d\varphi.$$

From the relations

$$A^*(0)\,B^*(0) = A^*\,B^*(0), \quad A(0)\,B(0) = A\,B(0),$$
$$|A(e^{i\varphi})| = |A^*(e^{i\varphi})|, \quad |B(e^{i\varphi})| = |B^*(e^{i\varphi})|$$

and from (8) we now infer:

$$\frac{1}{2\pi}\int_0^{2\pi} \log\left|\frac{A(e^{i\varphi})}{A(0)}\right| d\varphi + \frac{1}{2\pi}\int_0^{2\pi} \log\left|\frac{B(e^{i\varphi})}{B(0)}\right| d\varphi \geqq \frac{1}{2\pi}\int_0^{2\pi} \log\left|\frac{A\,B(e^{i\varphi})}{A\,B(0)}\right| d\varphi.$$

Now application of JENSEN's theorem leads to (7).

More general inequalities can be deduced from theorem 7.

Theorem 8: *If $\psi(x)$ is a convex function of the real variable x, we have, if $\lambda>0$, $\mu>0$,*

$$(\lambda+\mu)\sum_{\nu=1}^{n}\psi\left(\frac{\log|\gamma_\nu|}{\lambda+\mu}\right) \leqq \lambda\sum_{\nu=1}^{n}\psi\left(\frac{\log|\alpha_\nu|}{\lambda}\right) + \mu\sum_{\nu=1}^{n}\psi\left(\frac{\log|\beta_\nu|}{\mu}\right).$$

Proof: Putting $r_1 = e^{\lambda t}$, $r_2 = e^{\mu t}$ in (7), we find, if

$$h_t(x) = \mathrm{Max}\,(t-x, 0),$$

$$(\lambda+\mu)\sum_{\nu=1}^{n} h_t\left(\frac{\log|\gamma_\nu|}{\lambda+\mu}\right) \leqq \lambda\sum_{\nu=1}^{n} h_t\left(\frac{\log|\alpha_\nu|}{\lambda}\right) + \mu\sum_{\nu=1}^{n} h_t\left(\frac{\log|\beta_\nu|}{\mu}\right).$$

This inequality is also true (with the sign of equality), if $h(x)$ is a linear function of x, owing to the relation

$$\prod_{\nu=1}^{n}|\alpha_\nu| \cdot \prod_{\nu=1}^{n}|\beta_\nu| = \prod_{\nu=1}^{n}|\gamma_\nu|.$$

We can construct (cf. the argument in I, theorem 7) a linear combination

of a linear function and functions $h_t(x)$, with positive weights, which attains the same value as $\psi(x)$ in the $3n$ points

$$x=\frac{\log|\alpha_\nu|}{\lambda}\quad,\quad x=\frac{\log|\beta_\nu|}{\mu}\quad,\quad x=\frac{\log|\gamma_\nu|}{\lambda+\mu}\qquad(\nu=1,2,\ldots,n).$$

The theorem follows from this remark.

Theorem 9: *If* $p\geq 0$ [11]), $k\geq 1, \frac{1}{k}+\frac{1}{k'}=1$, *we have*

$$\sum_{\nu=1}^{n}|\gamma_\nu|^p\leqq\Big(\sum_{\nu=1}^{n}|\alpha_\nu|^{pk}\Big)^{\frac{1}{k}}\Big(\sum_{\nu=1}^{n}|\beta_\nu|^{pk'}\Big)^{\frac{1}{k'}}. \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Proof: For $\psi(x)=e^{px}$ the previous theorem yields

$$(\lambda+\mu)\sum_{\nu=1}^{n}|\gamma_\nu|^{\frac{p}{\lambda+\mu}}\leqq\lambda\sum_{\nu=1}^{n}|\alpha_\nu|^{\frac{p}{\lambda}}+\mu\sum_{\nu=1}^{n}|\beta_\nu|^{\frac{p}{\mu}}. \quad . \quad . \quad . \quad . \quad (9)$$

Let u be a positive parameter. The inequality (9) being true for any two polynomials $A(z)$ and $B(z)$, we can also apply it to $A(\frac{z}{u})$ and $B(uz)$, so that

$$(\lambda+\mu)\sum_{\nu=1}^{n}|\gamma_\nu|^{\frac{p}{\lambda+\mu}}\leqq\lambda u^{\frac{p}{\lambda}}\sum_{\nu=1}^{n}|\alpha_\nu|^{\frac{p}{\lambda}}+\mu u^{-\frac{p}{\mu}}\sum_{\nu=1}^{n}|\beta_\nu|^{\frac{p}{\mu}}.$$

For shortness we write this as

$$(\lambda+\mu)c\leqq\lambda a\cdot u^{\frac{p}{\lambda}}+\mu b\cdot u^{-\frac{p}{\mu}}.$$

The right-hand side of this inequality attains its minimum if $u=\left(\frac{b}{a}\right)^{\frac{\lambda\mu}{p(\lambda+\mu)}}$ and for this value of u the inequality becomes

$$c\leqq a^{\frac{\lambda}{\lambda+\mu}}\cdot b^{\frac{\mu}{\lambda+\mu}}$$

or

$$\sum_{\nu=1}^{n}|\gamma_\nu|^{\frac{p}{\lambda+\mu}}\leqq\Big(\sum_{\nu=1}^{n}|\alpha_\nu|^{\frac{p}{\lambda}}\Big)^{\frac{\lambda}{\lambda+\mu}}\cdot\Big(\sum_{\nu=1}^{n}|\beta_\nu|^{\frac{p}{\mu}}\Big)^{\frac{\mu}{\lambda+\mu}}.$$

On taking $\lambda=\frac{1}{k}$, $\mu=\frac{1}{k'}$ we obtain (3).

The inequalities obtained above are symmetrical in $A(z)$ and $B(z)$. There also are inequalities in which $B(z)$ does not occur explicitly. We state

Theorem 10: *Let* $\psi(x)$ *be a convex function of the real variable* x. *Then we have*

$$\sum_{\nu=1}^{n}\psi(\log|\gamma_\nu|)\leqq\sum_{\nu=1}^{n}\psi(\log|\alpha_\nu|).$$

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[11]) The theorem is also valid for $p<0$ but then we must suppose no α_ν or β_ν to be zero.

in the following cases:

a) all zeros of $B(z)$ lie on the circle $|z| = 1$;

b) all zeros of $B(z)$ lie in $|z| \geq 1$ and $\psi(x)$ is monotonically non-increasing;

c) all zeros of $B(z)$ lie in $|z| \leq 1$ and $\psi(x)$ is monotonically non-decreasing.

Proof: In case a) theorem 7 yields ($r_1 = r$, $r_2 = 1$)

$$\prod_{|\gamma_\nu| \leq r} \frac{r}{|\gamma_\nu|} \leq \prod_{|\alpha_\nu| \leq r} \frac{r}{|\alpha_\nu|} \qquad (r > 0)$$

or

$$\sum_{\nu=1}^{n} \varphi_r(\log|\gamma_\nu|) \leq \sum_{\nu=1}^{n} \varphi_r(\log|\alpha_\nu|) \quad . \quad . \quad . \quad . \quad . \quad (10)$$

if

$$\varphi_r(t) = \begin{cases} \log r - t & t \leq \log r \\ 0 & t \geq \log r \end{cases}.$$

Furthermore, (10) holds also for linear functions $\varphi(t)$ with the sign of equality. By the argument of theorem 8 it then follows that (10) holds for any convex function.

In case *b*) (10) is also valid. Besides, it holds also for non-increasing functions $\varphi(t) \equiv at + b$, which follows from

$$\sum_{\nu=1}^{n} \log|\gamma_\nu| = \sum_{\nu=1}^{n} \log|\alpha_\nu| + \sum_{\nu=1}^{n} \log|\beta_\nu| \geq \sum_{\nu=1}^{n} \log|\alpha_\nu|.$$

Now, by linear superpositions with positive weights, the result again follows (cf. I, theorem 7).

Case *c*) can be derived from *b*) by the transformation $z_1 = \frac{1}{z}$.

On taking $B(z) \equiv nz(1+r)^{n-1}$, we find $AB(z) \equiv zA'(z)$. Therefore we can apply the preceding theorems to deduce some new results concerning the derivative of a polynomial, which were not obtained in I and II.

Theorem 11: *If the polynomial $f(z)$ has the zeros $\xi_1, \ldots, \xi_n$ and if $\eta_1, \ldots, \eta_{n-1}$ are the zeros of its derivative, then*

$$\sum_{\nu=1}^{n-1} |\eta_\nu|^p \leq (n-1)^{1-\frac{1}{k}} \left(\sum_{\nu=1}^{n} |\xi_\nu|^{pk}\right)^{\frac{1}{k}} \qquad (k \geq 1, p \geq 0).$$

Theorem 12: *Under the assumptions of theorem 11 we have*

$$\sum_{\nu=1}^{n-1} \psi(\log|\eta_\nu|) \leq \sum_{\nu=1}^{n} \psi(\log|\xi_\nu|)$$

where $\psi(x)$ is a convex function of the real variable x which is non-decreasing and for which $\psi(-\infty) \geq 0$.

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Mathematics. — *An outer limit of nonconformalness, for which* PICARD's *theorem still holds.* By R. J. WILLE. (Communicated by Prof. W. VAN DER WOUDE.)

(Communicated at the meeting of September 27, 1947.)

1. Let $w = f(z)$ be a function, which involves a $1 - 1$ correspondence between the finite z-plane and a, possibly multiply sheeted, mapping surface on the w-plane. By projection of this mapping surface on a sphere with radius $\frac{1}{2}$, touching the w-plane at zero with its southpole and using its northpole as centre of projection, we obtain a mapping surface F on the sphere.

PICARD's theorem on exceptional points does not only hold in case of conformal mapping, in which infinitesimal circles on F correspond to infinitesimal circles in the z-plane, but also in case of quasi-conformal mapping, in which infinitesimal circles on F correspond to infinitesimal ellipses in the z-plane, with eccentricities uniformly bounded by $e < 1$, as L. AHLFORS has shown (Eindeutige Analytische Funktionen, by R. NEVANLINNA, chapter XIII, § 8).

Let $F(r)$ be the mapping surface of $|z| \leq r$; we suppose the eccentricities $\varepsilon(z)$ of the infinitesimal ellipses, corresponding to the infinitesimal circles at points on $F(r)$, to have the upper bound $e(r) < 1$. Then we propose to show the validity of PICARD's theorem, when $e(r)$ tends not too rapidly to 1 (r tending to infinity) and to indicate also the order of rapidity, for which the theorem just holds. The precise theorem, we shall prove, is the following:

If the integral

$$\int^{\infty} \frac{\sqrt{1-e(r)^2}}{r}\, dr$$

is divergent, there are two exceptional points at the most. On the contrary, if the integral is convergent, a mapping exists even with a whole circle of exceptional points.

2. In order to give the proof we memorise the following inequality, deduced by L. AHLFORS:

$$h\,L(r) + \sum_{i=1}^{3} n(a_i, r) \geq \frac{O(r)}{\pi}, \quad \ldots\ldots\ldots \quad (1)$$

where h is a constant, independent of the choice of $f(z)$; a_1, a_2, a_3 three fixed points on the sphere; $n(a_i, r)$ the number of times $F(r)$ covers a_i; $L(r)$ and $O(r)$ contourlength and area of $F(r)$.

If there are three exceptional points, we take them as a_1, a_2, a_3; the inequality (1) then reduces to:

$$\frac{L(r)}{O(r)} \geq \frac{1}{\pi h}. \quad \ldots\ldots\ldots \quad (2)$$

In the first place, supposing the integral

$$\int^{\infty} \frac{\sqrt{1-e(r)^2}}{r}\, dr \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

divergent, we may find, as we shall prove, values

$$r_1 < r_2 < \ldots < r_\nu < \ldots \to \infty$$

so, that

$$\lim_{\nu \to \infty} \frac{L(r_\nu)}{O(r_\nu)} = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad : \quad (4)$$

which is in contradiction with (2), so that PICARD's theorem then holds.

In order to prove the existence of the values r_ν we need a second inequality, deduced by AHLFORS:

$$L(r)^2 \frac{dr}{r} \leqq 4\pi K(r)\, dO(r), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

where $K(r)$ means:

$$K(r) = \frac{1}{2}\left\{\sqrt{1-e(r)^2} + \frac{1}{\sqrt{1-e(r)^2}}\right\}. \quad . \quad . \quad . \quad . \quad . \quad (6)$$

As

$$K(r) \leqq \frac{1}{\sqrt{1-e(r)^2}}$$

inequality (5) reduces to:

$$L^2 \frac{dr}{r} \leqq 4\pi \frac{1}{\sqrt{1-e^2}}\, dO. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

By multiplying both sides of (7) with $\frac{\sqrt{1-e^2}}{O^2}$ and integrating from r_μ to $r_{\mu+1}$, values to be defined further on, we obtain:

$$\int_{r_\mu}^{r_{\mu+1}} \frac{L^2}{O^2} \frac{\sqrt{1-e^2}}{r}\, dr \leqq 4\pi \int_{r_\mu}^{r_{\mu+1}} \frac{dO}{O^2} = 4\pi \left\{\frac{1}{O(r_\mu)} - \frac{1}{O(r_{\mu+1})}\right\}. \quad . \quad (8)$$

As $\frac{\sqrt{1-e^2}}{r} > 0$ and $\frac{L^2}{O^2}$ continuous, there exist values r_ν, with

$$r_\mu \leq r_\nu \leq r_{\mu+1},$$

so that

$$\left\{\frac{L(r_\nu)}{O(r_\nu)}\right\}^2 \int_{r_\mu}^{r_{\mu+1}} \frac{\sqrt{1-e^2}}{r}\, dr = \int_{r_\mu}^{r_{\mu+1}} \frac{L^2}{O^2} \frac{\sqrt{1-e^2}}{r}\, dr.$$

Introducing this in (8) we find:

$$\left\{\frac{L(r_\nu)}{O(r_\nu)}\right\}^2 \leqslant \frac{4\pi}{\int_{r_\mu}^{r_{\mu+1}} \frac{\sqrt{1-e^2}}{r}\,dr}\left\{\frac{1}{O(r_\mu)} - \frac{1}{O(r_{\mu+1})}\right\}. \quad . \quad . \quad . \quad (9)$$

As $\int_{r_\mu}^{\infty} \frac{\sqrt{1-e^2}}{r}\,dr$ is supposed to be divergent we may choose the values $r_\mu \to \infty$, so that

$$\lim_{\mu\to\infty} \int_{r_\mu}^{r_{\mu+1}} \frac{\sqrt{1-e^2}}{r}\,dr = \infty. \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

From (9) and (10) follows (4).

E.g. in case of representations, for which the order of $\sqrt{1-e^2}$ is $\frac{1}{\log r}$, the integral (3) diverges and so PICARD's theorem holds.

In the second place we suppose $e(r)$ given in such a way that

$$\int^{\infty} \frac{\sqrt{1-e^2}}{r}\,dr$$

converges; then we shall show, that there exists a corresponding mapping with a whole circle of exceptional points.

Before proving this, we give an example of a mapping for which the above integral converges (see fig. 1).

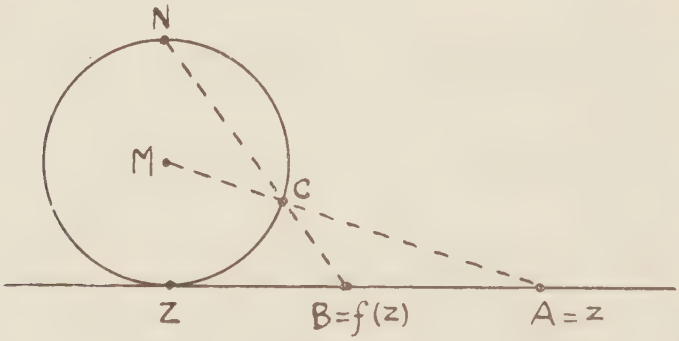


Fig. 1.

We project A to C on the sphere (rad. $\frac{1}{2}$), with its centre M as centre of projection. Then we reproject C to B on the plane, but now with the northpole N as centre. The function $w = f(z)$ will be defined as the 1 — 1 correspondence A to B. It is clear that the points C form the mapping

surface called above F. An infinitesimal circle at C, with radius ds, corresponds to an infinitesimal ellipse at A, with major axis a:

$$a = (1 + 4r^2)\, ds, \qquad (|z| = r)$$

and with minor axis b:

$$b = \sqrt{1 + 4r^2}\, ds.$$

We find

$$\sqrt{1 - e^2} = \frac{b}{a} = \frac{1}{\sqrt{1 + 4r^2}},$$

hence

$$\int^{\infty} \frac{\sqrt{1 - e^2}}{r}\, dr = \int^{\infty} \frac{dr}{r\sqrt{1 + 4r^2}},$$

which integral is convergent.

We now examine a more general mapping, defined by an arbitrary function $f(r)$, but again depending only on r (see fig. 2).

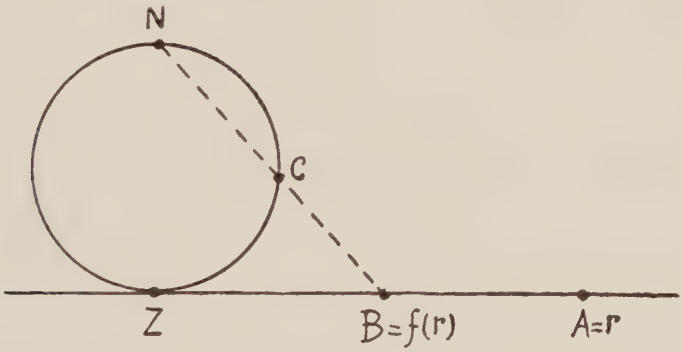


Fig. 2.

An infinitesimal circle at C, with radius ds, corresponds to an infinitesimal ellipse at A with one axis a_1, along the radius r:

$$a_1 = \frac{1 + f^2}{f'}\, ds$$

and the other axis a_2:

$$a_2 = \frac{1 + f^2}{f}\, r\, ds.$$

a_1 is major or minor axis according to $\frac{r f'}{f} < 1$ or $\frac{r f'}{f} > 1$, so that:

$$\sqrt{1 - e^2} \text{ or } \frac{1}{\sqrt{1 - e^2}} = \frac{a_2}{a_1} = r\frac{f'}{f}.$$

After these preliminaries we prove the second part of our theorem. Let $e(r) < 1$ be a given function, with $e(r) \to 1$ for $r \to \infty$, so that

$$\int^{\infty} \frac{\sqrt{1-e^2}}{r}\, dr$$

is convergent.

Putting

$$r\frac{f'}{f} = \sqrt{1-e^2}, \quad \ldots\ldots\ldots \quad (11)$$

we are sure of finding a mapping to which corresponds the given $e(r)$.

From (11) follows at once ($r_0 > 0$):

$$\log f(r) = C + \int_{r_0}^{r} \frac{\sqrt{1-e^2}}{r}\, dr,$$

from which it follows, that $f(r)$ is a bounded function. Hence the mapping, only depending on r, which we may construct by means of this function, will have a whole circle of exceptional points round infinity.

(Evidently the mapping has no exceptional points near zero, for $\frac{\sqrt{1-e^2}}{r}$ being aequivalent to $\frac{1}{r}$ near zero, $\log f(0) = -\infty$ and $f(0) = 0$.)

If, instead of (11), we had put $r\frac{f'}{f} = \frac{1}{\sqrt{1-e^2}}$, we should have found a mapping for which a_1, is the minor axis of the ellipse, and for which infinity is the only exceptional point.

Mathematics. — *Non-homogeneous binary quadratic forms.* IV (continued). By H. DAVENPORT. (Communicated by Prof. J. G. VAN DER CORPUT.)

(Communicated at the meeting of June 28, 1947.)

5. By the remarks at the beginning of § 4, and by Lemmas 4, 5, 6, there remains for consideration only the possibility

$$\frac{1}{\eta_m} > \bar{\beta} > \frac{1}{\eta_{m+1}}, \quad \frac{1}{\xi_m} < \alpha < \frac{1}{\xi_{m+1}}, \quad . \quad . \quad . \quad . \quad . \quad (29)$$

where m is an odd positive integer. Our aim will be to show that α, $\bar{\beta}$ necessarily have the values

$$\alpha = \alpha_m, \qquad \bar{\beta} = -\alpha'_m,$$

where α_m is defined by (5). That these values do in fact satisfy (29) follows from the simple inequalities

$$\frac{\theta}{1+\theta^{-n+3}} < \frac{\theta^{n+1}-1}{\theta^n+1} < \frac{\theta}{1+\theta^{-n}},$$

$$\frac{1}{1-\theta^{-n}} > \frac{1-\theta^{-n-1}}{1-\theta^{-n}} > \frac{1}{1-\theta^{-n-3}}.$$

Lemma 7. *If $\alpha = \alpha_m$, $\beta = \alpha'_m$, where m is an odd positive integer, then*

$$|(\alpha\xi - 1)(\beta\xi' - 1)| = 1$$

for the following values of ξ:

$$\theta, \quad \xi_m, \quad \xi_{m+1}, \quad -\eta'_m, \quad -\eta'_{m+1}.$$

Proof. We have to show that $N(\alpha_m \xi - 1) = \pm 1$ for the above values of ξ. In fact

$$\alpha_m \theta - 1 = \frac{2\theta(\theta^{n+1}-1)}{\theta^n+1} - 1 = \frac{(2\theta^2-1)\theta^n - (2\theta+1)}{\theta^n+1} = \theta^3\left(\frac{\theta^n-1}{\theta^n+1}\right),$$

and since n is odd, this has norm 1. Similarly we find that

$$\alpha_m \xi_m - 1 = \theta^{-n+2}\left(\frac{\theta^n-1}{\theta^n+1}\right),$$

$$\alpha_m \xi_{m+1} - 1 = -\theta^{-n-1},$$

$$\alpha_m(-\eta'_m) - 1 = -\theta^{n+1},$$

$$\alpha_m(-\eta'_{m+1}) - 1 = \theta^{n+4}\left(\frac{\theta^n-1}{\theta^n+1}\right),$$

and all these have norm 1.

Lemma 8. *If* (29) *holds, where* m *is an odd positive integer, then*

$$|\alpha-\alpha_m|\leqq C_m\,\varepsilon, \qquad |\beta-\alpha'_m|\leqq C_m\,\varepsilon, \quad . \quad . \quad . \quad . \quad (30)$$

where C_m *depends only on* m.

P r o o f. We write $\alpha=\alpha_m+\gamma$, $\beta=\alpha'_m+\delta$. The inequality (13), with $\xi=\xi_m$, becomes

$$|\{(\alpha_m+\gamma)\,\xi-1\}\{(\alpha'_m+\delta)\,\xi'-1\}|\geqq 1-\varepsilon.$$

Dividing throughout by $|(\alpha_m\,\xi_m-1)\,(\alpha'_m\,\xi'_m-1)|$, which is 1 by Lemma 7, we obtain

$$|(1+\varrho_1\gamma)(1+\sigma_1\delta)|\geqq 1-\varepsilon, \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

where (using the result in the proof of the preceding Lemma)

$$\left.\begin{aligned}\varrho_1&=\frac{\xi_m}{\alpha_m\,\xi_m-1}=\tfrac{1}{2}\,\theta^{n-3}\,\frac{(1+\theta^{-n+3})\,(1+\theta^{-n})}{(1-\theta^{-n})},\\ \sigma_1&=\varrho'_1=-\tfrac{1}{2}\,\frac{(1+\theta^{-n+3})\,(1-\theta^{-n})}{(1+\theta^{-n})}.\end{aligned}\right\} \quad . \quad . \quad . \quad (32)$$

That $1+\varrho_1\gamma>0$ is plain from the fact that $\alpha\,\xi_m-1$ has the same sign as $\alpha_m\,\xi_m-1$, since α and α_m both lie in the interval specified in (29). Also $|\sigma_1|<1$ since $n\geqq 3$, and

$$|\delta|<\eta_m^{-1}-\eta_{m+1}^{-1}<\theta^2-2<1,$$

since $\eta_m^{-1}\leqq\theta^2$ and $\eta_{m+1}^{-1}>2$. Thus $1+\sigma_1\,\delta>0$. By the inequality of the arithmetic and geometric means, (31) implies

$$1+\tfrac{1}{2}(\varrho_1\gamma+\sigma_1\delta)\geqq\sqrt{1-\varepsilon}\geqq 1-\varepsilon,$$

or

$$\varrho_1\gamma+\sigma_1\delta\geqq-2\varepsilon. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (33)$$

Similarly, from (13) with $\xi=\xi_{m+1}$ or $-\eta'_m$ or $-\eta'_{m+1}$, we obtain three other inequalities of the same form as (32), with the following values in place of ϱ_1, σ_1:

$$\left.\begin{aligned}\varrho_2&=\frac{\xi_{m+1}}{\alpha_m\,\xi_{m+1}-1}=-\tfrac{1}{2}\,(\theta^n+1),\\ \sigma_2&=\varrho'_2=-\tfrac{1}{2}\,(1-\theta^{-n}),\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (34)$$

$$\left.\begin{aligned}\varrho_3&=\frac{\eta'_m}{\alpha_m\,\eta'_m+1}=\frac{1}{2\theta}\,(1+\theta^{-n}),\\ \sigma_3&=\varrho'_3=\tfrac{1}{2}\,\theta^{n+1}\,(1-\theta^{-n}),\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (35)$$

$$\left.\begin{aligned}\varrho_4&=\frac{\eta'_{m+1}}{\alpha_m\,\eta'_{m+1}+1}=\frac{1}{2\theta}\,\frac{(1-\theta^{-n-3})\,(1+\theta^{-n})}{(1-\theta^{-n})},\\ \sigma_4&=\varrho'_4=-\tfrac{1}{2}\,\theta^{n+4}\,\frac{(1-\theta^{-n-3})\,(1-\theta^{-n})}{(1+\theta^{-n})}.\end{aligned}\right\} \quad . \quad . \quad . \quad (36)$$

We observe that ϱ_1, ϱ_3, σ_3, ϱ_4 are positive, and σ_1, ϱ_2, σ_2, σ_4 are negative. We observe also that

$$|\varrho_2|\,\sigma_3 - |\sigma_2|\,\varrho_3 = \tfrac{1}{4}\,\theta^{2n+1}\,(1+\theta^{-n})\,(1-\theta^{-n}) - \tfrac{1}{4}\,\theta^{-1}\,(1-\theta^{-n})\,(1+\theta^{-n}) > 0.$$

We multiply the second inequality of the type (33) by σ_3 and the third by $-\sigma_2$, and add. This gives

$$-(|\varrho_2|\,\sigma_3 - |\sigma_2|\,\varrho_3)\,\gamma \geqq -2\,\varepsilon\,(\sigma_3 + |\sigma_2|),$$

which implies $\gamma \leq C\,\varepsilon$, where C depends only on m. Again, on multiplying the second inequality by ϱ_3 and the third by $-\varrho_2$ and adding, we obtain

$$(|\varrho_2|\,\sigma_3 - |\sigma_2|\,\varrho_3)\,\delta \geqq -2\,\varepsilon\,(\varrho_3 + |\varrho_2|).$$

This implies $\delta \mathrel{Æ} -C'\varepsilon$. These two results, substituted in the first and fourth inequalities of the type (33), give $\delta \leq C''\varepsilon$ and $\gamma \geq -C'''\varepsilon$. This proves the Lemma.

Lemma 9. *If* (29) *holds, where* m *is an odd positive integer, then* $\alpha = a_m$ *and* $\beta = a'_m$.

Proof. We begin by defining, for any rational integer r (positive or negative) integers X_r, Y_r of $k(\theta)$, which will all satisfy

$$|N(a_m\,\xi - 1)| = 1$$

when used as values for ξ. The definitions are obtained by a natural generalization of the formulae which occurred in the proof of Lemma 7. We define X_r, Y_r (for fixed m) by

$$a_m\,X_r - 1 = \theta^{-r(n+1)+3}\left(\frac{\theta^n - 1}{\theta^n + 1}\right), \quad \ldots\ldots \quad (37)$$

$$a_m\,Y_r - 1 = -\,\theta^{-r(n+1)}. \quad \ldots\ldots\ldots \quad (38)$$

It is plain that X_r, Y_r have the property which we have stated. It has to be proved, however, that the definitions are such as to make them integers. Now

$$\frac{1}{a_m} = \left(\frac{\theta^n + 1}{2}\right)(\theta^{n+1} - 1)^{-1}.$$

The first factor is an integer (since n is a multiply of 3), and it is clear from (39) that Y_r is an integer, since $1 - \theta^{r(n+1)}$ is divisible by $1 - \theta^{n+1}$. As regards X_r, it will suffice to verify that

$$(\theta^n + 1) + \theta^{-r(n+1)+3}\,(\theta^n - 1)$$

is divisible by $1 - \theta^{n+1}$. Now to the modulus $1 - \theta^{n+1}$, the last expression is congruent to

$$\theta^{-1} + 1 + \theta^3\,(\theta^{-1} - 1) = \theta + \theta^3\,(-\theta^{-2}) = 0.$$

Hence both X_r and Y_r are integers. On comparing (37), (38) with the formulae in the proof of Lemma 7, we see that X_r, Y_r reduce to ξ_m, ξ_{m+1} when $r = 1$, and to $-\eta'_{m+1}, -\eta'_m$ when $r = -1$.

We write $\alpha = \alpha_m + \gamma$, $\beta = \alpha'_m + \delta$ as in the proof of Lemma 8, and have

$$|\gamma| \leqslant C_m \varepsilon, \qquad |\delta| \leqslant C_m \varepsilon \quad . \quad . \quad . \quad . \quad . \quad . \quad (39)$$

by (30). By (13) with $\xi = X_r$,

$$|\{(\alpha_m + \gamma) X_r - 1\}\{(\alpha'_m + \delta) X'_r - 1\}| \geqslant 1 - \varepsilon.$$

Dividing throughout by $|(\alpha_m X_r - 1)(\alpha'_m X'_r - 1)|$, which is 1, we obtain

$$|(1 + R_r \gamma)(1 + R'_r \delta)| \geqslant 1 - \varepsilon, \quad . \quad . \quad . \quad . \quad . \quad . \quad (40)$$

where

$$\left.\begin{aligned} R_r &= \frac{X_r}{\alpha_m X_r - 1} = \frac{1}{\alpha_m}\left\{1 + \theta^{r(n+1)-3}\left(\frac{\theta^n + 1}{\theta^n - 1}\right)\right\}, \\ R'_r &= \frac{1}{\alpha'_m}\left\{1 + \theta^{-r(n+1)+3}\left(\frac{\theta^n - 1}{\theta^n + 1}\right)\right\}. \end{aligned}\right\} \quad . \quad . \quad (41)$$

We shall deduce from (40) that $\gamma \geq 0$ and $\delta \leq 0$, always on the understanding that ε is less than a suitable positive number depending only on m. It is plain that $|R'_r|$ is absolutely bounded for $r \geq 1$, and that R_r is positive and increases with r. Moreover

$$\frac{R_{r+1}}{R_r} < D_m, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (42)$$

where D_m depends only on m. Also $-R_1 \gamma \leq R_1 C_m \varepsilon < 1$ by (39) and our hypothesis concerning ε. If $\gamma < 0$, we can, by (42), choose a positive integer r for which

$$\frac{1}{D_m} \leqslant R_r(-\gamma) < 1.$$

We then have

$$0 < 1 + R_r \gamma \leqslant 1 - D_m^{-1},$$
$$|1 + R'_r \delta| \leqslant 1 + K C_m \varepsilon$$

where K is an upper bound for $|R'_r|$ for $r \geq 1$. Hence

$$|(1 + R_r \gamma)(1 + R'_r \delta)| \leqslant (1 - D_m^{-1})(1 + K C_m \varepsilon),$$

and this contradicts (40) if ε is less than a suitable positive number depending only on m. Hence $\gamma \geq 0$.

Similarly, if we use (40) with negative values of r, in which case $-R'_r$ increases to $+\infty$ as $r \to -\infty$, and satisfies a condition similar to (42), we obtain $\delta \leq 0$.

Finally, the inequality analogous to (40) derived from Y_r instead of X_r is

$$|(1+S_r\gamma)(1+S'_r\delta)| \geqq 1-\varepsilon, \quad . \quad . \quad . \quad . \quad . \quad . \quad (43)$$

where

$$S_r = \frac{Y_r}{\alpha_m Y_r - 1} = -\frac{1}{\alpha_m}(\theta^{r(n+1)} - 1),$$

$$S'_r = \frac{1}{\alpha'_m}(1-\theta^{-r(n+1)}).$$

As $r \to -\infty$, $S'_r \to +\infty$ and S_r is bounded, whereas as $r \to +\infty$, $S_r \to -\infty$ and S'_r is bounded. Arguing as before, we deduce from (43) that $\delta \leq 0$ and $\gamma \geq 0$. Hence $\gamma = \delta = 0$, which proves the Lemma.

6. *Proof of Theorem 1.* We have proved that if α, β satisfy (12), (13), (14) then either $\alpha = \beta = 2$ or $\alpha = \alpha_m$, $\beta = \alpha'_m$, where m is an odd positive integer. The condition (14) was ensured by modifying α, β using the two operations specified at the time. Reversing these operations, the original α and β defined by (11) are either of the form 2τ, $2\tau'$ or of the form $\alpha_m\tau$, $\alpha'_m\tau'$ (in either order), where τ is a unit of $k(\theta)$. Hence a, b are either of the form (1) or of the form (4), where ξ_0 is an integer of $k(\theta)$. The proof of Theorem 1 is now complete.

7. *Proof of Theorem 2.* If a, b are of the form (1), we have

$$(\xi - a)(\xi' - b) = \tfrac{1}{4} N(2(\xi - \xi_0) - \tau),$$

and so has numerical value at least $\frac{1}{4}$, since $2(\xi - \xi_0) - \tau$ is an integer of $k(\theta)$ which is not zero. Moreover there are an infinity of units of $k(\theta)$ of the form $2\eta - 1$, where η is an integer of $k(\theta)$, e.g. all units of the form θ^{3h}, where h is a rational integer. If we define ξ by

$$\xi - \xi_0 = \tau\eta,$$

we obtain an infinity of integers ξ of $k(\theta)$ for which

$$|(\xi - a)(\xi' - b)| = \tfrac{1}{4}|N(2\eta - 1)| = \tfrac{1}{4}.$$

Similar arguments apply when a, b are of the form (4) provided that $m = 1$. As we saw in § 1, $\alpha_1 = \sqrt{5}$, so that

$$\left|N\left(\xi - \left(\frac{\tau}{\alpha_1} + \xi_0\right)\right)\right| = \tfrac{1}{5}|N(\sqrt{5}(\xi - \xi_0) - \tau)| \geqq \tfrac{1}{5},$$

and there are an infinity of units of the form $\sqrt{5}\,\eta - 1$.

The result is by no means so obvious when $m > 1$. We have to prove that

$$\left|N\left(\xi - \xi_0 - \frac{\tau}{\alpha_m}\right)\right| \geqq \frac{1}{|N\alpha_m|}$$

for all integers ξ of $k(\theta)$. Writing $(\xi-\xi_0)\tau^{-1}=\eta$, this is equivalent to the assertion that

$$|N(\alpha_m\eta-1)|\geqq 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (44)$$

for all integers η of $k(\theta)$. We have also to show that there are infinitely many integers for which equality occurs. The latter follows at once from the proof of Lemma 9, where we found that equality occurred for $\eta=X_r$ or Y_r where r is any rational integer. It remains to prove (44), with $m\geqq 3$ and odd.

The substitution

$$\alpha_m\eta-1=\theta^{n+1}(\alpha_m\zeta-1) \qquad (n=3m) \quad . \quad . \quad . \quad . \quad (45)$$

transforms integers η into integers ζ, and vice versa. For

$$\eta=\theta^{n+1}\zeta-\frac{\theta^{n+1}-1}{\alpha_m},$$

and the last term is an integer by the definition (5) of α_m, since $\theta^n\equiv 1 \pmod 2$. When η, ζ are connected by (45),

$$N(\alpha_m\eta-1)=N(\alpha_m\zeta-1),$$

and we see that by repeated application of (45) it will suffice to prove (44) when η satisfies

$$\theta^{-n-1}\leqq\left|\frac{\alpha_m\eta-1}{\alpha'_m\eta'-1}\right|<\theta^{n+1}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (46)$$

We suppose there exists an integer η of $k(\theta)$ satisfying (46) for which

$$|(\alpha_m\eta-1)(\alpha'_m\eta'-1)|<1, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (47)$$

and deduce a contradiction. It follows from (46) and (47) that

$$|\alpha_m\eta-1|<\theta^{\frac{1}{2}(n+1)}, \qquad |\alpha'_m\eta'-1|<\theta^{\frac{1}{2}(n+1)}. \quad . \quad . \quad . \quad . \quad (48)$$

Also one of these two must be less than 1.

We record first some obvious facts about the numbers α_m and α'_m. α_m increases with m, and tends to 2θ, so that

$$2\theta>\alpha_m\geqq\alpha_3=\frac{11}{4\theta-3}=3{\cdot}168\ldots. \quad . \quad . \quad . \quad . \quad (49)$$

Also $-\alpha'_m$ decreases as m increases, and tends to 2, so that

$$2<-\alpha'_m\leqq-\alpha'_3=\frac{11}{4\theta^{-1}+3}=2{\cdot}010\ldots. \quad . \quad . \quad . \quad (50)$$

Case 1. Suppose that (47) holds, and that

$$|\alpha_m\eta-1|<1, \qquad |\alpha'_m\eta'-1|<\theta^{\frac{1}{2}(n+1)}. \quad . \quad . \quad . \quad . \quad (51)$$

The first inequality implies that

$$0<\eta<\frac{2}{\alpha_m}<\frac{2}{3}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (52)$$

It also implies that

$$|\alpha_m \eta - 1| \geqq |2\theta\eta - 1| - (2\theta - \alpha_m)\eta.$$

Using this, together with the second half of (51), in (47), we obtain

$$|(2\theta\eta - 1)(\alpha'_m \eta' - 1)| < 1 + (2\theta - \alpha_m)\eta\,\theta^{\frac{1}{2}(n+1)}.$$

Now $|N(2\theta\eta - 1)| \geq 1$, since $2\theta - 1$ is a non-zero integer of $k(\theta)$. Hence the last inequality implies

$$\left|\frac{\alpha'_m \eta' - 1}{2\theta'\eta' - 1}\right| < 1 + (2\theta - \alpha_m)\eta\,\theta^{\frac{1}{2}(n+1)}. \quad . \quad . \quad . \quad . \quad (53)$$

Since η is a non-zero integer of $k(\theta)$, we have

$$|\eta'| \geqq |\eta|^{-1} > \tfrac{3}{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (54)$$

by (52). The expression on the left of (53) is either

$$\frac{|\alpha'_m|\,|\eta'| - 1}{2\theta^{-1}|\eta'| - 1} \quad \text{or} \quad \frac{|\alpha'_m|\,|\eta'| + 1}{2\theta^{-1}|\eta'| + 1},$$

according as $\eta' < 0$ or $\eta' > 0$. The latter is the smaller, since $|\alpha'_m| > 2 > 2\theta^{-1}$ by (50). Hence (53) implies

$$\frac{|\alpha'_m|\,|\eta'| + 1}{2\theta^{-1}\eta'| + 1} < 1 + (2\theta - \alpha_m)\eta\,\theta^{\frac{1}{2}(n+1)}. \quad . \quad . \quad . \quad . \quad (55)$$

By the definition of α_m in (5),

$$2\theta - \alpha_m = \frac{2(\theta + 1)}{\theta^n + 1} = \frac{2\theta^2}{\theta^n + 1} < 2\theta^{-n+2}.$$

Also $|\alpha'_m| > 2$ and $0 < \eta < \frac{2}{3}$. Hence (55) implies

$$\frac{2|\eta'| + 1}{2\theta^{-1}|\eta'| + 1} < 1 + \tfrac{4}{3}\,\theta^{\frac{1}{2}(5-n)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (56)$$

This gives a contradiction if $n \geq 15$; for since $|\eta'| > \frac{3}{2}$ by (54) the left hand side is greater than

$$4\theta/(3\theta^{-1} + 1) > 2,$$

and the right hand side is less than $1 + \frac{1}{3}\theta^{-5} < 2$.

There remains the case $n = 9$. Here (56) gives

$$\frac{2|\eta'| + 1}{2\theta^{-1}|\eta'| + 1} < 1 + \tfrac{4}{3}\,\theta^{-2} < 1 \cdot 51,$$

whence

$$2|\eta'|(1 - 1 \cdot 51\,\theta^{-1}) < 0 \cdot 51,$$

$$|\eta'| < (0 \cdot 51)/(0 \cdot 1335) < 4.$$

Hence $|N\eta| < \frac{8}{3}$, and since 2 is not a norm in $k(\theta)$, it follows that η is a unit. The only units satisfying $0 < \eta < \frac{2}{3}$, $|\eta'| < 4$ are θ^{-1} and θ^{-2}. These values of η do not satisfy (47), since

$$\alpha_3 \theta^{-1} - 1 = \frac{11}{\theta(4\theta - 3)} - 1 = \frac{7 - \theta}{\theta(4\theta - 3)},$$

with norm $\frac{41}{19}$, and

$$\alpha_3 \theta^{-2} - 1 = \frac{11}{\theta^2(4\theta - 3)} - 1 = \frac{10 - 5\theta}{\theta^2(4\theta - 3)}$$

with norm $-\frac{25}{19}$.

Case 2. Suppose that (47) holds, and that

$$|\alpha_m \eta - 1| < \theta^{\frac{1}{2}(n+1)}, \qquad |\alpha'_m \eta' - 1| < 1. \quad . \quad . \quad . \quad . \quad . \quad (57)$$

The second inequality implies that

$$0 < -\eta' < \frac{2}{|\alpha'_m|} < 1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (58)$$

It also implies that

$$|\alpha'_m \eta' - 1| \geqq |2\eta' + 1| - |\alpha'_m + 2|\,|\eta'|.$$

Using this, together with the first half of (57), in (47), we obtain

$$|(\alpha_m \eta - 1)(2\eta' + 1)| < 1 + |\alpha'_m + 2|\,|\eta'|\,\theta^{\frac{1}{2}(n+1)}.$$

Since $|(2\eta + 1)(2\eta' + 1)| \geq 1$, and $|\eta'| < 1$, this implies

$$\left|\frac{\alpha_m \eta - 1}{2\eta + 1}\right| < 1 + |\alpha'_m + 2|\,\theta^{\frac{1}{2}(n+1)}. \quad . \quad . \quad . \quad . \quad . \quad (59)$$

Using (58), we have

$$|\eta \geqq |\eta'|^{-1} > 1.$$

If $\eta < -1$, (59) implies

$$\frac{\alpha_m |\eta| + 1}{2|\eta| - 1} < 1 + |\alpha'_m + 2|\,\theta^{\frac{1}{2}(n+1)}.$$

Since $\alpha_m > 3$, the left hand side is greater than $\frac{3}{2}$. Now

$$|\alpha'_m + 2| = 2\left|\frac{1 - \theta^{-n-1}}{1 - \theta^{-n}} - 1\right| = \frac{2\theta^{-n-2}}{1 - \theta^{-n}}.$$

Hence, since $n \geqq 9$,

$$\tfrac{3}{2} < 1 + 2\frac{\theta^{-\frac{1}{2}(n+3)}}{(1 - \theta^{-n})} \leqq 1 + \frac{2\theta^{-6}}{1 - \theta^{-9}} < 1 \cdot 113,$$

a contradiction.

Hence we must have $\eta > 1$. The corresponding inequality now is

$$\frac{a_m \eta - 1}{2\eta + 1} < 1 \cdot 113.$$

Since $a_m \geq a_3 > 3.167$, this gives

$$\eta\,(3.167 - 2.226) < 2.113,$$

and so certainly $\eta < 2.5$. Since $|\eta'| < 1$, η must be a unit, and the only unit with $1 < \eta < 2.5$ is $\eta = \theta$. When $\eta = \theta$ equality occurs in (44), by Lemma 7. Hence there is again a contradiction to (47), and this completes the proof of Theorem 2.

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Mathematics. — *On the principles of intuitionistic and affirmative mathematics* [1]. I. By D. VAN DANTZIG. (Communicated by Prof. J. G. VAN DER CORPUT.)

(Communicated at the meeting of June 28, 1947.)

Ch. 1. BROUWER's *intuitionism.*

In 1907 in his thesis [1] [2] L. E. J. BROUWER developed the principles of intuitionistic mathematics. Some of the most important among the many original ideas he defended there and worked out in several later papers on the subject may be circumscribed about as follows.

Mathematics has to be regarded as a part of human activity, rather than as a system of books, theorems, words or symbols. It is (not a *result of* human experience, but) a method of *dealing with* human experience. In order to grasp its characteristic features we have to abstract from all qualitative properties of particular experiences, in the most elementary description of which the method usually has been applied already a good many times.

In its most elementary form it consists of fixing our attention upon a single one out of the totality of our perceptions, and of distinguishing this one from the rest of them. As the distinguishing of a single perception lies at the basis of the mathematical idea of discreteness, and the totality of perceptions at the basis of the notion of continuity, it presupposes both these notions, though, of course, in an unanalysed form. This mental process BROUWER calls the continuum-intuition, or the primordial intuition ("Urintuition") of mathematics, or also the time-intuition, as also the possibility of ordering our perceptions according to time is not reducible to a more elementary mental process [3].

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[1] The present paper has been written in February 1942 on bequest of the redaction of the Revista Mathematica Hispano Americana and was sent to that journal through an official Spanish instance in the (then occupied) Netherlands in March 1942. For some unknown reason the redaction of the journal did not publish the paper. Although the paper is not anymore entirely up to the present situation in mathematical logic nor to my own present state of mind, there might be some use in publishing it nevertheless without other than a few alterations, mostly of style. I hope to have leasure for publishing soon a closer examination of some questions raised here. Footnotes [1], [4], [16]—[19], [35], [38] have been added 1947.

[2] The numbers in square brackets refer to the bibliography at the end of the paper.

[3] "..... the primordial intuition of mathematics (and of any intellectual activity) is the substratum of all observations of change, stripped of all qualitative properties; a unity of continiuty and discreteness, a possibility of mentally joining several units, connected by a "between" which never is exhausted by intercalation of new units." (L. E. J. BROUWER [1] p 8.)

"This neo-intuitionism considers as the elementary occurrence of muman intellect: the splitting up of moments of life into qualitatively different parts, which can be reunited

The term "mathematics" is used by BROUWER for every mental process which can be conceived as being built up out of such elementary processes, hence in a much more general sense than it is done usually, which includes, not only what HILBERT calls "metamathematics", but also logic, and science in general. The principal result obtained by BROUWER is, that for the foundation of mathematics (in the ordinary, restricted sense) it is sufficient as well as necessary to consider systems composed of sequences obtained by applying an unlimited number of times (limited only for finite arithmetics, etc.) such a single mental process, without paying attention to the qualitative differences which in such processes effectively may occur.

The sufficiency BROUWER proved by undertaking the laborious work, which was later continued by his disciples, M. J. BELINFANTE [4]) and A. HEYTING, to derive explicitly the fundamental parts of arithmetics, set-theory, analytic geometry, analysis, etc. The necessity he showed by giving critical analyses of the most prominent rival theories, and proving that they always presuppose explicitly or implicitly indefinite repetitions of a single process, in particular the idea of order and (as far as infinite systems are concerned) complete induction.

Once one is convinced of the necessity and sufficiency, it becomes natural to demand that mathematical considerations shall be restricted to such "constructions", i.e. complexes, consisting of indefinite repetitions of a single act, and BROUWER reserves the word "mathematics" to developments satisfying this condition.

With regard to the form in which mathematical statements are given as well as in the way they are proved, this standpoint has several peculiar consequences. First, a demonstration in this sense is not a method of "convincing" a reader or auditor in a more or less indirect way of the "truth" of a statement, viz. by the application of certain aprioristic "logical principles", but it is (or ought to be) the construction itself, the possibility of which is stated in the theorem. The only form of demonstration admitted here is "showing by doing". As, however, the process, according to BROUWER, is a mental one, it cannot be directly observed by a reader or auditor, so that the teacher has to describe it in words. But, as a description of an experience never is "adequate" [5]), never determines this

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only if separated in the time, and it sees as the elementary occurrence of mathematical thinking: the process of stripping this splitting up from every emotional content until the intuition of abstract di-unity remains." (L. E. J. BROUWER [2] III p. 12.)

Here and in the other quotations from BROUWER [1] and [2] an entirely literal translation could not be reached. Although I have tried to paraphrase in Ch. 1 BROUWER's own ideas without letting my sometimes somewhat deviating opinions interfere with them unless where they seem to me to be in accordance with his, I am not quite certain that I always have succeeded in representing BROUWER's ideas correctly. Anyhow, the responsibility for the representation rests with me alone.

[4]) Since this was written Dr. M. J. BELINFANTE was killed with his family in a german concentration-camp in Poland.

[5]) This should not be misunderstood: of course we can not mention any part of an

experience completely and uniquely, also the construction under consideration cannot be described "exactly", in the sense in which this was formerly meant to be possible, nor is there any "certainty" in an absolute sense, that misunderstanding is excluded [6]).

A demonstration therefore rests essentially on "suggestion", not of the "truth" of the statement, but of the method of construction itself. An able mathematician may "suggest" to a gifted student, how to built his constructions, like an able musician may suggest a gifted scolar, how to compose a sonata; in both cases by showing and describing as well as he can, how de does it, by telling the principles in accordance with which (as he detects *a posteriori*) he has built them, and by criticising the student's exercises. Hence the logics "underlying" a construction, like the harmonics "underlying" a symphony, is (or ought to be!): not a system of rules, given a priori and followed dogmatically, but a system of regularities, observed a posteriori in one's own or somebody else's constructions [7]).

The question whether a logical principle is "trustworthy" or not, then means: if a mathematician, in order to save time, skips over some part of his construction on account of such a regularity, observed in previous constructions, may he then "reasonably expect" that he (or some one else) can later make it complete? Of course this question can only be answered by experience. The answer of such mathematicians as have much experience with this kind of work is: sometimes yes, sometimes no. In particular it is often "no", if the regularity under consideration is the so-called principium tertii exclusi, applied to infinite systems. This is not astonishing at all,

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experience which can *not* be described in words. This would imply a direct contradiction. But — even apart from the emotion of "insufficiency" we have with regard to many descriptions —, with respect to every given description we can mention *afterwards*, numerous elements of the experience, *not* involved in the description which show the given description to be incomplete.

[6]) "To the question where mathematical exactness *does* exist, both parties give different answers: the intuitionist says: in the mind of men, the formalist: on paper." (L. E. J. BROUWER [2] III p. 7.)

"Nun gibt es aber für Willensübertragung, insbesondere für durch die Sprache vermittelte Willensübertragung, weder Exaktheid, noch Sicherheit. Und diese Sachlage bleibt unvermittelt bestehen, wenn die Willensübertragung sich auf die Konstruktion reinmathematischer Systeme bezieht. *Es gibt also für die reine Mathematik keine sichere Sprache*, d.h. keine Sprache welche in der Unterhaltung Missverständnisse ausschliesst, und bei der Gedächtnisunterstützung vor Fehlern schützt." (L. E. J. BROUWER [3] p. 157.)

[7]) "Theoretical logic as well as logistics therefore are *empirical sciences*, and *applications* of mathematics, which never can teach us anything about the organisation of human intellect, and must be regarded to belong to *ethnography* rather than to *psychology*.

And the language of logical arguments is no more an *application of theoretical logic* than the human body is an application of anatomy". (L. E. J. BROUWER [1] p. 130.)

"Moreover, in arguments concerning empirical facts spanned upon mathematical systems, the logical principles are not directories, but regularities discovered afterwards in the accompanying language ....." (L. E. J. BROUWER [2] I p. 7.)

since by more general and less specific experience we know that, if we have ascertained that a certain construction can not be impossible, we need not have the slightest guarantee that we — or someone else — may succeed in carrying it out.

We pass here over the further consequences of these ideas, in particular with regard to set theory, where especially BROUWER's criticism of the comprehension-axiom should be mentioned, and refer the reader to BROUWER's original publications and to HEYTING's [2] monography, where also further litterature can be found.

An abstract scheme for a way of dealing with human experience, we called mathematics before. In dealing with our experiences we always single out a finite number of them, disregarding all other ones as belonging to the "continuous background" of our perceptions. Among them we see certain sequences, ordered in time, which to a certain degree are identified with each other, and give rise to a common substratum, called a "causal sequence" by BROUWER.

More or less characteristic for the behaviour of men is, as BROUWER calls it, the "mathematical action" or the "replacement of aim by means". "And the behaviour of men shows a tendency to observe as many as possible of those mathematical sequences, in order to choose the earlier element as a directory for their actions, always when interference with reality seems to be more succesfull at an earlier element than at a later one, even if instinct is only affected by the latter." (L. E. J. BROUWER [1] p. 81). The construction "in advance" of mathematical systems by abstraction from the qualitative differences of the systems of causal sequences on which they may be applied, is itself an example of such a "jump from aim to means" [8]).

We shall not go here further into BROUWER's ideas concerning physical science. We also end herewith our reference of a part of BROUWER's thesis and ulterior papers on intuitionism, although we could mention only a small portion of the ideas contained in them.

In the beginning BROUWER's ideas met with great resistance. In 1919 [2] he still had to admit that "the ideas defended here still have found only few adherents". In fact, during a long time H. WEYL was one of the few under the leading mathematicians who, to a large extent, accepted BROUWER's principles. Since about ten or twelve years however the situation has greatly changed.

In Amsterdam Prof. G. MANNOURY following a train of thought partly deviating from, to a large extent however in accordance with BROUWER's lines, had often expressed the opinion that a formal description of the regularities occurring in the intuitionistic way of reasoning must be possible.

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[8]) "Selbstverständlich besitzt eine kausale Folge keine weitere *Existenz* ausser als Korrelat einer mathematische Handlungen hervorrufenden Einstellung des menschlichen Willens, und kann von der Existenz eines kausalen Zusammenhangs der Welt unabhängig vom Menschen keine Rede sein." (L. E. J. BROUWER [3] p. 154.)

A prize-question to this purpose, published by the mathematical society of Amsterdam, was answered in an excellent way by BROUWER's disciple A. HEYTING, under the characteristic motto "stones instead of bread". His difficult and laborious, but successfull work [1], and later his clear exposition of BROUWER's ideas [2] have greatly contributed not only to the interest in, but also to the understanding of intuitionism, in particularly among logicians. Moreover GÖDEL [1] showed that every sufficiently extended non-contradictory logistic system, satisfying certain simple conditions, allowes the formulation of problems, unsolvable *within the system*. Among them occurs the question after the formal non-contradictority of the system itself. Finally the "multi-valued logics" of the Polish school of LUKASIEWICZ and TARSKI contained important contributions to a better understanding of the logical structure of intuitionism. Several papers of HERBRAND, SKOLEM, CHURCH and his disciples, GENTZEN, I. JOHANNSSEN, etc. worked in the same direction, and showed that nowadays at least intuitionistic *logic* is completely recognised by the great majority of leading logicians.

Among mathematicians, however, there still is a certain resistance or indifference with regard to the intuitionistic way of reasoning.

There are some possible causes, which may, at least partly, explain this attitude.

1°. Many mathematicians are not particularly interested in philosophy, nor even in logic. They desire to carry their science further instead of uprooting the very fundamentals. They often don't quite understand what should be wrong with their customary way of reasoning, and they don't see any but philosophical reasons (to which they often don't attribute great importance) that anything at all should be wrong. Like so many important discoveries however, intuitionism is independent of the particular philosophy of its maker; its good sense can be demonstrated on a purely technical base. In ch. 3 I shall try to show, that the very desire of every genuine mathematician, viz. to prove his theorems as rigorously and as "economically" as he can, if consequently followed, leads almost automatically to a special form of intuitionistic mathematics.

2°. Many mathematicians are of opinion that they have no need of a "constructive" mathematics. I shall deal with their standpoint in the beginning of ch. 3 and in ch. 4.

3°. There is a general dislike under mathematicians of the great number of "almost equivalent" notions which occur in intuitionistic mathematics, and the great number of cases which often have to be considered wounds their sense of elegance. The fact that there are e.g. no less than 10 countability-relations [9]) for a set conflicts with their desire "to kill n birds with one stone" [10]). This fact, which of course is

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[9]) L. E. J. BROUWER [5] I p. 255, A. HEYTING [3].
[10]) G. MANNOURY [2] p. 69.

of a less principal nature, can be avoided by restricting the number of notions and theorems by one of the two ways shown in Ch. 2 and 3.

4°. The same is true for some other incommodities of a still more accidental nature, e.g. concerning terminology and notations. One can for instance use the word "set" for what BROUWER calls a "Spezies", i.e. in a sense not very different from the ordinary notion of set, whereas BROUWER's entirely different notion of "Menge" may (and will here) be described by the term "Brouwerian set". The fact that then the notion of "Brouwerian set" is prior to "set" in general will hardly be considered as an inconvenient.

The purpose of the present paper is to show, how perhaps a better unterstanding between intuitionists and "ordinary mathematicians" could be reached. This might be possible in one of two ways. In Ch. 2 we shall mention a method by which the intuitionist might try to meet the ordinary mathematician half way. In this chapter we shall therefore start from the intuitionist's point of view. In Ch. 3 on the contrary we shall take the standpoint of the classical ("formal") mathematician and see which way leads him to intuitionistic (i.e. "affirmative") mathematics. In Ch. 4 finally some remarks of a more general nature concerning formalism and intuitionism will be made.

Ch. 2. *The weak interpretation. Stable mathematics.* (*Classical mathematics from an intuitionistic standpoint.*)

We here take the intuitionist's standpoint and ask, in how far we can meet the ordinary mathematician's demands. The latter may remark that many misunderstandings arise from the fact that BROUWER interprets the ordinary mathematical theorems much "stronger" than he (the ordinary mathematician) intends them to be. If BROUWER rejects the theorem that every function $f(x)$, continuous for $0 \leqq x \leqq 1$, has at least one maximum in this interval, he does so, because he can not construct the abscissis of this maximum. But the mathematician might already be content if the supposition that $f(x)$ "has" (whatever this may mean!) a maximum, were exempt of contradiction. He may take POINCARÉ's point of view, according to which a mathematical entity "exists" if it is free from contradiction.

The intuitionist might meet this remark as follows. If A denotes any statement in the intuistionistic sense, then $\neg A$ (non-A) is its negation, i.e. $\neg A$ denotes that A would imply a contradiction (that A is "absurd", according to BROUWER's terminology). The classical mathematician therefore demands only $\neg\neg A$ instead of A to be proven. The intuitionist might therefore help the classical mathematician by replacing every statement A by $\neg\neg A$. This might seem hopeless (as $\neg\neg A$ would have to be replaced by $\neg\neg\neg\neg A$ for the same reason), were it not that BROUWER has proved [6] that $\neg\neg\neg A$ always implies and therefore is equivalent with $\neg A$, though $\neg\neg A$ generally not with A). Calling

a statement *stable* if it is equivalent with its double negation, we see that every *negative* statement (i.e. the negation of any statement) and only such a one is stable.

If A and B are stable, then also $\neg A$ and $A \wedge B$ (i.e. "A and B") are. But $A \vee B$ (i.e. "A or B") and $A \supset B$ (i.e. "A implies B"), interpreted in BROUWER's "strong" sense are not. We can however interpret these relations in a weaker sense if we *define* them by

$$A \vee B \overset{\text{df}}{\equiv} \neg (\neg A \wedge \neg B)$$

$$A \supset B \overset{\text{df}}{\equiv} \neg (A \wedge \neg B)$$

with these definitions $A \vee B$ and $A \supset B$ become stable. If A depends upon a variable x and for every x is stable, then also $\mathfrak{A}_x A(x)$ (i.e. $A(x)$ holds for all x) is stable (as generally $\neg \neg \mathfrak{A}_x A(x)$ implies $\mathfrak{A}_x \neg \neg A(x)$), but $\mathfrak{E}_x A(x)$ (i.e. an x with $A(x)$ exists), if interpreted in the strong sense, in general is not. If, however, we *define* the latter symbol by

$$\mathfrak{E}_x A(x) \overset{\text{df}}{\equiv} \neg \mathfrak{A}_x \neg A(x)$$

then the statement becomes stable. With these definitions we obtain a system of formulae, closed with respect to the elementary logical operations, forming a part of intuitionistic logic and satisfying the formal rules of classical logic, including the principium tertii exclusi. In fact, with the above definitions we have not only $\neg \neg A \supset A$, but even $A \vee \neg A$ for every statement A. All this was essentially found by K. GÖDEL [2].

Passing from logic to mathematics, we have, of course, to take care that all definitions of mathematical objects are given in a stable form. At first, viz as long as we are concerned with natural (or rational) numbers only, no difficulties arise, as the fundamental relations between these numbers, equality and inequality, are stable.

The introduction of real numbers, however, leads to difficulties. Of course we have to avoid here BROUWER's definition, which certainly is too strong for our present purpose. We try CANTOR's definition by means of fundamental sequences of rational numbers, CAUCHY's criterium $\mathfrak{A}_\varepsilon \mathfrak{E}_n \mathfrak{A}_m | a_{n+m} - a_n | \leqq \varepsilon$ [11]) [12]) of course has to be interpreted in the weak sense. After elimination of the defined symbols $\vee$, $\supset$ and $\mathfrak{E}$ this reads:

$$\mathfrak{A}_\varepsilon \neg \mathfrak{A}_n \neg \mathfrak{A}_m | a_{n+m} - a_n | \leqq \varepsilon$$

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[11]) We omit for abbreviation the condition that n, m etc. are natural and ε positive rational numbers, etc. Zero is considered as a natural number.

[12]) For real numbers the relations $x \leqq y$ and $x = y$ are stable; $x < y$ and $x \neq y$, if interpreted in the strong sense, are not. They become stable if they are interpreted in the weak sense:

$$x < y \overset{\text{df}}{\equiv} \neg (y \leqq x); \qquad x \neq y \overset{\text{df}}{\equiv} \neg (x = y).$$

A sequence $\{a_n\}$ of rational numbers a_n satisfying this condition may be called a *weak fundamental sequence*. In the same way a *weak null sequence* may be defined by

$$\mathfrak{V}_\varepsilon \neg \mathfrak{V}_n \neg \mathfrak{V}_m |a_{n+m}| \leqq \varepsilon.$$

If, however, we wish to define e.g. the quotient-sequence of two weak fundamental sequences $\{a_n\}$, $\{b_n\}$, the latter not being a weak null sequence, we must take account of the fact that the a_n/b_n need not be rational numbers, as some of the b_n may be zero. We can avoid this difficulty by working with integers instead of rational numbers only, viz with the numerators p_n, r_n and denominators q_n, s_n of the rational numbers a_n and b_n. Then the relation $|a_{n+m} - a_n| < \varepsilon$ becomes $|p_{n+m} q_n - p_n q_{n+m}| < \varepsilon |q_n q_{n+m}|$. We take here $< \varepsilon$ instead of $\leqq \varepsilon$ in order to exclude the trivial solution $q_n = q_{n+m} = 0$ of the inequality. Now it implies $|q_n| \geqq 1$, $|q_{n+m}| \geqq 1$. Now let $\{r_n, s_n\}$ be a weak fundamental non-null sequence, determined by the same relations as those for $\{p_n, q_n\}$, together with inequalities $\neg(|p_{n+m}| < \varepsilon |q_{n+m}|)$. Then the quotient-sequence $\{p_n s_n, q_n r_n\}$ is a weak fundamental sequence.

This is proved by showing that the conjunction of the four statements $\Omega_1, \Omega_2, \Omega_3, \Omega_4$, (which are easily seen to be stable) leads to a contradiction [13]), where the following abbreviations have been introduced:

$$\begin{array}{llll}
\Omega_1 \equiv \mathfrak{V}_\varepsilon \; \Omega_1(\varepsilon) & \Omega_1(\varepsilon) \equiv \neg \mathfrak{V}_n \neg \Omega_1(\varepsilon, n) & \Omega_1(\varepsilon, n) \equiv \mathfrak{V}_m \; \Omega_1(\varepsilon, n, m) \\
\Omega_2 \equiv \mathfrak{V}_\varepsilon \; \Omega_2(\varepsilon) & \Omega_2(\varepsilon) \equiv \neg \mathfrak{V}_n \neg \Omega_2(\varepsilon, n) & \Omega_2(\varepsilon, n) \equiv \mathfrak{V}_m \; \Omega_2(\varepsilon, n, m) \\
\Omega_3 \equiv \neg \mathfrak{V}_\varepsilon \neg \Omega_3(\varepsilon) & \Omega_3(\varepsilon) \equiv \mathfrak{V}_n \; \Omega_3(\varepsilon, n) & \Omega_3(\varepsilon, n) \equiv \neg \mathfrak{V}_m \neg \Omega_3(\varepsilon, n, m) \\
\Omega_4 \equiv \neg \mathfrak{V}_\varepsilon \neg \Omega_4(\varepsilon) & \Omega_4(\varepsilon) \equiv \mathfrak{V}_n \; \Omega_4(\varepsilon, n) & \Omega_4(\varepsilon, n) \equiv \neg \mathfrak{V}_m \neg \Omega_4(\varepsilon, n. m)
\end{array}$$

$$\begin{aligned}
\Omega_1(\varepsilon, n, m) &\equiv |p_n q_{n+m} - p_{n+m} q_n| < \varepsilon |q_n q_{n+m}| \\
\Omega_2(\varepsilon, n, m) &\equiv |r_n s_{n+m} - r_{n+m} s_n| < \varepsilon |s_n s_{n+m}| \\
\Omega_3(\varepsilon, n, m) &\equiv |r_{n+m}| \geqq \varepsilon |s_{n+m}| \\
\Omega_4(\varepsilon, n, m) &\equiv |p_n q_{n+m} r_{n+m} s_n - p_{n+m} q_n r_n s_{n+m}| \geqq \varepsilon |q_n q_{n+m} r_n r_{n+m}|.
\end{aligned}$$

In fact, the system of relations

$$\Omega_1(\varepsilon_0, k_0, n-k_0),\ \Omega_1(\varepsilon_1, k, n-k),\ \Omega_1(\varepsilon_1, k, n+m-k),\ \Omega_2(\varepsilon_0, l_0, n+m-l_0),$$
$$\Omega_2(\varepsilon_2, l, n-l),\ \Omega_2(\varepsilon_2, l, n+m-l),\ \Omega_3(\varepsilon_3 j, i),\ \Omega_4(\varepsilon_4, n, m)$$

with

$$j = k_0 + l_0 + k + l, \quad n = j + i,$$

implies a relation of the form

$$\varepsilon_3^2 \varepsilon_4 < 2(A_1 \varepsilon_1 + A_2 \varepsilon_2)$$

with coefficients

$$A_1 = \varepsilon_0 + |r_{l_0}/s_{l_0}|, \; A_2 = \varepsilon_0 + |p_{k_0}/q_{k_0}| + \varepsilon_3 \varepsilon_4,$$

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[13]) If statements of the form $\neg \mathfrak{V}_\varepsilon \neg A(\varepsilon)$ are used, we must of course take care, not to use demonstrations of the type "Let ε be a number satisfying $A(\varepsilon)$" as this goes beyond the weak interpretation.

independent of m, i, k, l. The same remains the case if these variables are successively dropped by applying; first $\neg \mathfrak{V}_m \neg$ to the last relation and $\mathfrak{V}_m$ to the other ones; then $\neg \mathfrak{V}_i \neg$ to the second last relation and $\mathfrak{V}_i$ to the other ones, and finally $\neg \mathfrak{V}_k \neg$ and $\neg \mathfrak{V}_l \neg$ to the second ($=$ third) and fifth ($=$ sixth) relation respectively, and $\mathfrak{V}_j$ to the seventh one [14]). This however can not be true for all positive ε_1 and ε_2. Hence the system

$$\Omega_1(\varepsilon_0, k_0) \wedge \Omega_1 \wedge \Omega_2(\varepsilon_0, l_0) \wedge \Omega_2 \wedge \Omega_3(\varepsilon_3) \wedge \Omega_4(\varepsilon_4) \wedge \varepsilon_0 > 0 \wedge \varepsilon_3 > 0 \wedge \varepsilon_4 > 0$$

(which implies the relations obtained in this way) implies a contradiction viz $\mathfrak{V}_{\varepsilon_1} \mathfrak{V}_{\varepsilon_2} \; \varepsilon_3^2 \varepsilon_4 < 2(A_1 \varepsilon_1 + A_2 \varepsilon_2)$ with $A_1, A_2, \varepsilon_3, \varepsilon_4$ all positive and independent of $\varepsilon_1, \varepsilon_2$. This remains so after successive application of the operators

$$\neg \mathfrak{V}_{k_0} \neg, \neg \mathfrak{V}_{l_0} \neg, \neg \mathfrak{V}_{\varepsilon_3} \neg, \neg \mathfrak{V}_{\varepsilon_4} \neg, \mathfrak{V}_{\varepsilon_0},$$

so that the system $\Omega_1 \wedge \Omega_2 \wedge \Omega_3 \wedge \Omega_4$ is contradictory, which proves the theorem.

We have worked out this example, in order to show that proofs of this type do *not run* entirely along the customary lines, as the ordinary proofs usually consists of rather inconsequent mixtures of weak and strong interpretations.

There is a further question we have to consider. Is the relation $x \in N$ stable? Apparently this question is meaningless, as long as the negation of the statement $x \in N$ has no definite meaning (i.e. we can not conclude anything from it). This however becomes different after the real numbers have been introduced. Let us write $x \in N^*$ if x is a set of weak fundamental sequences $\{p_n, q_n\}$, weakly concurrent with a constant sequence $\{m, 1\}$. hence defined by

$$\neg \mathfrak{V}_m \neg \mathfrak{V}_\varepsilon \neg \mathfrak{V}_n \neg \mathfrak{V}_k \, | \, p_{n+k} - m \, q_{n+k} | < \varepsilon \, | \, q_{n+k} |.$$

Evidently $x \in N^*$ has a definite negation and is stable.

It should also be noted that we may meet the relation $x \in N^*$ where we would superficially expect $x \in N$. We show this by an example. Let $F(n)$ be a "fugitive property" [15]) of natural numbers $n \in N$, i.e. let it be decidable for every $n \in N$ whether $F(n)$ or $\neg F(n)$ holds. Let for those numbers which have been investigated $\neg F(n)$ have been proved, though no proof of $\mathfrak{V}_n \neg F(n)$ be known.

On the contrary, we suppose that $\neg \mathfrak{V}_n \neg F(n)$ has been proved. Let

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[14]) This follows from the properties that $A(x)$ implies $\neg \mathfrak{V}_x \neg A(x)$ that $\mathfrak{V}_x (A(x) \supset B)$, where B is independent of x, implies $\neg \mathfrak{V}_x \neg (A(x) \supset B)$, (provided $\neg \mathfrak{V}_x \neg A(x)$), and that for natural h and k $\mathfrak{V}_h A(h)$ implies $\mathfrak{V}_h A(h+k)$.

[15]) L. E. J. BROUWER [3] p. 161.

us then define $\mu\,[F]$ as the smallest "natural" number n for which $F\,(n)$ holds. Then we can not state $\mu\,[F] \in N$. In fact, defining the characteristic function $\iota_n\,[F]$ of F by

$$\iota_n\,[F] = \begin{cases} 0 \text{ if } & F(n) \\ 1 \text{ if } & \neg F(n), \end{cases}$$

then $n \in N \supset \iota_n\,[F] \in N$. Moreover, if $p_n = \mu_n\,[F]$, $q_n = 1$, where

$$\mu_n\,[F] = \sum_0^n{}_i \prod_0^i{}_k \iota_k\,[F]$$

then $\mu_n\,[F]$ is the smallest number $k \leqq n$ with $F\,(k)$ if such a one exists, and otherwise $\mu_n\,[F] = n + 1$. Evidently $n \in N \supset \mu_n\,[F] \in N$. Then $\{p_n, q_n\}$ is a weak fundamental sequence because of $\neg\, \mathfrak{V}_n \neg F\,(n)$, and therefore determines a weak real number $\mu[F]$. If $\{p_n, q_n\}$ were not concurrent with any natural number $m \in N$, then for every n p_n would be $> n$

$$\text{(as } |\mu_{n+m}\,[F] - \mu_n\,[F]| \geqq \varepsilon > 0 \supset \mu_{n+m}\,[F] > \mu_n\,[F] \supset \mu_n\,[F] > n),$$

hence $\mathfrak{V}_n\, \iota_n\,[F] = 1$ contradicting $\neg\, \mathfrak{V}_n \neg F(n)$ and the definition of $\iota_n\,[F]$. Formally we may of course write

$$\mu\,[F] = \lim \mu_n\,[F] = \sum_0^\infty{}_i \prod_0^i{}_k \iota_k\,[F]. \qquad \text{Hence } \mu\,[F] \in N^*.$$

All these questions have to be considered with greater care and precision than I could give to them here [16].

BROUWER's "strong real numbers" form a Brouwerian set [17] which is

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[16] The following lines (till the end of the chapter) replace some rather hesitating remarks in the original MS.

[17] In terms somewhat different from BROUWER's [5 I] ones, a Brouwerian set ("Menge") may be defined as a law which 1^0 allows to distinguish certain finite sequences $a_1, \ldots\ldots, a_n$ of natural numbers as "allowed ("ungehemmt") sequences" from other ones, such that 1*a*) for every n if $a_1, \ldots\ldots, a_{n-1}$ is an allowed sequence, then for every natural x it can be decided whether or not $a_1, \ldots\ldots, a_{n-1}, x$ is an allowed sequence, 1*b*) for every n and every allowed $a_1, \ldots\ldots, a_{n-1}$ at least one natural x can be found such that $a_1, \ldots\ldots, a_{n-1}, x$ is allowed, and which 2^0 for every n and every allowed sequence $a_1, \ldots\ldots, a_n$ determines an n^{th} symbol (or sequence of symbols, "Zeichenreihe") σ_n.

An infinite sequence $\sigma_1, \sigma_2, \ldots$ obtained in this way is called an *element* of the Brouwerian set.

The Brouwerian set will be called *special* if for every n a natural k_n is determined, such that $a_1, \ldots\ldots, a_n$ can only be admitted if for every $i \leqq n$ $a_i \leqq k_n$. (BROUWER's term "finite Menge" is not very well chosen, as the set itself need neither be finite nor even enumerable. Example: if the rational numbers are enumerated in a definite way, then all sequences of rational numbers form a Brouwerian set, where for every n all natural numbers a_n are allowed, and $\sigma\,(a_n)$ is the $a_n{}^{th}$ rational number. All sequences of rational numbers $r_1, r_2 \ldots$ with $r_n = \frac{m_n}{n}$, $0 \leqq m_n \leqq n$ form a special Brouwerian sets with $k_n = n$.

special [18]) if they are restricted to a finite interval (e.g. $\geqq 0$ and $\leqq 1$).

These examples make it, I believe, sufficiently clear that:

1°. no contradiction can occur between intuitionistic and classical mathematics [19]) provided the latter is consistently interpreted as a system of *stable* statements;

A *set* ("Spezies") of zero order is a Brouwerian set or an element of such a one. All sets of n^{th} order possessing some "well defined" ("begrifflich fertig definierte"; this definition, of course, is not sufficiently clear) property form a set ("Spezies") of $(n+1)^{th}$ order. Cf. BROUWER [1], p. 135; [2] III p. 15 seq.; [7] p. 1421.

All sequences of rational numbers form a Brouwerian set. All weak fundamental sequences form a set of order 1; a real number, defined as a set of all weak fundamental sequences concurrent with one of them also is a set of order 1; all real numbers form a set of order 2. An (unordered) pair (or triple, etc.) of real numbers also is a set of order 2; an ordered pair, being defined as an unordered triple, two of the elements of which are equal (hence $(x, y) = (x, x, y) = (x, y, x) = (y, x, x)$) is a set of order 3. A ("weakly defined") *function* may be defined as a set S of ordered pairs $P = (x, y)$ of real numbers (x, y) which for every x contains one and only one y (of course in weak interpretation:

$$\mathfrak{V}_x \neg \mathfrak{V}_y \neg_{\mathrm{I}} (x, y) \in S_{\mathrm{I}}\ [18]) \text{ and } \mathfrak{V}_x \mathfrak{V}_y \mathfrak{V}_{z\,\mathrm{III}} (x, y) \in S_{\mathrm{I}} \wedge_{\mathrm{I}} (x, z) \in S_{\mathrm{II}} \supset_{\mathrm{I}} y = z_{\mathrm{II}};$$

(For the meaning of the strokes cf. the next chapter.) A function is a set of order 4, all of them form a set of order 5.

[18]) From a formal point of view it may be of importance to remark that each successive passage to a set of higher order introduces a new type of all-symbol, which can not be defined by means of the previous ones: first we have $\mathfrak{V}_n{}^{\mathbf{N}}$ where n runs through the natural (or integer or rational, etc.) numbers, then $\mathfrak{V}_\xi{}^{\mathbf{S}}$ with ξ running through all (or all weak fundamental) sequences, then $\mathfrak{V}_x{}^{\mathbf{R}}$ where x runs through all real numbers, etc. Of course, the use of one single all-symbol by writing e.g. explicity $\mathfrak{V}_{x\,\mathrm{II}} x \in \mathrm{R}_{\mathrm{I}} \supset \Omega(x)_{\mathrm{I}}$, instead of $\mathfrak{V}_x{}^{\mathbf{R}} \Omega(x)$ does not alter the logical situation.

[19]) Cf. L. E. J. BROUWER [2 I]: "Still, in unjustified use [of the principium tertii] one will never be checked by a contradiction and discover in *this* way the unfoundedness of one's argument". .

[3] "Denn auf der Basis der intuitionistischen Einsichten lassen sich ausser den unabhängig from Prinzip des ausgeschlossenen Dritten entwickelbaren *richtigen Theorien* auch unter Heranziehung dieses Prinzips [for finite sets of properties] *nichtkontradiktorische* Theorien herleiten, mit denen sich von der bisherigen Mathematik ein viel grösserer Teil als mit den richtigen Theorien umfassen lässt. Eine geeignete Mechanisierung der Sprache dieser intuitionistisch-nichtkontradiktorischen Mathematik müsste also gerade liefern, was die formalistische Schule sich zum Ziel setzt.

Dagegen kann die gleichzeitige Aussage des Prinzips des ausgeschlossenen Dritten für *beliebige Spezies* von Eigenschaften sehr wohl kontradiktorisch sein. So lässt sich von der der folgenden Aussage die Kontradiktorität beweisen: Alle reelle Zahlen sind entweder rational oder irrational."

The contradiction with classical mathematics, however, is only apparent, because BROUWER uses the words "sind entweder ... oder..." in his strong sense (it can be *decided* whether ... or ...) whereas classical mathematics mean them to be interpreted in the weak sense (it is *not* true that neither ... nor ...). Moreover, it seems to me that BROUWER's higher valuation of the non-stable part of intuitionistic mathematics, which I completely share, should rather not be expressed by the term "richtig".

2⁰. on the contrary, with this interpretation classical mathematics becomes a *part* of intuitionistic mathematics;

3⁰. the main importance of BROUWER's work may be seen in the fact that a *stronger* interpretation of the classical statements than the stable one is possible, and (as we show in Ch. 3) in many respects of considerably greater interest; it may then be expected that the most important interpretation will be the strongest one, which leads to the *affirmative* (Ch. 3) and through it to a consistent *finitistic* interpretation of mathematics.

Mathematics. — *Bericht über die verschiedenen Methoden zur Lösung eines Systems linearer Gleichungen mit reellen Koeffizienten.* I. By E. BODEWIG. (Communicated by Prof. J. G. VAN DER CORPUT.)

(Communicated at the meeting of September 27, 1947.)

Systeme linearer Gleichungen mit vielen Unbekannten treten bei allen grösseren *statischen Berechnungen,* und bes. bei allen *Ausgleichungen* nach der Methode der kleinsten Quadrate auf. Daher gehören sie seit GAUSS insbes. zur täglichen Arbeit der Geodäten und Astronomen. Daneben haben sie in neuerer Zeit eine wichtige Anwendung gefunden bei der Lösung von *Integralgleichungen,* und obwohl sie hier im Prinzip keine exakte Lösung ergeben, wie es etwa die LIOUVILLE–NEUMANNsche Reihe tut, sondern stets nur Näherungslösungen, die auch nicht verbessert werden können, wäre die — wissenschaftlich genommen — so grobe Approximation durch lineare Gleichungen jenen feinen Methoden in der Praxis doch weit überlegen, wenn es gelänge, lineare Gleichungen mit vielen Unbekannten in „endlicher" Zeit zu lösen. Ja, einer der hervorragendsten Vertreter der Anwendung mathematischer Methoden hat — wohl im Hinblick auf die Integralgleichungen — sich dahin ausgesprochen, dass „die schnelle Lösung eines Systems von etwa 100 linearen Gleichungen eine grossen Fortschritt der Technik und Zivilisation zur Folge haben würde".

In der Tat ist aber die Lösung linearer Gleichungen *eines der lästigsten Probleme* des numerischen Rechnens. Anders jedoch als bei sonstigen Problemen, insbes. der Lösung von Gleichungen überhaupt, ist die Schwierigkeit nicht prinzipieller Art — denn man weiss sehr wohl, wie man lineare Gleichungen zu lösen hat —, sondern *sie entsteht erst durch die hohe Zahl von Gleichungen* (und Unbekannten) und wächst dann allerdings schnell. Dadurch versagen eine Reihe von Methoden, die bei andern Gleichungstypen zur ersten Approximation angewendet werden, insbes. die grafischen Methoden, aber auch alle gebräuchlichen Approximationsmethoden. Ueberhaupt nützt einem hier anders als sonst eine erste Approx. nichts, um daraus eine folgende zu finden. Und man hat, wenn man sukzessive approximieren will, erst spezielle Approximationsmethoden für sie zu entwickeln.

Andere Schwierigkeiten entstehen gewissermassen gerade dadurch, dass die Gleichungen im Wesen so einfach sind. Während man sich z.B. die Lösung eines Systems beliebiger Gleichungen, von einer angenäherten Lösung her kommend, vereinfachen kann, indem man das System durch ein einfacheres (nämlich ein lineares) ersetzt, ist bei linearen Gleichungen keine weitere Vereinfachung möglich. Es ist sogar eigenartig, dass die

meisten Methoden geradezu das Bestreben haben, die prinzipiellen Schwierigkeiten des Problems zu erhöhen, wohl in der Hoffnung, dass dann gleichzeitig die aus der Anzahl der Gleichungen entstehenden Schwierigkeiten abnehmen werden. Man benutzt z.B. Iterationsmethoden, obwohl diese viel tiefer liegende Begriffe und Untersuchungen verlangen, als für das Verständnis und die Lösung des Problems nötig sind. Das geht so weit, dass manche Begriffe, wie die Konvergenz, eingeführt werden, die dem Problem eigentlich fremd sind. Oder man arbeitet mit der Inversen einer Matrix, oder mit der Darstellung einer Matrix als Produkt zweier Dreiecksmatrizen usw.

Trotz aller dieser Kunstgriffe und Methoden kann man die *Ursache aller Schwierigkeiten des Problems, die hohe Zahl von Gleichungen, nicht beseitigen.* Mit ihr hat man bei allen Methoden zu rechnen. Es ist zwecklos, sie überwinden zu wollen.

Es kann sich also nur darum handeln, *die vorhandenen grossen Schwierigkeiten zu verkleinern,* d.h. nach der *relativ besten Methode* zu suchen.

Bei solchen weitläufigen Rechnungen muss man natürlich *Rechenmaschinen* benutzen. Die neuerdings berühmt gewordenen Elektronen-Rechenmaschinen lösen solche Gleichungen sogar ganz automatisch, so dass es hier auf die Berechnungsmethode nicht ankommt. Der Preis solcher Maschinen ist aber so hoch, dass jeder der beiden Maschinentypen nur in einem einzigen Exemplar existiert (oder existieren wird). An allen übrigen mathematischen Instituten und Berechnungsstätten der Welt, muss man sich mit den üblichen Rechenmaschinen begnügen. Und bei ihnen ist die Berechnungsmethode durchaus nicht gleichgültig. Auch bei einer Rechenmaschine kommt es sehr darauf an, ob man zur Lösung eines Problems 300000 oder eine Million Multiplikationen auszuführen, d.h. ½ oder zwei Millionen Einstellungen vorzunehmen hat. Und diese Zahl von Operationen — oder eine noch weit grössere Zahl, je nach der Methode — hat man bei der Lösung von 100 linearen Gleichungen nötig. Es handelt sich dabei nicht nur um einen Gewinn an Zeit, sondern vor allem um die Vermeidung grober Fehler durch falsches Einstellen, was natürlich bei einer solch grossen Zahl von Operationen sehr leicht auftritt.

Der *Hauptzweck* der vorliegenden Uebersicht über die wichtigsten Lösungsmethoden ist daher: *für jede Methode die Anzahl der zur Lösung notwendigen Operationen genau anzugeben.* Damit ergibt sich von selber die beste Lösungsmethode.

Daneben sollen *die verschiedenen Methoden klassifiziert und auf ihren eigentlichen Kern zurückgeführt* werden. Es wird sich dabei herausstellen, dass äusserlich ganz verschiedene Methoden im Wesen identisch sind und sich entweder nur durch die Reihenfolge der Operationen unterscheiden oder sogar nur durch die Bezeichnung. So z.B. erscheint die Methode von BOLTZ nur als eine andere Berechnung der inversen Matrix nach der Relation von FROBENIUS–SCHUR. Ferner ergibt sich, dass die zweckmässigste Iterationsmethode nicht eine der gebräuchlichen von

MISES, SEIDEL u.a. ist, die alle nur linear konvergieren, sondern die Entwicklung in eine FROBENIUS–NEUMANNsche Reihe — eine Methode, die bisher noch nicht beschrieben wurde. Obwohl auch diese Methode nur linear konvergiert, ist sie schon in dieser Form den anderen iterativen Methoden überlegen. Es kommt aber hinzu, dass man bei ihr sukzessive 2, 4, 8, 16, ..., 2^n, ... Glieder durch je eine einzige Operation addieren und damit die Konvergenz zu einer quadratischen machen kann.

Die *wirksamste aller Methoden* aber ist eine Modifikation der GAUSSschen Methode, bei welcher der GAUSSsche Algorithmus iterativ benutzt wird.

Den Anlass zu dem vorliegenden Bericht gab eine Reihe von Unklarheiten und Fehlern in der Literatur. So urteilt z.B. JEFFREYS in seinem neuesten, sonst so vorzüglichen Buche „Methods of mathematical Physics" (Cambridge, 1946) abfällig über die GAUSSsche Methode und empfiehlt statt dessen die Iterationsmethode von SEIDEL. Er „beweist" diese Behauptung an einem Beispiel mit drei Unbekannten, das er auf beide Arten löst. Geht man aber beide Beschreibungen näher durch, so stellt man mit Erstaunen fest: Bei der GAUSSschen Methode benötigt er 10 Multiplikationen und hat die Wurzeln auf 4 Ziffern. Diese Zahl 10 kann man ihm also auch bei der anderen Methode als „Kredit" geben. Aber bevor er diese Methode anwendet, dividiert er jede der drei Gleichungen durch einen bestimmten Koeffizienten. Er scheint ganz zu übersehen, dass er dazu schon 9 Multiplikationen (Divisionen) braucht, dass er also den ganzen Kredit schon verausgabt hat, bevor er mit der neuen Methode überhaupt anfängt! Darauf beginnt er, nach SEIDEL zu iterieren. Nach fünf Iterationen hat er glücklich das Ergebnis auf 2—3 Dezimalen genau. Jede Iteration aber kostet ihn 6 Multiplikationen, im ganzen also 30 Multiplikationen, obwohl er keinen „Kredit" mehr hat. Es kommt aber hinzu, dass man bei GAUSS stets mit 10 Multiplikationen auskommt, auch wenn man das Resultat auf 6 Dezimalen haben will. Wollte JEFFREYS aber dieselbe Genauigkeit mittels der von ihm empfohlenen Methode SEIDELs erreichen, so hätte er sicherlich 7 weitere Iterationen nötig, also 42 neue Multiplikationen, im ganzen somit 81 Multiplikationen! Dieser Umstand scheint ihm nicht völlig entgangen zu sein, denn er entschuldigt die schwache Annäherung damit, dass sein Beispiel „nicht günstig gewählt" sei, begeht aber damit einen neuen Irrtum. Denn abgesehen davon, dass man sich ja in der Praxis seine Beispiele überhaupt nicht „wählen" kann, ist sein Beispiel sogar sehr günstig gewählt, weil die Diagonalelemente vorherrschen und das Verfahren daher konvergiert. Dieser Ausnahmefall findet sich aber nur selten verwirklicht. Man müsste also erst eine Methode haben, um ein gewöhnliches Gleichungssystem auf eine solches mit vorherrschender Diagonale zu transformieren, ein Gesichtspunkt, der aber weder bei JEFFREYS noch bei den meisten anderen Autoren auftritt (nur CESARI bildet eine Ausnahme). Wenn er trotz dieser vielfachen Irrtümer die SEIDELsche Methode der GAUSSschen vorzieht, so lässt sich

dies wohl nur psychologisch erklären, indem die GAUSSsche Methode die gewöhnliche, auf der Schule schon gebrauchte Methode ist und daher wegen ihrer Einfachheit nicht so elegant erscheint wie die Iterationsmethoden und darum ohne nähere Ueberlegung, „instinktiv" gegenüber letzteren zurückgesetzt wird. Man braucht die Methode von GAUSS heute, wo man grössere Systeme von linearen Gleichungen als zu seiner Zeit vor sich hat, nicht mehr unbesehen hinzunehmen. Doch sollte man mit Kritik gerade bei ihm sehr vorsichtig sein. Denn man wird wohl annehmen dürfen, dass auch GAUSS einige Studien über lineare Gleichungen angestellt hat, bevor er seine Methode (für Normalgleichungen) veröffentlichte. Jedenfalls Gleichungen mit drei Unbekannten konnte auch GAUSS zweckmässig lösen.

Bezeichnungen.

Das gegebene Gleichungssystem in den Unbekannten $x_1, x_2, \ldots, x_n$ sei

$$G_i\colon\ a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n = v_i, \qquad i = 1, \ldots, n$$

oder als Matrizengleichung geschrieben:

$$\mathfrak{A}\mathfrak{x} = \mathfrak{v}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\mathrm{I}, 1)$$

Im Folgenden bezeichnen grosse gothische Buchstaben stets eine Matrix, kleine gothische Buchstaben einen Vektor, also eine $(1, n)$-Matrix einen Zeilenvektor und eine $(n, 1)$-Matrix einen Spaltenvektor.

Eine Matrix, die auf einer Seite der Diagonalen aus lauter Nullen besteht, heisst *Dreiecksmatrix*. Liegen die Nullen unterhalb der Diagonalen, also links davon, so heisst sie eine linke Dreiecksmatrix $\mathfrak{D}_l$, ähnlich rechte Dreiecksmatrix $\mathfrak{D}_r$. Man deutet sie oft an durch $\diagdown\!|$ bzw. $|\!\diagdown$.

Eine Matrix, die sowohl oberhalb als unterhalb der Diagonale lauter Nullen hat, heisst Diagonalmatrix. Wir bezeichnen sie mit $\mathfrak{D}$.

Eine Matrix, die überhaupt aus lauter Nullen besteht, heisst *Nullmatrix* und wird mit $\mathfrak{O}$ bezeichnet. Aehnlich der *Nullvektor* mit $\mathfrak{o}$.

Man sagt, die Matrix $\mathfrak{A}$ transformiere den Vektor $\mathfrak{x}$ in den Vektor $\mathfrak{y}$ wenn zwischen $\mathfrak{x}$ und $\mathfrak{y}$ die Beziehung besteht,

$$\mathfrak{A}\mathfrak{x} = \mathfrak{y}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\mathrm{I}, 2)$$

Die charakteristischen Wurzeln von $\mathfrak{A}$ nennen wir kurz die *Wurzeln* von $\mathfrak{A}$ und bezeichnen sie mit griechischen Buchstaben: $\lambda_1, \ldots, \lambda_n$. Sie erfüllen die Gleichung

$$|\mathfrak{A} - \lambda E| = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\mathrm{I}, 3)$$

Die linke Seite von (3) ist die Gleichungsdeterminante des homogenen linearen Systems:

$$(\mathfrak{A} - \lambda E)\mathfrak{x} = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\mathrm{I}, 4)$$

Für jedes λ_k, welches (3) genügt, lässt sich das Gleichungssystem (4), also

$$a_{i1}x_1 + \ldots + a_{in}x_n = \lambda_k x_i$$

„nicht-trivial" lösen, d.h. so, dass $\mathfrak{x}$ nicht der Nullvektor ist. Jeder Lösungsvektor $\mathfrak{x}$ von (4) heisst ein *Eigenvektor von* $\mathfrak{A}$.

Einteilung der Lösungsmethoden.

Bei den Lösungsmethoden des Systems (1) gibt es *direkte Methoden* und sukzessive *Approximationsmethoden*. Die direkten führen bei einmaliger Anwendung zur (theoretisch exakten) Lösung.

Ferner heisst eine Methode „*bestimmt*", wenn sie den Lösungsvektor $\mathfrak{x}$ selber und direkt liefert, „*unbestimmt*", wenn sie den Wert der reziproken Matrix $\mathfrak{A}^{-1}$ liefert, aus der sich $\mathfrak{x}$ selber erst durch Multiplikation mit $\mathfrak{v}$ findet: $\mathfrak{x} = \mathfrak{A}^{-1}\,\mathfrak{v}$.

Die *unbestimmten Lösungen* haben den Vorteil, dass sie die Auflösung auch jedes anderen Gleichungssystems $\mathfrak{A}\,\mathfrak{x} = \mathfrak{w}$ mit derselben Gleichungsmatrix $\mathfrak{A}$ einfach zu berechnen gestatten. Dies kann in verschiedenen Fällen zweckmässig sein, etwa, wenn man mehrere Systeme mit derselben Matrix zu lösen hat, oder wenn der Vektor $\mathfrak{v}$, also die rechten Seiten des Systems, erst später bekannt werden oder sich durch spätere Messungen verändern. Dies kommt vor allem bei geodätischen Messungen vor, wo die Matrix $\mathfrak{A}$ ohne Messungen aufstellbar ist und wo man die Diagonalglieder von $\mathfrak{A}^{-1}$ zur Bestimmung des mittleren Fehlers der Unbekannten braucht, ebenso bei allen Ausgleichungsrechnungen.

Im Gegensatz dazu löst eine *bestimmte Lösung* nur jenes eine System (1) auf. Nun lässt sich freilich *jede bestimmte Lösungsmethode auch zur Berechnung von* $\mathfrak{A}^{-1}$ *verwenden*. Man braucht sie nur anzuwenden auf die n Gleichungen, deren rechte Seiten nacheinander die Vektoren $\mathfrak{e}_i$ sind, die in den n Spalten der Einheitsmatrix E stehen:

$$\mathfrak{A}\,\mathfrak{x}_1 = \mathfrak{e}_1, \ldots, \mathfrak{A}\,\mathfrak{x}_n = \mathfrak{e}_n.$$

In der Tat ist $\mathfrak{x}_i = \mathfrak{A}^{-1}\,\mathfrak{e}_i$ die i-te Spalte von $\mathfrak{A}^{-1}$, so dass

$$\mathfrak{A}^{-1} = (\mathfrak{x}_1, \mathfrak{x}_2, \ldots, \mathfrak{x}_n). \quad \ldots\ldots\ldots \quad (\mathrm{I}, 5)$$

I. Direkte Methoden. Bestimmte Lösungen.

Die Methode von Gauss.

Sie ist die schulmässige. Sie löst das System

$$S_n: \quad \mathfrak{A}\,\mathfrak{x} = \mathfrak{v}$$

der n Gleichungen G_i durch *sukzessive Elimination*. Zunächst wird etwa x_1 eliminiert. Dadurch entsteht ein System

$$S_{n-1}: \quad \mathfrak{A}^{(1)}\,\mathfrak{x} = \mathfrak{v}^{(1)}$$

von $n-1$ Gleichungen $G_i^{(1)}$, wobei die erste Spalte von $\mathfrak{A}^{(1)}$ aus lauter

Nullen besteht, so dass S_{n-1} nur die $n-1$ Unbekannten $x_2, \ldots, x_n$ enthält. S_n ist also transformiert in das System bestehend aus dem System S_{n-1} und einer beliebigen Gleichung G_i.

Das System S_{n-1} wird jetzt ähnlich reduziert, d.h. man eliminiert x_2 aus je zwei seiner Gleichungen und bekommt ein System

$$S_{n-2}: \quad \mathfrak{A}^{(2)}\,\mathfrak{x} = \mathfrak{v}^{(2)}$$

von $n-2$ Gleichungen $G_i^{(2)}$, wobei die beiden ersten Spalten von $\mathfrak{A}^{(2)}$ aus lauter Nullen bestehen, so dass S_{n-2} nur die $n-2$ Unbekannten $x_3, \ldots, x_n$ enthält. S_n ist damit transformiert auf das System bestehend aus dem System S_{n-2}, einer beliebigen Gleichung G_i und einer beliebigen Gleichung $G_i^{(1)}$.

So geht man weiter bis zum System $S_1 \equiv G_1^{(n-1)}$, das nur die Unbekannte x_n enthält. Damit ist S_n transformiert auf das System $\overline{S_n}$, welches man erhält, wenn man aus jedem System S_i eine Gleichung herausgreift:

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n &= v_1 \\ a_{22}^{(1)}x_2 + a_{23}^{(2)}x_3 + \ldots + a_{2n}^{(1)}x_n &= v_2^{(1)} \\ a_{33}^{(2)}x_3 + \ldots + a_{3n}^{(2)}x_n &= v_3^{(2)} \\ \ldots\ldots\ldots\ldots & \\ a_{nn}^{(n-1)}x_n &= v_n^{(n-1)} \end{aligned}\right\} \quad . \quad . \quad \text{(I, 6)}$$

Und dieses System lässt sich leicht von unten her auflösen.

Anzahl der Operationen.

Das System S_{n-1} erhält man aus S_n auf dieselbe Weise, wie man die Determinante D_{n-1} aus D_n nach dem Kombinationssatz (WHITTAKERs „Regel von CHIÒ") bekommt. Vgl. Lit. 1. D.h. man subtrahiert von jeder Gleichung G_i die mit a_{i1}/a_{11} multiplizierte erste Gleichung G_1. Doch hat man jetzt natürlich die Operationen auch auf die rechten Seiten v_i auszudehnen. Die Reduktion von S_n auf S_{n-1} erfordert demnach für jede der $n-1$ Gleichungen : $n+1$ Multiplikationen, worunter eine Division, und n Additionen, im ganzen daher n^2-1 bzw. n^2-n. Die Reduktion von S_n auf das System (6) erfordert somit

$$\left.\begin{aligned} &\sum_2^n (n^2-1) = \tfrac{1}{6}\,n\,(n-1)\,(2n+5) \text{ Multiplikationen und} \\ &\sum_2^n (n^2-n) = \tfrac{1}{3}\,n\,(n^2-1) \text{ Additionen.} \end{aligned}\right\} \quad . \quad \text{(I, 7a)}$$

Nunmehr muss das Dreieckssystem (6) von unten aufgelöst werden. Dabei ist in jeder Gleichung nur eine Unbekannte. Jede Gleichung erfordert daher genau so viele Multiplikationen (worunter eine Division), wie

es Koeffizienten auf der linken Seite gibt, und in jeder Zeile eine Addition weniger. Die *Auflösung eines Dreieckssystems* kostet also

$$\left.\begin{array}{r}\Sigma n = \frac{1}{2} n (n+1) \text{ Multiplikationen und } \Sigma (n-1) = \\ = \frac{1}{2} n (n-1) \text{ Additionen.}\end{array}\right\} \quad (\text{I}, 7b)$$

Ergebnis. Die Gesamtlösung des Systems S_n erfordert so viel Operationen wie (7*a*, *b*) zusammen, d.h.

$$\frac{1}{3} n (n^2-1) + n^2 \text{ Multiplikationen und } \frac{1}{6} n (n-1)(2n+5) \text{ Additionen.} \quad (\text{I}, 7)$$

Fall der symmetrischen Matrix.

Bekanntlich hat GAUSS seine Methode aus den Bedürfnissen der Ausgleichung von Beobachtungen, wo sich zum ersten Male grössere Systeme von linearen Gleichungen ergaben, entwickelt, also für die sogen. „*Normalgleichungen*". Hier ist die quadratische Gleichungsmatrix $\mathfrak{N}$ das Produkt der rechteckigen (ursprünglichen) Gleichungsmatrix $\mathfrak{B}$ mit ihrer Transponierten: $\mathfrak{N} = \mathfrak{B}' \mathfrak{B}$. Dieses $\mathfrak{N}$ ist unter anderem symmetrisch, d.h. gleich seiner Transponierten: $\mathfrak{N} = \mathfrak{N}'$ oder $n_{ik} = n_{ki}$.

Nicht nur bei Normalgleichungen, sondern bei allen symmetrischen Gleichungen vereinfacht sich natürlich die Berechnung des Systems (6), weil nicht nur das Ausgangssystem S_n symmetrisch ist, sondern jedes der folgenden Systeme S_j ebenfalls. Denn da das System S_{n-1} aus den Gleichungen $G_i^{(1)} \equiv -\frac{a_{i1}}{a_{11}} G_1 + G_i$ besteht, so sind seine Koeffizienten $a_{ik}^{(1)} = -\frac{a_{i1}}{a_{11}} a_{1k} + a_{ik}$, und dies ist wegen der Symmetrie der a_{ik} gleich $a_{ki}^{(1)}$.

Man braucht also von jedem System S_j nur die Koeffizienten in der Diagonale und auf einer Seite davon zu berechnen, die andern sind dazu spiegelbildlich gleich. Wir haben daher ungefähr halb so viel Operationen vorzunehmen wie im allgemeinen Falle (1). Genauer erfordert jetzt der Uebergang von S_n zu S_{n-1}:

$$3 + 4 + \ldots + (n+1) = \tfrac{1}{2}(n^2 + 3n - 4) \text{ Multiplikationen und}$$
$$2 + 3 + \ldots + n = \tfrac{1}{2}(n^2 + n - 2) \text{ Additionen,}$$

die Berechnung des Systems (6) also

$$\frac{1}{6} n (n-1)(n+7) \text{ Multiplik. und } \frac{1}{6} n (n-1)(n+4) \text{Additionen,} \quad (\text{I}, 7a')$$

also etwa die Hälfte von (7*a*). Die Lösung des ganzen Gleichungssystems kostet also

$$\frac{1}{6} n (n^2 + 9n - 4) \text{ Multiplik. und } \frac{1}{6} n (n-1)(n+7) \text{ Additionen.} \quad (\text{I}, 7')$$

Der Kern der GAUSS*schen Methode.*

Zunächst besteht das *Resultat* (6) der verschiedenen Operationen darin, dass die Gleichungsmatrix $\mathfrak{A}$ in eine linke Dreiecksmatrix $\mathfrak{D}_l$ übergegangen

ist, d.h. dass das beliebige System $\mathfrak{A}\,\mathfrak{x} = \mathfrak{v}$ ersetzt wurde durch das dreieckige System

$$\mathfrak{D}_l\,\mathfrak{x} = \mathfrak{V}. \quad \ldots\ldots\ldots\ldots \quad (\mathrm{I}, 8)$$

Was das *Resultat der Operationen einzeln* angeht, so besteht das System S_{n-1} aus den Gleichungen $G_i^{(1)}$, wo

$$G_i^{(1)} \equiv -\frac{a_{i1}}{a_{11}}\,G_1 + G_i, \qquad i = 2, \ldots, n.$$

Schreiben wir von jeder Gleichung nur die Koeffizienten, also die Matrix, so entsteht die Matrix der $G_i^{(1)}$ durch linksseitige Multiplikation von $\mathfrak{A}$ mit der $(n-1, n)$-Matrix

$$\mathfrak{M}^{(1)} = \begin{bmatrix} -\frac{a_{21}}{a_{11}} & 1 & 0 & 0 & \ldots & 0 \\ -\frac{a_{31}}{a_{11}} & 0 & 1 & 0 & \ldots & 0 \\ -\frac{a_{41}}{a_{11}} & 0 & 0 & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ -\frac{a_{n1}}{a_{11}} & 0 & 0 & 0 & \ldots & 1 \end{bmatrix} \equiv (\mathfrak{a}^{(1)}\,E),$$

wo $\mathfrak{a}^{(1)}$ der Vektor ist mit den Komponenten $-\frac{a_{i1}}{a_{11}}$.

Das neue Gleichungssystem bestehend aus der ersten Gleichung G_1 von S_n und dem System S_{n-1} lautet also

$$\overline{\mathfrak{A}}^{(1)}\,\mathfrak{x} = \overline{\mathfrak{v}}^{(1)}, \text{ wo } \overline{\mathfrak{A}}^{(1)} = \overline{\mathfrak{D}}_r^{(1)}\,\mathfrak{A}, \quad \overline{\mathfrak{v}}^{(1)} = \overline{\mathfrak{D}}_r^{(1)}\,\mathfrak{v}$$

und

$$\overline{\mathfrak{D}}_r^{(1)} = \begin{bmatrix} 1 & 0 & 0 & \ldots & 0 \\ -\frac{a_{21}}{a_{11}} & 1 & 0 & \ldots & 0 \\ -\frac{a_{31}}{a_{11}} & 0 & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ -\frac{a_{n1}}{a_{11}} & 0 & 0 & \ldots & 1 \end{bmatrix} \equiv \begin{bmatrix} 1 & 0 \\ \mathfrak{a}^{(1)} & E \end{bmatrix}$$

eine rechte Dreiecksmatrix ist.

Die Ableitung des Systems S_{n-2} aus S_{n-1} geschieht auf dieselbe Art, d.h. S_{n-2} besteht aus den Gleichungen $G_i^{(2)}$, wo

$$G_i^{(2)} \equiv -\frac{a'_{i2}}{a'_{22}}\,G_2^{(1)} + G_i^{(1)}, \qquad i = 3, \ldots, n.$$

Die Matrix von S_{n-2} entsteht also aus der von S_{n-1} durch linksseitige Multiplikation mit der $(n-2,\ n-1)$-Matrix

$$\mathfrak{M}^{(2)} = \begin{bmatrix} -\frac{a'_{32}}{a'_{22}} & 1 & 0 & \dots & 0 \\ -\frac{a'_{42}}{a'_{22}} & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ -\frac{a'_{42}}{a'_{22}} & 0 & 0 & \dots & 0 \end{bmatrix} = (\mathfrak{a}^{(2)}\, E),$$

wo $\mathfrak{a}^{(2)}$ der Vektor mit den Komponenten $-a'_{i2}/a'_{22}$ ist.

Das neue System besteht aus der ersten Gleichung G_1 von S_n, aus der ersten Gleichung $G_2^{(1)}$ von S_{n-1} und dem System S_{n-2}, lautet also

$$\overline{\mathfrak{A}}^{(2)}\, \mathfrak{x} = \overline{\mathfrak{v}}^{(2)},\ \text{wo}\ \overline{\mathfrak{A}}^{(2)} = \overline{\mathfrak{D}}_r^{(2)}\, \overline{\mathfrak{A}}^{(1)},\ \overline{\mathfrak{v}}^{(2)} = \overline{\mathfrak{D}}_r^{(2)}\, \mathfrak{v}^{(1)}$$

und

$$\overline{\mathfrak{D}}_r^{(2)} = \begin{bmatrix} 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & -\frac{a'_{32}}{a'_{21}} & 1 & 0 & \dots & 0 \\ 0 & -\frac{a'_{42}}{a'_{22}} & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & -\frac{a'_{n2}}{a'_{22}} & 0 & 0 & \dots & 1 \end{bmatrix}$$

In dieser Weise geht man weiter, bis man zur Matrix $\mathfrak{A}^{(n-1)}$ kommt, bei welcher unterhalb der Diagonale lauter Nullen stehen und die gleich der Matrix $\mathfrak{D}_l$ von (8) ist. Es ist also in (8):

$$\mathfrak{D}_l = \overline{\mathfrak{D}}_r^{(n-1)} \dots \overline{\mathfrak{D}}_r^{(2)}\, \overline{\mathfrak{D}}_r^{(1)}\, \mathfrak{A}. \qquad \text{(I, 9)}$$

Nun ist aber das Produkt zweier gleichgerichteter Dreiecksmatrizen wieder eine Dreiecksmatrix derselben „Richtung". Also

$$\mathfrak{D}_l = \overline{\mathfrak{D}}_r\, \mathfrak{A},\ \text{wo}\ \overline{\mathfrak{D}}_r = \overline{\mathfrak{D}}_r^{(n-1)} \dots \overline{\mathfrak{D}}_r^{(2)}\, \overline{\mathfrak{D}}_r^{(1)}, \qquad \text{(I, 9a)}$$

Somit haben wir das

Ergebnis. Das GAUSSsche Verfahren kommt darauf hinaus, dass die Gleichungsmatrix $\mathfrak{A}$ mit derjenigen rechten Dreiecksmatrix $\overline{\mathfrak{D}}_r$ mit den Diagonalelementen 1 multipliziert wird, dass das Produkt eine linke Dreiecksmatrix $\mathfrak{D}_l$ ergibt, symbolisch:



Die Methode von Cholesky.

Für ein *symmetrisches System* von Gleichungen kann man eine Methode des Franzosen Cholesky (gest. 1918) benutzen, die dieser eigentlich nur für spezielle symmetrische Gleichungen, nämlich für *Normalgleichungen*, angegeben hat und die von vielen heutzutage als die beste Methode angesehen wird, obwohl man in den 25 Jahren Zeit genug gehabt hätte, diese Behauptung näher zu prüfen. Wir wollen sehen, ob und inwieweit jene Meinung gerechtfertigt ist.

Während Gauss das System mit einer rechten Dreiecksmatrix multipliziert, um eine linke Dreiecksmatrix zu bekommen: $\mathfrak{D}_r \mathfrak{A} = \mathfrak{D}_l$, deren zugehöriges Gleichungssystem einfach aufzulösen ist, zieht Cholesky offenbar aus jener Matrizengleichung die Folgerung $\mathfrak{A} = \overline{\mathfrak{D}}_r \mathfrak{D}_l$, wo $\overline{\mathfrak{D}}_r$ hier die Reziproke von $\mathfrak{D}_r$ bezeichnen soll, und sucht dann die Koeffizienten der beiden Matrizen so zu bestimmen, dass sich im Produkt die a_{ik} ergeben.

Ist nun $\mathfrak{A}$ eine symmetrische Matrix $\mathfrak{S}$, so führt Cholesky noch zusätzlich die Bedingung ein, dass die rechte Dreiecksmatrix die Transponierte der linken sein muss:

$$\mathfrak{S} = \mathfrak{D}_l' \mathfrak{D}_l.$$

Er muss also auf die Annehmlichkeit von Gauss verzichten, dass die Diagonalkoeffizienten von $\mathfrak{D}_r$ gleich 1 sind. Es treten somit im Produkt $\mathfrak{D}_l' \mathfrak{D}_l$ in der Diagonale Quadratsummen auf, mit der Folge, dass die Diagonalkoeffizienten von $\mathfrak{D}_l$ das Ziehen von Quadratwurzeln erfordert.

Im einzelnen verläuft dann die Bestimmung von $\mathfrak{D}_l$ wie folgt. Das Produkt der Matrizen $\mathfrak{D}_l' \mathfrak{D}_l$

$$\begin{pmatrix} l_{11} & 0 & 0 & 0 & \dots & 0 \\ l_{12} & l_{22} & 0 & 0 & \dots & 0 \\ l_{13} & l_{23} & l_{33} & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ l_{1n} & l_{2n} & l_{3n} & l_{4n} & \dots & l_{nn} \end{pmatrix} \begin{pmatrix} l_{11} & l_{12} & l_{13} & l_{14} & \dots & l_{1n} \\ 0 & l_{22} & l_{23} & l_{24} & \dots & l_{2n} \\ 0 & 0 & l_{33} & l_{34} & \dots & l_{3n} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & l_{nn} \end{pmatrix}$$

muss die Matrix $\mathfrak{S}$ liefern. Die i-te Zeile des Produktes liefert die i-te Zeile von $\mathfrak{D}_l$, nämlich die l_{ik}. Hiervon sind aber die ersten l_{ik} schon bekannt, nämlich $l_{i1} = l_{1i}$, $l_{i2} = l_{2i}$, ..., $l_{i,i-1} = l_{i-1,i}$. Die übrigen hingegen müssen bestimmt werden. Nun ist a_{ii} eine Summe von i Quadraten:

$$a_{ii} = l_{1i}^2 + \dots + l_{ii}^2,$$

wovon nur das letzte unbekannt ist. Hingegen ist

$$a_{ik} = l_{1i} l_{1k} + l_{2i} l_{2k} + \dots + l_{ii} l_{ik}, \quad k > i.$$

Hieraus ergibt sich l_{ik}, wenn alle vorhergehenden l der Zeile i bekannt sind, und zwar mittels i Multiplikationen und $i-1$ Additionen, während l_{ii} eine Quadratwurzel, $i-1$ Multiplikationen und ebenso viele Additionen erfordert. Die ganze i-te Zeile kostet also eine Wurzel, $i-1+i(n-i)$

Multiplikationen und $(i-1)(n-i+1)$ Additionen. Summiert man dies von $i=1$ bis n, so erfordert die Bestimmung der Matrix $\mathfrak{D}_l$:

$$\left.\begin{array}{r} n \text{ Wurzeln, } \frac{1}{6}n(n-1)(n+4) \text{ Multiplikationen und} \\ \frac{1}{6}n(n-1)(n+1) \text{ Additionen.} \end{array}\right\} . \quad \text{(I, 10)}$$

Dies ist beinahe schon so viel wie die gesamte Auflösung (eines symmetrischen Systems) nach GAUSS. Vgl. (I, 7a).

Das System $\mathfrak{S}\mathfrak{x}=\mathfrak{r}$ ist also übergegangen in $\mathfrak{D}_l'\mathfrak{D}_l\mathfrak{x}=\mathfrak{r}$. Der *Kunstgriff von* CHOLESKY besteht nun in seinem Kern darin, dieses transformierte System in zwei Systeme zu spalten:

$$\mathfrak{D}_l'\mathfrak{y}=\mathfrak{r} \quad , \quad \mathfrak{y}=\mathfrak{D}_l\mathfrak{x}.$$

Das erste System wird nach $\mathfrak{y}$ aufgelöst, das zweite nach $\mathfrak{x}$. Da beides dreieckige Systeme sind, erfordert jedes nach (I, 7b): $n(n+1)/2$ Multiplikationen und $n(n-1)/2$ Additionen, welche Anzahlen zu (10) hinzukommen. Somit haben wir:

Ergebnis. 1. Die Methode von CHOLESKY ist nur auf symmetrische Systeme anwendbar.
2. Sie erfordert dann an Operationen:
n Quadratwurzeln, $\frac{1}{6}n(n^2+9n+2)$ Multiplikationen und $\frac{1}{6}n(n-1)(n+7)$ Additionen.
3. Dies ist mehr als nach der Methode von GAUSS. Die erwähnte Ansicht von der Ueberlegenheit der Methode CHOLESKYs ist also falsch.
4. Die Methode hat allerdings den Vorteil einer geringeren Schreibarbeit, und dies kann sehr wichtig sein.

II. Unbestimmte Lösungen.

Berechnung mittels der Relation von SCHUR.

In formeller Hinsicht am einfachsten geschieht die Berechnung von $\mathfrak{A}^{-1}$ durch die Relation von SCHUR. Dazu wird die Matrix $\mathfrak{A}$ in vier Teilmatrizen zerlegt, indem man in der linken oberen Ecke eine quadratische Matrix $\mathfrak{P}$ abtrennt und die Trennungsstriche durchzieht:

$$\underset{(n,n)}{\mathfrak{A}} = \begin{pmatrix} \underset{(p,p)}{\mathfrak{P}} & \underset{(p,n-p)}{\mathfrak{Q}} \\ \underset{(n-p,p)}{\mathfrak{R}} & \underset{(n-p,n-p)}{\mathfrak{S}} \end{pmatrix} \quad \text{Z. B. ist } \mathfrak{Q} \text{ eine } (p, n-p)\text{-Matr.}$$

Dann ist nach SCHUR (Lit. 1):

$$\mathfrak{A}^{-1} = \begin{pmatrix} \mathfrak{P}^{-1}+\mathfrak{T}\mathfrak{U} & -\mathfrak{T} \\ -\mathfrak{S}_1^{-1}\mathfrak{U} & \mathfrak{S}_1^{-1} \end{pmatrix} . \quad . \quad . \quad . \quad . \quad . \quad \text{(II, 1)}$$

wo

$$\mathfrak{S}_1=\mathfrak{S}-\mathfrak{R}\mathfrak{P}^{-1}\mathfrak{Q} \quad , \quad \mathfrak{T}=\mathfrak{P}^{-1}\mathfrak{Q}\mathfrak{S}_1^{-1} \quad , \quad \mathfrak{U}=\mathfrak{R}\mathfrak{P}^{-1}.$$

Auf einen Beweis von (1) verzichten wir an dieser Stelle, da er sich weiter unten bei der Methode von BOLTZ von selber ergibt.

Der Wert der Formel beruht darauf, dass, wenn man die Reziproke von $\mathfrak{P}$ kennt, die Berechnung der Reziproken von $\mathfrak{A}$ sich bedeutend vereinfacht. Man kann also, von $\mathfrak{P}$ ausgehend, die Reziproke von $\mathfrak{A}$ durch sukzessive Ränderung finden. In einer früheren Arbeit (Lit. 1, p. 55/6) habe ich gezeigt, dass die Anzahl der zur Bildung von $\mathfrak{A}^{-1}$ nötigen Operationen unabhängig ist von der Ordnung p der abgetrennten Matrix $\mathfrak{P}$, und zwar gleich

$$n^3 \text{ Multiplikationen und } n^3 - 2n^2 + 2n \text{ Additionen} \quad . \quad . \quad \text{(II, 2)}$$

Man nimmt daher am einfachsten $p = 1$, also $\mathfrak{P}$ gleich einer Zahl (oder, was natürlich auf dasselbe hinauskommt, $\mathfrak{S}$ gleich einer Zahl) und berechnet dann $\mathfrak{A}^{-1}$, indem man $\mathfrak{S}$ sukzessive mit einer Zeile und einer Spalte rändert, bis $\mathfrak{S}$ gleich der Adjungierten von a_{11} in $\mathfrak{A}$ geworden ist.

Vergleicht man die Zahlen in (2) mit denen in (I, 7), so stellt man fest, dass erstere für grosse n etwa dreimal so gross und für $n \leqq 10$ etwa doppelt so gross sind wie letztere. Es ist also nicht richtig, wenn SCHULZ (Lit. 10, p. 57) behauptet, dass die Berechnung von $\mathfrak{A}^{-1}$ „n-mal so viel Rechenarbeit" erfordere wie die Lösung des Gleichungssystems S_n, da erstere darauf hinauslaufe, dass man n Gleichungssysteme S_n aufzulösen hat. (Vgl. (I, 5).)

Man darf auch nicht in den Fehler verfallen zu glauben, wenn man mehr als drei Systeme mit derselben Matrix $\mathfrak{A}$ aufzulösen hat, die Benutzung der SCHURschen Identität vorteilhafter sei als die Methode von GAUSS. Denn wir werden unten sehen, dass bei Systemen mit derselben Matrix jedes neue Gleichungssystem nur $2n^2$ zusätzliche Multiplikationen und $2n^2$ zusätzliche Additionen benötigt, während die Dreiecksmatrix $\mathfrak{D}_l$ unverändert bleibt. Erst, wenn die Zahl k der Gleichungssysteme grösser als $n/3$ geworden ist, wird die Benutzung der Relation von SCHUR vorteilhafter.

Ergebnis. 1. Die Berechnung von $\mathfrak{A}^{-1}$ nach SCHUR erfordert n^3 Multiplikationen und $n^3 - 2n^2 + 2n$ Additionen. Die Lösung des Gleichungssystems $\mathfrak{A}\,\mathfrak{x} = \mathfrak{r}$ kostet nochmals n^2 Multiplikationen und $n^2 - n$ Additionen.

2. Hat man k Gleichungssysteme mit derselben Matrix $\mathfrak{A}$ zu lösen, so ist das GAUSSsche Verfahren immer noch vorteilhafter als die Lösung mittels SCHUR, solange k etwa unterhalb $n/3$ liegt. Erst für $k > n/3$ ist es umgekehrt.

Mathematics. — *A matric representation of binary modular congruence groups of degree m.* (First communication.) By F. VAN DER BLIJ. (Communicated by Prof. J. G. VAN DER CORPUT.)

(Communicated at the meeting of September 27, 1947.)

In this publication we continue the study of the behaviour of general theta functions of degree m under substitutions of the modular group of degree m [1]).

In chapter 1 we mention without proofs some theorems concerning the behaviour of the theta functions. For the proofs we may refer to our thesis. The theta functions are functions of the complex elements of a symmetric matrix T. We replace this matrix by $(AT + B)(CT + D)^{-1}$, where $\mathbf{U} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is a modular matrix, that is to say the matrix $\mathbf{U}$ satisfies the equation $\mathbf{U}'\mathbf{I}\mathbf{U} = \mathbf{I}$ where $\mathbf{I} = \begin{pmatrix} N & E \\ -E & N \end{pmatrix}$.

In chapter 2 we normalize the general theta functions. Under certain assumptions about the parameters of the theta functions, we obtain a set of functions with the property that the function obtained from a function of the set by applying a modular substitution on T, can be represented as a linear aggregate of functions of this set.

In chapter 3 we deduce from the results of chapter 2 a matric representation of the group $G(\varepsilon_n)$ of modular substitutions $\mathbf{U} \equiv \mathbf{E} \pmod{\varepsilon_n}$. In this representation the unity element corresponds with all substitutions $\mathbf{U} \equiv \mathbf{E} \pmod{\nu\varepsilon_n}$. The representation of the quotient group $G(\varepsilon_n)/G(\nu\varepsilon_n)$ gives a matric representation of the binary modular congruence groups of degree m modulo ν, since this group is simply isomorphic with this quotient group. We restrict ourselves to odd moduli ν.

In chapter 4 we prove in a direct way, that the formulas of chapter 3 define a matric representation of the binary modular congruence group of degree m and to the modulus ν.

1. 1. *Definitions and notation.*

A matrix will be called integral, if its elements are rational integers. An integral matrix will be said to be divisible by a given integer, if all its elements are divisible by this integer. Two matrices A and B will be called congruent modulo an integer ν, if their difference is divisible by ν. The well-known notation $A \equiv B \pmod{\nu}$ will often be abbreviated to $A \equiv B(\nu)$.

The transposed matrix of a matrix A will be denoted by A'. The usual

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[1]) See F. VAN DER BLIJ, Theta functions of degree m. Thesis, Leiden 1947.

definition of the product of two matrices will be used consistently. If a and b are n-vectors (that is to say matrices of one column and n rows), the product $a'b \equiv b'a$ is a scalar and the product ab' (or ba') is a matrix of n rows and n columns.

The matrix which we obtain by writing the columns of a matrix B to the right of those of a matrix A is denoted by (AB). Analogous meaning have the symbols $\begin{pmatrix} A \\ B \end{pmatrix}$ and $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$.

A matrix A with n rows and m columns will also be denoted by $A = A^{(n,\,m)}$. The matrix $A = A^{(n,\,n)} = A^{(n)}$, in the special case $n = m$, will be called a square matrix of degree n. The value of the determinant of the square matrix A will be denoted by $|A|$.

The following letters will retain the same meaning hereafter:

n and m	positive integers.
N	a matrix the elements of which are all zero.
E	the unity matrix.
Q	an integral, symmetric, positive (definite) matrix of degree n.
Δ	the determinant $\|Q\|$ of Q.
T	a symmetric matrix of degree m with variable (complex) elements. The imaginary part of the matrix T will supposed to be positive.
Z	a matrix $Z^{(n,\,m)}$ of complex variable elements.
ν	a positive integer.
ν_0	$\nu_0 = \Delta\nu$.

An integral matrix A with n rows is called *special* (with respect to Q) if the matrix QA is divisible by Δ.

The letters

P, R, S, L, G, H, F	denote henceforth special matrices of n rows and m columns.
U and V	denote integral matrices with determinant unity.

The trace $\sigma(A)$ of a matrix A is the sum of its diagonal elements. It can be seen readily that $\sigma(A) = \sigma(A')$ and $\sigma(AB) = \sigma(BA)$.

For abbreviation we write

$$e[x] = e^{\pi i x} \text{ and } e\{X\} = e^{\frac{\pi i}{\Delta}\sigma(QX)}.$$

In this notation we have the following identities

$$e\{X+Y\} = e\{X\}.e\{Y\} \text{ and } e\{X'\} = e\{X\}.$$

If a summation must be extended over special matrices only, we put an asterisk after the sign of summation (Σ^*).

The column vector, the elements of which are the diagonal elements of a square matrix K is denoted by $\langle K \rangle'$. The column vector v is defined by $Qv = \Delta \langle Q \rangle'$.

1. 2. *Theta functions and the sums* $\varphi(P, R)$.

General theta functions of degree m were introduced by the following definition:

$$\theta_{GH}(Z|\,T;P,\nu)=$$

$$(1.01)\qquad \sum_{M\equiv P(\nu_0)} e\left\{\frac{(M-P)H'}{\nu_0}\right\} e\left\{\frac{(M+\tfrac{1}{2}G)T(M+\tfrac{1}{2}G)'}{\nu_0}\right\} e\{2Z(M+\tfrac{1}{2}G)'\}.$$

Here the summation must be extended over all integral matrices $M = M^{(n,m)}$, which are a multiple of ν_0 plus P. Under the assumptions made about T and Q the series converges absolutely and uniformly in Z in each finite domain, uniformly in T in those finite domains, the closure of which contains only points with matrices T with positive imaginary part.

The following relations are easily established:

$$(1.02)\qquad \theta_{G+2L,H}(Z|\,T;P,\nu)=\theta_{GH}(Z|\,T;P+L,\nu),$$

$$(1.03)\qquad \theta_{G,H+2L}(Z|\,T;P,\nu)=\theta_{GH}(Z|\,T;P,\nu).$$

Let σ denote a complete system of non-congruent special matrices $P^{(n,m)}$ mod ν_0. If P runs through a system σ, the corresponding theta functions are linearly independent.

Modular matrices of degree m were introduced by C. L. SIEGEL [2]). We consider the matrix

$$(1.04)\qquad \mathfrak{U}=\begin{pmatrix} A & B\\ C & D\end{pmatrix},$$

which is composed of four integral square matrices A, B, C and D of degree m. This matrix is termed modular if it satisfies the equation

$$(1.05)\qquad \mathfrak{U}'\,\mathfrak{I}\,\mathfrak{U}=\mathfrak{I} \text{ where } \mathfrak{I}=\begin{pmatrix} N & E\\ -E & N\end{pmatrix}.$$

We can prove by a straightforward calculation that

$$(1.06)\qquad \begin{aligned} &A'D-C'B=D'A-B'C=AD'-BC'=DA'-CB'=E,\\ &A'C-C'A=AB'-BA'=DC'-CD'=D'B-B'D=N.\end{aligned}$$

Hereafter we shall reserve the letters A, B, C and D for the elements of a modular matrix $\mathfrak{U}$.

The sums φ_{FGH} are multiple sums which are closely connected with a generalization of the ordinary Gaussian sums.

If C and $\overline{C}$ are non-singular, integral, square matrices of degree m and if γ is a positive integer, such that $C\overline{C}=\gamma E$, the sums φ are defined by

$$\varphi_{FGH}(P,R;\nu,\mathfrak{U})=\sum_{\substack{X\overline{C}'\bmod\gamma\nu_0\\ X\equiv P(\nu_0)}} e\left\{\frac{(X-P)H'}{\nu_0}\right\}$$

$$(1.07)\qquad e\left\{\frac{(X+\tfrac{1}{2}G)A\overline{C}(X+\tfrac{1}{2}G)'-2(R+\tfrac{1}{2}F)\overline{C}(X+\tfrac{1}{2}G)'+(R+\tfrac{1}{2}F)\overline{C}D(R+\tfrac{1}{2}F)'}{\gamma\nu_0}\right\}.$$

[2]) C. L. SIEGEL, Ueber die analytische Theorie der quadratischen Formen. Ann. of Math. (2) 36, 527—605 (1935).

Here the parameters are supposed to satisfy the relation

$$(1.08) \qquad Q(F + GA + HC + \nu v \langle A'C \rangle) \equiv 0 \pmod{2\triangle}.$$

The matrices X run through a system of integral matrices, all congruent P mod ν_0, such that the matrices $X\overline{C}'$ run through a system of non-congruent matrices mod $\gamma\nu_0$.

1. 3. *The behaviour of theta functions under the modular group.*

If F is a function of Z and T, we define the left-hand operator $\mathbf{U}$ by

$$(1.09) \qquad \mathbf{U}F(Z|T) = F(Z(CT+D)^{-1} | (AT+B)(CT+D)^{-1}).$$

If $\mathbf{U}$ *is a modular matrix with non-singular matrix* C, *it can be proved that*

$$(1.10) \qquad \mathbf{U}\theta_{GH}(Z|T;P,\nu) = W \sum_{S \bmod \nu_0}^{*} \varphi_{G_1 GH}(P,S;\nu,\mathbf{U})\,\theta_{G_1 H_1}(Z|T;S,\nu).$$

Here the symbols G_1 and H_1 have the following meaning

$$(1.11) \qquad \begin{aligned} G_1 &= GA + HC + \nu v \langle A'C \rangle, \\ H_1 &= GB + HD + \nu v \langle B'D \rangle. \end{aligned}$$

If $\lambda_1, \lambda_2, \ldots, \lambda_m$ are the roots of the equation $|\Re(T) - \lambda J(T)| = 0$ we define

$$\sqrt{|-iT|} = \sqrt{|J(T)|} \prod_{k=1}^{m} \sqrt{(1-i\lambda_k)} \text{ where } \Re(\sqrt{}) > 0.$$

With this definition of the square root we have

$$(1.12) \qquad W = \frac{\{\sqrt{|-i(T+C^{-1}D)|}\}^n}{\nu^{\frac{1}{2}nm}\,\triangle^{\frac{1}{2}m}}\, e\{\nu_0 Z(CT+D)^{-1}CZ'\}.$$

The formula (1.10) is a special case of a more general transformation formula of the theta functions under modular substitutions.

If $\mathbf{U} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ *is a modular matrix with a matrix* C *of rank* r, *the effect of the operator* $\mathbf{U}$ *is given by* (we suppose either $\nu \equiv 1$ or $v \equiv 0 \pmod 2$.):

$$(1.13) \qquad \mathbf{U}\theta_{GH}(Z|T;P,\nu) = W\sigma_\nu(\mathbf{U}) \sum_{XC \bmod \nu_0}^{*} \eta_{GH}(P,X;\nu,\mathbf{U})\,\theta_{G_1 H_1}(Z|T;R,\nu).$$

Here we write as an abbreviation $R = PA + XC - \frac{1}{2}\nu v \langle A'C \rangle$; and the sum over X runs over special matrices X if the matrix R is integral, otherwise the sum runs over special matrices $2X$ with $2XC \equiv \nu v \langle A'C \rangle \pmod 2$.

G_1 and H_1 are defined by (1.11);

$$(1.14) \qquad \eta_{GH} = e\left\{\frac{(P+\tfrac{1}{2}G)AB'(P+\tfrac{1}{2}G)' + 2(P+\tfrac{1}{2}G)BC'(X+\tfrac{1}{2}H)' + (X+\tfrac{1}{2}H)DC'(X+\tfrac{1}{2}H)'}{\nu_0}\right\}.$$

$$(1.15) \qquad \sigma_\nu(\mathbf{U}) = \gamma^{-n(m-r)} \sum_{\substack{Y \bmod \gamma \\ YC' \equiv 0 (\gamma)}} e\left\{\frac{\nu_0 YA'CY'}{\gamma^2}\right\}.$$

(the positive integer γ is a multiple of the discriminant of C, that is to say of the greatest common divisor of all minors of degree r from C.)

$$(1.16) \qquad W = \frac{\omega^n (CT+D)}{\nu^{\frac{1}{2}nr} \triangle^{\frac{1}{2}r}} e\{\nu_0 Z(CT+D)^{-1} CZ'\}.$$

In order to define $\omega(CT+D)$ we need the following lemma [3]:

There exist unimodular matrices U and V such that

$$UCV^{-1} = \begin{pmatrix} C_1 & N \\ N & N \end{pmatrix}; \quad U(CT+D)V' = \begin{pmatrix} C_1 T_1 + D_1 & * \\ N & E \end{pmatrix},$$

where $|C_1^{(r)}| \neq 0$ and where the asterisk must be replaced by a non-interesting matrix. Then we define

$$\omega(CT+D) = \sqrt{|-i(T_1 + C_1^{-1} D_1)|}.$$

2. 1. *Construction of a special set of theta functions.*

Now we determine a necessary and sufficient condition which must be satisfied by a modular matrix $\mathbf{U}$ in order that we obtain in (1. 13) a sum of one term only. Then every special matrix X must satisfy $XC \equiv N \pmod{\nu_0}$. Denoting the largest elementary divisor of Q by ε_n, we find the necessary and sufficient condition: $C \equiv N \pmod{\nu\varepsilon_n}$.

In order to deduce a representation of the binary modular congruence group we must determine a set of theta functions with variable P, such that the function obtained from a function of the set by the application of a modular substitution can be represented as a linear aggregate of functions of the set.

Thus we have to determine the matrices G and H such that for all modular matrices $\mathbf{U}$ the congruences

$$(2.01) \qquad \begin{aligned} G &\equiv GA + HC + \nu v \langle A'C \rangle \pmod 2, \\ H &\equiv GB + HD + \nu v \langle B'D \rangle \pmod 2 \end{aligned}$$

are satisfied. Hence, if we denote the elements of the vector v by v_i $(1 \leq i \leq n)$, we have to determine vectors g_i and h_i such that for $i = 1, \ldots n$:

$$(2.02) \qquad \begin{aligned} g_i' &\equiv g_i' A + h_i' C + \nu v_i \langle A'C \rangle \pmod 2, \\ h_i' &\equiv g_i' B + h_i' D + \nu v_i \langle B'D \rangle \pmod 2. \end{aligned}$$

If $m > 1$, this system of congruences, where $\mathbf{U} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ runs through all modular matrices of degree m, modulo 2, is incompatible unless $\nu v \equiv 0 \pmod 2$.

In order to prove this we use the following lemma:

If m is an integer > 1, an integral matrix A of degree m can be constructed, such that $|A| = \pm 1$ and $|A - E| = \pm 1$.

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[3]) l.c. [1]) lemma 5, pag. 32.

Construction:

First let m be even, $m = 2k$. The matrix

$$A = \begin{pmatrix} E^{(k)} & E^{(k)} \\ E^{(k)} & N^{(k)} \end{pmatrix}; \quad A-E = \begin{pmatrix} N^{(k)} & E^{(k)} \\ E^{(k)} & -E^{(k)} \end{pmatrix}$$

satisfies the conditions.

If $m = 2k+1$ the following matrix satisfies the conditions

$$A = \begin{pmatrix} E^{(k-1)} & N^{(k-1,3)} & E^{(k-1)} \\ N^{(3,k-1)} & X^{(3,3)} & N^{(3,k-1)} \\ E^{(k-1)} & N^{(k-1,3)} & N^{(k-1)} \end{pmatrix}; \quad A-E = \begin{pmatrix} N_{(k-1)} & N^{(k-1,3)} & E^{(k-1)} \\ N^{(3,k-1)} & Y^{(3,3)} & N^{(3,k-1)} \\ E^{(k-1)} & N^{(k-1,3)} & -E^{(k-1)} \end{pmatrix}$$

Here

$$X = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \text{ and } Y = \begin{pmatrix} 0 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix}.$$

Now we shall prove the system (2.02) to be incompatible, unless $\nu v \equiv 0 \pmod 2$. The proof consists of two parts.

I. We choose the integral matrix A such that $|A| = \pm 1$ and $|A-E| = \pm 1$. Further we take $B = C = N$ and $D' = A^{-1}$. These four matrices are the "elements" of a modular matrix, since $AD' - BC' = E$ and $AB' = BA'$, $DC' = CD'$. The first congruence of (2.02) can now be written

$$(2.03) \quad g_i'(A-E) + h_i' C + \nu v_i \langle A'C \rangle = g_i'(A-E) \equiv 0 \pmod 2.$$

Hence $g_i \equiv 0 \pmod 2$.

II. We choose the "elements" of the modular matrix by

$$A = B = D = E, \; C = N.$$

The second congruence of (2.02) can now be written

$$(2.04) \quad g_i' B + h_i'(D-E) + \nu v_i \langle B'D \rangle = g_i' + \nu v_i (1, 1, \ldots, 1) \equiv 0 \pmod 2.$$

Thus we have $g_i \equiv \nu v_i (1, 1, \ldots 1)' \pmod 2$.

From this it is quite obvious that the system (2.02) is incompatible unless $\nu v \equiv 0 \pmod 2$.

Henceforth we suppose $v \equiv 0 \pmod 2$ *and* $\nu \equiv 1 \pmod 2$. Since Δ must be an odd integer, the diagonal elements of Q must be even. *Thus the matrix* Q *must be of even degree.* (Q is a skew symmetric matrix modulo 2.)

2. 2. *The normalized functions* $X(P)$.

We consider the behaviour of certain functions $X(Z \mid T; P)$ under substitutions of the modular group of degree m, which belong to the principal

congruence group of "Stufe" ε_n, — ε_n is the largest elementary divisor of Q —,

$$\mathbf{U} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \equiv \begin{pmatrix} E & N \\ N & E \end{pmatrix} \pmod{\varepsilon_n}. \tag{2.05}$$

We define the symbol $\mathbf{U}$, apart from its meaning as a left-hand operator, as a right-hand operator on functions F of Z and T by

$$F(Z|T)\,\mathbf{U} = |CT+D|^{\frac{1}{2}n(\nu-1)} F(Z(CT+D)^{-1}|(AT+B)(CT+D)^{-1}) \tag{2.06}$$

Now we introduce the function

$$X(P) = X(Z|T;P) = \frac{\theta_{NN}(Z|T;P,\nu)}{\theta_{NN}^{\nu}(Z|T;N,1)}. \tag{2.07}$$

If we use the formula (1.13) we get

$$\begin{aligned} &\mathbf{U}\,\theta_{NN}(Z|T;P,\nu) = \\ &\quad = W\,\sigma_\nu(\mathbf{U}) \sum_{XC \bmod \nu_0}^{*} \eta_{NN}(P,X;\nu,\mathbf{U})\,\theta_{NN}(Z|T;PA+XC,\nu). \end{aligned} \tag{2.08}$$

And in the special case $\nu = 1$, $P = N$ we have

$$\begin{aligned} &\mathbf{U}\,\theta_{NN}(Z|T;N,1) = \\ &\quad = \frac{\omega^n(CT+D)}{\triangle^{\frac{1}{2}r}}\, e\{\triangle Z(CT+D)^{-1}CZ'\}\,\sigma_1(\mathbf{U})\,\theta_{NN}(Z|T;N,1). \end{aligned} \tag{2.09}$$

We obtain from (2.07), (2.08) and (2.09)

$$X(P)\,\mathbf{U} = \sum_{R \bmod \nu_0}^{*} A(P,R)\,X(R), \tag{2.10}$$

where

$$\begin{aligned} A(P,R) = \nu^{-\frac{1}{2}nr}\,\triangle^{\frac{1}{2}r(\nu-1)}\,i^{\frac{1}{2}nr(\nu-1)}\,(disr\,C)^{\frac{1}{2}n(\nu-1)}\,\sigma_\nu(\mathbf{U})\,\sigma_1^{-\nu}(\mathbf{U}) \\ e\left\{\frac{PAB'P'+2PBC'X'+XDC'X'}{\nu_0}\right\}. \end{aligned} \tag{2,11}$$

if there exists a special matrix X with $R \equiv PA + XC \pmod{\nu_0}$.

$A(P,R) = 0$ otherwise.

Here *disr* C denote the product of the non-vanishing elementary divisors of C.

Now we calculate the value of $\sigma_\nu(\mathbf{U})\,\sigma_1^{-\nu}(\mathbf{U})$ under the assumption that $\mathbf{U}$ belongs to the principal congruence group of "Stufe" ε_n.

First we write the sum $\sigma_\nu(\mathbf{U})$ in a somewhat more convenient form. It is possible to determine matrices V and W of determinant unity such that

$$W^{-1}CV' = \begin{pmatrix} C_1 & N \\ N & N \end{pmatrix} \quad ; \quad W'AV' = \begin{pmatrix} A_1 & N \\ N & E \end{pmatrix},$$

where $|C_1| = |C_1^{(r)}| \neq 0$.

After an obvious modefication of the matrix of summation in the definition of $\sigma_\nu(\mathbf{U})$ we get

$$\sigma_\nu(\mathbf{U}) = \sum_{\substack{Y \bmod \gamma \\ YC_1' \equiv 0(\gamma)}} e\left\{\frac{\nu_0\,YC_1'A_1\,Y'}{\gamma^2}\right\}. \tag{2.12}$$

It is possible to determine a matrix A_1^* with $(|A_1^*|, \gamma) = 1$ such that for every integral matrix Y satisfying $Y C_1' \equiv 0 (\gamma)$ we have

$$Y C_1' A_1 Y' \equiv Y C_1' A_1^* Y' \pmod{\gamma^2}.$$

Let p be a prime, $p \mid \gamma$. If $C_1 = N \pmod p$ it follows $|A_1| \not\equiv 0 \pmod p$ and we can determine an integral matrix K with $(|A_1 + KC_1|, p) = 1$. If C_1 has rank r_p ($r_p > 0$) modulo p there exist p-adic unimodular matrices V and W such that

$$W^{-1} C_1 V' = \begin{pmatrix} C_2 & N \\ N & N \end{pmatrix} \quad ; \quad W' A_1 V' = \begin{pmatrix} A_2 & N \\ N & E \end{pmatrix}$$

and $|C_2^{(r_p)}| \not\equiv 0 \pmod p$. Now it may be seen readily that there can be found an integral matrix K with $(|A_1 + KC_1|, p) = 1$.

The sum $\sigma_\nu (\mathbf{U})$ may then be written as

$$\sigma_\nu (\mathbf{U}) = \sum_{\substack{Y \bmod \gamma \\ Y C_1' A_1^* \equiv 0 (\gamma)}} e\left\{\frac{\nu_0 Y C_1' A_1^* Y'}{\gamma^2}\right\}. \tag{2.13}$$

We suppose now γ to be equal to disr C.

In order to calculate $\sigma_\nu \sigma_1^{-\nu}$ we suppose $\gamma = 2^x \gamma_0$, $(\gamma_0, 2) = 1$. Then we have

$$\sigma_\nu = \sigma_\nu^* \, , \, \sigma_\nu^\circ$$

where

$$\sigma_\nu^* = \sum_{\substack{X \bmod 2^x \\ X C_1' A_1^* \equiv 0 (2^x)}} e\left\{\frac{\nu_0 X C_1' A_1^* X'}{2^{2x}}\right\},$$

$$\sigma_\nu^\circ = \sum_{\substack{X \bmod \gamma_0 \\ X C_1' A_1^* \equiv 0 (\gamma_0)}} e\left\{\frac{\nu_0 X C_1' A_1^* X'}{\gamma_0^2}\right\}.$$

Now there exist γ_0-adic unimodular matrices U and V such that $U'QU$ is a diagonal matrix, the elements of which are $\alpha_i \varepsilon_i$, $(\alpha_i, \gamma_0) = 1$, $\varepsilon_i \mid \gamma_0$; and $V C_1' A_1^* V'$ is a diagonal matrix, the elements of which are

$$\beta_j \gamma_j \quad , \quad (\beta_j, \gamma_0) = 1 \quad , \quad \gamma_j \mid \gamma_0 \quad , \quad (j = 1, 2, \ldots r).$$

This follows at once from a theory of MINKOWSKI [4]).

If we replace the matrix of summation X by UXV we get

$$\sigma_\nu^\circ = \prod_{j=1}^{r} \prod_{i=1}^{n} \sum_{z \bmod \gamma_j} e\left[\frac{\nu \alpha_i \beta_j \varepsilon_i z^2}{\gamma_j}\right].$$

We supposed $C \equiv N \pmod{\varepsilon_n}$ thus it may be seen readily that $\gamma_j \equiv 0$

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[4]) H. MINKOWSKI. Grundlagen für eine Theorie der quadratischen Formen mit ganzzahligen Koeffizienten. Gesam. Abh. I. Nr. 1, S. 18—25.

(mod ε_i). *Now we suppose* $(\Delta, \nu) = 1$ and introduce $(\nu, \gamma_j) = \delta_j$ and

$$\nu = \delta_j \nu_j \quad ; \quad \prod_{j=1}^{r} \delta_j = \delta \quad ; \quad \nu = \nu_+ \delta.$$

Then we have

$$\sigma_\nu^\circ = \prod_{j=1}^{r} \prod_{i=1}^{n} \varepsilon_i \delta_j \sum_{z \bmod \gamma_j/\varepsilon_i \delta_j} e\left[\frac{\nu_j \alpha_i \beta_j z^2}{\gamma_j \varepsilon_i^{-1} \delta_j^{-1}}\right].$$

Using the well-known formula for the ordinary Gaussian sums, we obtain

$$\sigma_\nu^\circ = \Delta^{\frac{1}{2}r} \delta^{\frac{1}{2}n} \gamma_0^{\frac{1}{2}n} \prod_{j=1}^{r} \prod_{i=1}^{n} \left(\frac{\alpha_i \beta_j \nu_j}{\gamma_j \varepsilon_i^{-1} \delta_j^{-1}}\right) e\left[-\tfrac{1}{4}(\gamma_j \varepsilon_i \delta_j - 1)\right],$$

where $(—)$ denotes the symbol of JACOBI from the theory of quadratic residues. We have

$$\left(\frac{\alpha_i \beta_j \nu_j}{\gamma_j \varepsilon_i^{-1} \delta_j^{-1}}\right) e\left[-\tfrac{1}{4}(\gamma_j \varepsilon_i \delta_j - 1)\right] =$$

$$= \left(\frac{\alpha_i \beta_j}{\gamma_j \varepsilon_i^{-1} \delta_j^{-1}}\right) \left(\frac{\gamma_j \varepsilon_i^{-1} \delta_j^{-1}}{\nu_j}\right) e\left[-\tfrac{1}{4}(\nu_j - 1)(\gamma_j \varepsilon_i \delta_j - 1) - \tfrac{1}{4}(\gamma_j \varepsilon_i \delta_j - 1)\right],$$

$$= \left(\frac{\alpha_i \beta_j}{\gamma_j \varepsilon_i^{-1} \delta_j^{-1}}\right) \left(\frac{\gamma_j \varepsilon_i^{-1} \delta_j^{-1}}{\nu_j}\right) e\left[-\tfrac{1}{4}\nu_j (\gamma_j \varepsilon_i \delta_j - 1)\right].$$

If $\nu = 1$ we deduce

$$\sigma_1^\circ = \Delta^{\frac{1}{2}r} \gamma_0^{\frac{1}{2}n} \prod_{j=1}^{r} \prod_{i=1}^{n} \left(\frac{\alpha_i \beta_j}{\gamma_j \varepsilon_i^{-1}}\right) e\left[-\tfrac{1}{4}(\gamma_j \varepsilon_i - 1)\right].$$

We thus find, since n is even

$$(2.14) \qquad \sigma_\nu^\circ (\sigma_1^\circ)^{-\nu} = \Delta^{\frac{1}{2}r(1-\nu)} \gamma_0^{\frac{1}{2}n(1-\nu)} \delta^{\frac{1}{2}n} \left(\frac{\Delta}{\nu_+}\right) e\left[-\tfrac{1}{4} n \sum_j (\nu - \nu_j)\right].$$

Hereafter we calculate $\sigma_\nu^\bullet$.

It can be deduced from the theory of MINKOWSKI [4]) that there exists a 2-adic unimodular matrix V such that

$$V' C_1' A_1^* V = \begin{bmatrix} 2^{\varkappa_1} F_1 & N & \cdots & \cdots & \cdots \\ N & 2^{\varkappa_2} F_2 & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & 2^{\lambda_1} f_1 & 0 \cdots \\ \cdots & \cdots & \cdots & 0 & 2^{\lambda_2} f_2 \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}; \quad F_i = \begin{pmatrix} 2a_i & b_i \\ b_i & 2c_i \end{pmatrix},$$

where a_i, b_i and f_i are odd integers.

Replacing the matrix X in the definition formula of $\sigma_\nu^\bullet$ by XV' we get

$$\sigma_\nu^\bullet = \prod_i \sum_{x, y \bmod 2^{\varkappa_i}} e\left[\frac{2\nu(a_i x' Qx + b_i x' Qy + c_i y' Qy)}{2^{\varkappa_i}}\right]$$

$$\prod_j \sum_{z \bmod 2^{\lambda_j}} e\left[\frac{\nu f_j z' Qz}{2^{\lambda_j}}\right].$$

We consider the sum over x if y is fixed. If one of the elements of y is odd, it may be seen readily — by replacing x by $x + 2^{\varkappa_i - 1} k$, with an arbitrary integral vector k — that the sum over x vanishes. Thus we may replace y by $2 a_i y$ and if we suppose $\varkappa_i > 1$ we have, putting $d_i = a_i c_i - b_i^2$,

$$\sigma_\nu^* = \prod_i \sum_{y \bmod 2^{\varkappa_i - 1}} e\left[\frac{\nu a_i d_i y' Q y}{2^{\varkappa_i - 1}}\right] \sum_{x \bmod 2^{\varkappa_i}} e\left[\frac{2\nu a_i x' Q x}{2^{\varkappa_i}}\right] \prod_j \sum_{z \bmod 2^{\lambda_j}} e\left[\frac{\nu f_j z' Q z}{2^{\lambda_j}}\right].$$

Now we use the following formula [5])

$$\sum_{y \bmod 2^\lambda} e\left[\frac{a y' Q y}{2^\lambda}\right] = \left(\frac{2^\lambda}{\Delta}\right) 2^{\frac{1}{2}\lambda n}.$$

We use $\varkappa = 2\sum_i \varkappa_i + \sum_j \lambda_j$ and thus we have

$$\sigma_\nu^* = \prod_i \left(\frac{2^{\varkappa_i - 1}}{\Delta}\right) 2^{\frac{1}{2}(\varkappa_i - 1)n} \left(\frac{2^{\varkappa_i - 1}}{\Delta}\right) 2^{\frac{1}{2}(\varkappa_i + 1)n} \prod_j \left(\frac{2^{\lambda_j}}{\Delta}\right) 2^{\frac{1}{2}n\lambda_j},$$

$$\sigma_\nu^* = 2^{\frac{1}{2}n\varkappa} \left(\frac{2^\varkappa}{\Delta}\right).$$

It may be seen readily that this formula remains true if there exists an index i with $\varkappa_i \leq 1$. We get also

$$\sigma_1^* = 2^{\frac{1}{2}n\varkappa} \left(\frac{2^\varkappa}{\Delta}\right).$$

At last we write

$$\sigma_\nu^* (\sigma_1^*)^{-\nu} = 2^{\frac{1}{2}n\varkappa(1-\nu)}. \tag{2.15}$$

Thus we have proved

$$\sigma_\nu \sigma_1^{-\nu} = \Delta^{\frac{1}{2}r(1-\nu)} \gamma^{\frac{1}{2}n(1-\nu)} \delta^{\frac{1}{2}n} \left(\frac{\Delta}{\nu_+}\right) e\left[-\tfrac{1}{4}n \sum_{j=1}^{r} (\nu - \nu_j)\right]. \tag{2.16}$$

We introduce these results in (2.11). If there exists a special matrix X with $R \equiv PA + XC \pmod{\nu_0}$ we have

$$A(P, R) = \nu^{-\frac{1}{2}nr} \delta^{\frac{1}{2}n} \left(\frac{\Delta}{\nu_+}\right) e\left[\tfrac{1}{4}n \sum_{j=1}^{r} (\nu_j - 1)\right] e\left\{\frac{P A B' P' + 2 P B C' X' + X D C' X'}{\nu_0}\right\}.$$

And

$$A(P, R) = 0$$

otherwise.

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[5]) See H. D. KLOOSTERMAN, The behaviour of general theta functions under the modular group..... (I). Ann. of Math. (2) **47**, 339 (1946).

Physics. — *Reflection of light by rippled water surfaces.* By J. S. VAN WIERINGEN. (Communication from the Laboratories of the N.V. KEMA, Arnhem.) (Communicated by Prof. M. MINNAERT.)

(Communicated at the meeting of September 27, 1947.)

In this article some results are given of an investigation concerning the reflection of light from the sun or artificial lightsources by the rippled surface of the sea, canals and rivers. The following calculations complete the observations described by Prof. MINNAERT [1] [4].

The reflected image of a light source in a rippled water surface is not a single image of the source, as in the case of a flat surface, but a juxtaposition of a number of these images. They are not normal images as seen in a plane mirror, but they are deformed because of the curvature of the surface.

We consider two simple cases.

1. If the ripples are totally irregular, a light pillar is seen, extending over an area which generally is elongated in a vertical direction. In each

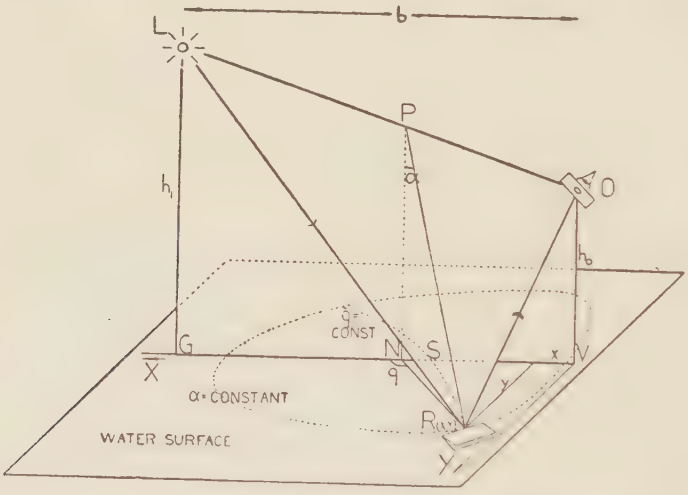


Fig. 1. The light of the point source L reaches the observer O via the points R of the water surface. The coordinates R are functions of the inclination α and the azimuth q of the surface element at R.

point R of the water surface there is a definite inclination α and an azimuth q for which light from the source L can be reflected towards the observer O (fig. 1). The observed reflection points are all lying within a closed contour, a limiting curve which will be shown to be a curve of

[1] MINNAERT, Natuurkunde van het Vrije Veld, I, p. 20.

the 6th degree. If the positions of the light source and of the observer are fixed, this curve is determined by the maximum angle of inclination α_m of the rippled surface. For the points *within* this contour, $|\alpha| < \alpha_m$.

2. If the waves are strictly parallel, al surface elements satisfy the condition $q = \text{const.}$; this penomenon occurs in canals and rivers where the waves by preference move in the direction of the canal and have their fronts perpendicular to the direction of propagation. The reflected images are now lying on a curve which generally is slanting and which will be shown to be of the 3d degree. This curve is entirely determined by the positions of the light source and the observer and by the direction of propagation of the waves. The maximum inclination α_m of the waves only determines whether the curve is illuminated over a greater or smaller extent. Of course the point of normal reflection S, where $\alpha = 0$, belongs to it.

Practically, in most cases an intermediate phenomenon is seen. The waves have all directions but with a preference for one of these. The reflection points are then lying within a closed curve which is neither the contour first mentioned nor the curve mentioned in the second case. Within this curve a number of images of the light source are seen. Just as in the other cases they are deformed because of the curvature of the surface, and they are constantly moving as a result of the propagation of the waves.

The light pillar observed on totally irregular waves.

As was shown above, the limiting curves are the curves $\alpha = \text{const.} = \alpha_m$. They can be calculated as follows (fig. 1).

If R (coordinates x and y, as in fig. 1) is a point which reflects light from the source L to the observer O, the line normal to the reflecting surface element in R should intersect the line LO at a point which we will call P. Besides, PR should divide $\angle LRO$ into two equal parts, or in other words:

$$PR = \sqrt{LR \cdot OR - LP \cdot OP} = \frac{PN}{\cos \alpha_m},$$

or

$$\sqrt{LR \cdot OR - \frac{LR \cdot OR}{(LR+OR)^2} \cdot LO^2} = h_o + (h_l - h_o)\frac{OR}{LR+OR} = \frac{h_o \cdot LR + h_l \cdot OR}{LR+OR}$$

Expressing LR and OR in terms of x, y, b, h_l and h_0, h_l and h_0 being the heights of L and O above the water surface and b their horizontal distance, the equation for (x, y), when $\alpha = \alpha_m$ results in:

$$\left.\begin{aligned} \cos\alpha_m \sqrt{\sqrt{(x^2+y^2+h_0^2)}\{(b-x)^2+y^2+h_l^2\}}\,[\{\sqrt{x^2+y^2+h_0^2}+\sqrt{(b-x)^2+y^2+h_l^2}\}^2-b^2-h_l-h_0)^2] = \\ = h_0\sqrt{(b-x)^2+y^2+h_l^2}+h_l\sqrt{x^2+y^2+h_0^2} \end{aligned}\right\} (1)$$

It is somewhat simplified if the source is at an infinite distance, as in the case of the sun. Then the direction of all incident rays is the same. The curve in this case is found by taking the limit of (1 for $b = \infty$. Then

$$\frac{\sqrt{(b-x)^2+y^2+h_l^2}}{b} = \frac{1}{\sin i} \quad (i = \text{angle of incidence}) \text{ and}$$

(1 is simplified to:

$$\cos \alpha_m \sqrt{2(x^2+y^2+h_0^2) - 2\sqrt{x^2+y^2+h_0^2}(x \cdot \sin i - h_0 \cdot \cos i)} = \left. \begin{array}{r} \\ = h_0 + \cos i \sqrt{x^2+y^2+h_0^2} \end{array} \right) \quad . \quad . \quad (2)$$

This is the limiting curve, *drawn in the plane of the reflecting water surface.*

The limiting curves are of the 6th degree in the general case of a source at a finite distance. The same result was found by PICCARD [2]) for the more special case that the heights of observer O and light source L to the water surface are equal ($h_0 = h_l$). In (1 and (2 the complete 6th and 4th degree curves include negative as well as positive roots. The negative roots describe the case of PR being external bissectrix of $\angle LRO$ and consequently that part of the complete curve has no physical meaning; it will not be considered further.

The equations (1 and (2 are too complicated for computation. It is much easier to find the curves experimentally. A lamp is placed above a table at L. A little mirror makes an angle α_m with the table and can be moved over it. By looking through a fixed hole at O we can seen the points where reflection occurs (fig. 1).

The curves found by this method prove to be well approximated by an ellipse, as was already found by PICCARD [2]). The observer looks at this pillar under an angle i, and sees the projection on a plane perpendicular to his direction of sight OS. The ellipse is therefore foreshortened.

We now consider the shape of the limiting curve projected on the plane perpendicular to OS. For general orientation it is sufficient to consider the case of a source at an infinite distance, the shape of the curve being mainly determined by i, the angle of incidence for normal reflection. The length of the short (horizontal) axis of the ellipse is repersented as an angle γ in fig. 2: $tg \frac{1}{2}\gamma = \cos i \,.\, tg\, 2\alpha$. The ratio $\frac{\gamma}{\alpha}$ of the short over the long axis is $\cos i$, if $90 - i > 2\alpha_m$. If $90 - i \leqq 2\alpha_m$ the pillar reaches the horizon and so the long axis is partly cut off.

Then the ratio is

$$\frac{\gamma}{a'} = \frac{2\ tg\ 2\alpha}{\frac{tg\ 2\alpha}{\cos i} + \frac{1}{\sin i}}. \qquad \text{(fig. 3).}$$

[2]) PICCARD, Arch. sc. phys. et nat. 21, 481 (1889).

It is remarkable that in general the ratio $\frac{\gamma}{\alpha}$, that is to say the shape of the ellipse is independent of the maximum inclination α_m of the waves. However, if the ellipse is cut off by the horizon, α_m comes into account. By measuring the arc of the pillar at the horizon the maximum inclination α_m of the waves can be easily determined [3]).

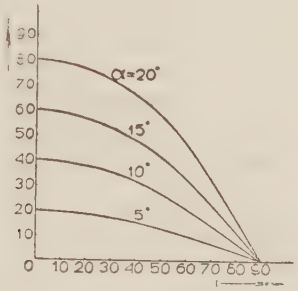


Fig. 2. The angle γ under which the horizontal axis of the pillar is seen for different angles of incidence i. Parameter: the maximum inclination α of the water surface. The light source is at an infinite distance.

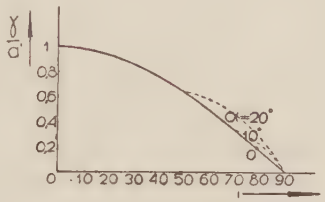


Fig. 3. The ratio of the angles under which the horizontal and the vertical axis of the pillar are seen as a function of the angle of incidence i (infinitely distant source). The dashed lines are corrections for the cutting-off effect of the horizon.

Properly speaking the cutting off does already occur before the reflected ray reaches the horizon. When grazing over the water surface, it could be withdrawn from the observer's eye by a wave top (fig. 4). This is the case if $90 - i \leqq 2\alpha + \beta$. Owing to the smallness of β ($tg\ \beta \approx \frac{2\ tg\ \alpha_m}{3\ \pi} \approx {}^1/_5\ tg\ \alpha_m$ for sine waves, so $\beta \approx 4°$), the correction due to this effect is not large.

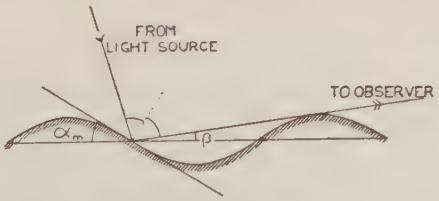


Fig. 4. Part of the pillar may be cut off by wave tops.

[3]) SPOONER, Corresp. du Baron de Zach, p. 331 (1822).

From figs. 2 and 3 it follows that in a reflected landscape the pillars are al about equally high, but that these of low objects are narrower than those of high objects.

The luminous line seen on plane waves.

The symbols h_o, h_l and b are the same as in the preceding paragraph and in fig. 1; the azimuth q is the angle between the direction of propagation of the waves and the vertical plane through L and O (fig. 1). The points $R\ (x, y)$ where reflection occurs are lying on a curve with $q =$ constant. Of course the point S of normal reflection is one of them ($\alpha = 0$). The terminal points of the curve are lying on the ellipse of the previous case. In the other points of the curve the azimuth q is the same, but $|\alpha| < \alpha_m$.

The calculation runs as follows (fig. 1):
As RP is the bissectrix of $\angle LRO$,

$$\frac{LR}{RO} = \frac{LP}{PO} = \frac{GN}{NV} \text{ or } LR \,.\, NV = RO \,.\, GN.$$

By expressing these in terms of x, y and q, the result obtained is:

$$(x \sin q - y \cos q) \sqrt{(b-x)^2 + y^2 + h_l^2} = \{(b-x) \sin q + y \cos q\} \sqrt{x^2 + y^2 + h_0^2} \quad . \quad (3)$$

This is the curve *drawn in the plane of the water surface.*

It is a curve of the third degree, for, taking the squares, the terms of the fourth degree disappear.

Again the equation becomes much simpler if we consider a light source at an infinite distance such as the sun. By taking the limit of (3 for $b = \infty$ and substituting $\frac{h_l - h_o}{b} = \operatorname{cotg} i$

it is found that:

$$\sin i \ \sin q \sqrt{x^2 + y^2 + h_0^2} = x \sin q - y \cos q \quad . \quad . \quad . \quad . \quad (4)$$

If the coordinate system is rotated in its plane over the angle q, the equation expressed in the new coordinates ξ and η is:

$$\eta = -\sin i \sin q \sqrt{\xi^2 + \eta^2 + h_0^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

So the images of a source at an infinite distance over a plane-wave surface are lying on one branch of a hyperbola (the root in (5 being positive only). They are lying on a straight line (the x-axis) if $q = 0$, that is to say if the waves are running in the direction of the observer; and also on a straight line (the ξ-axis) if $i = 0°$, when the sun is over the head of the observer. If at the same time $i = 90°$ and $q = 90°$ the curve consists of two imaginary lines of which only the intersection is observed, being the infinitely distant point of the x-axis.

The curves were computed for a light source at an infinite distance and for $i = 30°$ (fig. 5). From this figure it is seen that a gradual increase in

the azimuth q of the waves has only a small influence on the direction of the pillar as long as the observer is looking more or less perpendicularly to the waves. If the direction of observation exceeds 45°, the influence rapidly increases and becomes very strong near $q = 90°$: This effect is the more striking the greater the angle of incidence i.

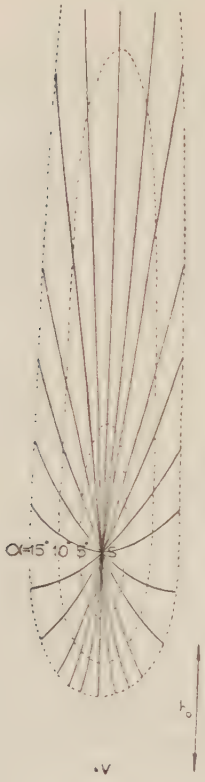


Fig. 5. Dashed: computed limiting curves of the pillar seen on an irregularly rippled surface with maximum inclination 5°, 10°, and 15°. Solid: the luminous lines seen on plane waves for different directions of propagation ($q = 0°$, 15°, 30°, 45°, 60°, 75° and 90°) for the case $i = 60°$. These lines are drawn in the plane of the reflecting water surface. The observer is at a point O directly over V at a height h_0. The infinitely distant soure L is at a height of 30° above the horizon ($i = 60°$).

However the computation is rather tedious and an experimental measurement is much simpler. It is easy to investigate the influence of the azimuth q by placing on the table a sheet of parallel-rippled glass, which is rotated in its plane. Here also the difference between a light source at a finite or an infinite distance proves to be small. In some cases with a source at at finite distance it is possible to see the inflexion point of the third degree curve.

In the reflections by rippled water surfaces it is possible to discover a curved pillar under very favorable conditions. However in most cases the pillar is practically straight, because the waves are not running exactly in one direction and the illuminated curve is more or less blurred. Therefore it is ordinarily sufficient, to know the mean inclination of the pillar. The angle p' in the surface between the curve $q =$ constant and the x-axis follows from $tg\, p' = tg\, q \cdot \cos^2 i$, as can be easily calculated from (3 or (4. The observer sees this angle projected on a plane perpendicular to his

direction of sight. This projection p of p' is found from $tg\ p = tg\ q.\ \cos i$. [4]) This formula describes completely the above mentioned effect: if q, the direction of the waves, is changed at an uniform rate, the inclination p of the pillar changes first at a slow, then at an increasingly quick rate; the effect being the more pronounced the larger i.

Summary.

It is shown that the reflected image of a light source in a rippled water surface is a light pillar which generally is limited by a curve of the 6th degree. If the source is at an infinite distance, the curve is of the 4th degree. The shape of the limiting curve is discussed. In parallel ripples the light pillar is a single curve which is generally slanting. It is a curve of the 3d degree, in the special case of an infinitely distant source of the 2d degree (cf. the solid lines in fig. 5). The influence of the direction of the ripples is discussed.

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[4]) MINNAERT, Physica **9**, 925 (1942).

Physical Geography. — *Theory on central rectilinear recession of slopes.* I. By J. P. BAKKER and J. W. N. LE HEUX. (Communicated by Prof. F. A. VENING MEINESZ.)

(Communicated at the meeting of May 31, 1947.)

Introduction.

After the early publications of OSMOND FISHER (1) [1]) and LAWSON (2), in which a brief mathematical derivation was used, the theories of OTTO LEHMANN (3) and ED. GERBER (4) were the first, in which a problem of physionomic geomorphology was more thoroughly treated from a quantatively exact point of view.

LEHMANN started from parallel recession of steep mountain slopes and constant height. GERBER accepted the former condition, but added a new one, assuming that the fault scarp due to crustal movements, or the valley-slope owing to vertical erosion increased in height during parallel recession. For various parts of the earth, especially in regions where permeable resistant sand- and limestones occur (the Dolomites in the Alps, many cuesta- and mesa-landscapes) there are several reasons for assuming parallel retreat of slopes as a real condition. (See fig. 4 and 5.)

Nevertheless we may note that as early as sixty years ago geomorphologists based deductions on the assumption of the recession of steep slopes with decreasing slope angle, the simplest case of which we find reproduced in PHILIPPSON's figure (5, II, 2, p. 63) for the development of denudation landscapes (our fig. 1). This figure, therefore, indicates that the intensity

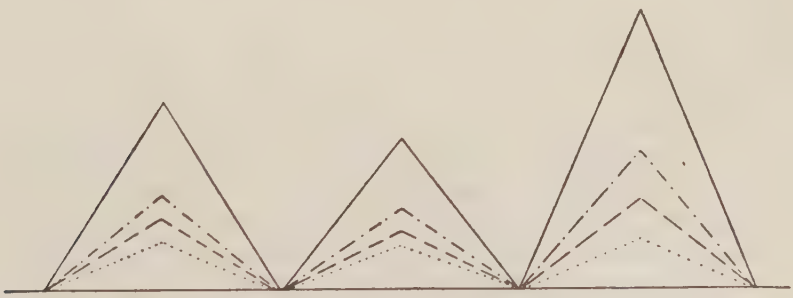


Fig. 1. (After PHILIPPSON). See text.

of weathering-removal increases rectiliniarly with the height of the wall, equalling zero at the basic point of the slope. We shall call this type of wall recession "*central rectilinear recession*". The basic point of the steep slope functions in this case as recession centre.

[1]) The numbers in parenthesis refer to the list of literature at the end of this article.

In the crests and peak regions of many high mountains, the rectilinearity of cross-profiles and the frequency of slope angles seem to indicate that straight-lined recession of steep slopes with decreasing slope angle exist in Nature. Again, the premiss of the stable or nearly stable position of the basic point is fairly frequently assumed in geomorphology. According to this two premisses the problem of central straight-lined retreat of slopes can be dealt with mathematically.

Some years ago BRYAN and others (6) pointed out many inadmissable generalisations and groundless deductions in several opinions on slope development. If such errors are to be avoided and if, within the given conditions, the resulting possibilities are to appear more clearly than in the generally too simple treatment of this type of problem, a mathematical basis is indispensable.

Our theory may be presented in various ways. The method given here is fairly simple and resembles LEHMANN's theory. For a more universal treatment of a theory of non-parallel slope recession a different method is preferable [2]). We hope to return to this subject on some future occasion, in cooperation with Dr. E. M. BRUINS, lecturer for mathematics at the University of Amsterdam.

Not only does our theory of central straight-lined recession of slopes offer some point of comparison with LEHMANN's theory, but it also is a step in the direction of the problem of convexity of mountain tops and drainage divides.

If we inspect the higher logarithmic curves of LEHMANN's theory, it is clear, that they have a different shape from the characteristic rounded summits of the European "Mittelgebirge". In LEHMANN's theory the strongest curvature is found near the basic point of the new mountain form underneath the screes; in our "Mittelgebirge", however, the rounding is found near the summits. It is evident that in most cases the problem of convexity of summits is not a question of parallel recession of slopes. Classic geomorphologists like GILBERT, DAVIS, ALBRECHT PENCK and PHILIPPSON already assumed from field observations that sharp crests and peaks with approximately rectilinear profiles can change into convex mountain forms, with nearly the same basic point. This conception was fairly generally accepted. The idea is found in GÖTZINGER (7, p. 113) and up to this day in HOWARD's publication on the Sacaton Mountains in Arizona (8, p. 100, fig. 20 B) [3]).

More interest is again being devoted to the whole question of the development of mountain slopes, as BLACHE's (9) very interesting but not mathematical treatment shows.

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[2]) This different method can also be applied in the treatment of rectilinear recession with unstable basic point (inselbergs etc.).

[3]) We are very grateful to Professor EARL B. SHAW, Worcester, Mass. and the Smithsonian Institute, Washington, for sending us American literature.

As an introduction to a theory on convexity of mountain tops, three points must first be investigated:

1. The influence of the central straight-lined recession on the shape of the curves.
2. The variations which the slope angles of mountains and those of the screes and alluvial cones can show in regions of different climatological and orographic types.
3. The limits within which the theories of parallel recession (LEHMANN) and central rectilinear retreat can be applied, without coming into conflict with their premisses.

These three points will be discussed in our theory.

Theory.

With the aid of fig. 2 we are able to state the following conditions. A steep slope *FS* of a known height *h* in a homogeneous rock is bordered at the top by a horizontal plateau *SR* and at the foot by an almost horizontal form *FR'*, on which falling debris may accumulate. The wall *FS*

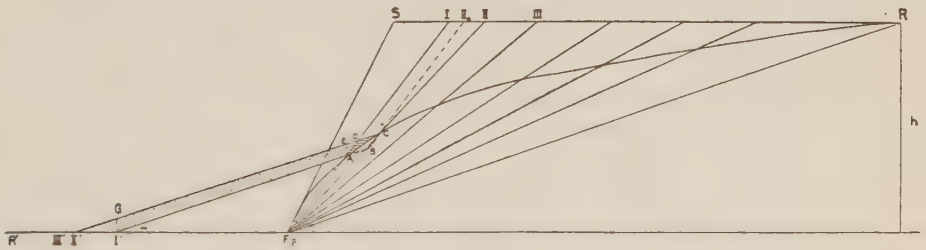


Fig. 2. Central rectilinear recession of the slope *FS* and the simultaneous formation of screes [α has been drawn slightly too large, as otherwise the diagram would become too wide].

is exposed exclusively to the free play of weathering-removal, without any direct effect of lateral erosion by rivers, glaciers etc. It is further assumed, that the screes of debris (slope angle α) are not deposited on the terrace *FR'* suddenly by rockfall and landslide, but little by little, and that the rock underneath the screes remains unchanged. From the steep rockwall, per unit of time, a part *SFAI, IACII* etc. is removed, while in the same period a screes volume *I'FA, II'I'AC* etc. is deposited at the foot. We further imagine the wall to be perpendicular to the plane of drawing, so that two-dimensional figures may be used. This enables us to substitute in our exposition parallelograms and triangles for the volumes of the corresponding quantities of solid rock and debris. In figure 2 we draw CII_0 parallel to *FI*. We assume that the stretch of rock removed per unit of time from the steep mountain slope (the sum of parallelogram $IABII_0$ and triangle II_0CII) and the corresponding stretch of debris ($I'II'CA$) are infinitely thin. At the same time, in order to obtain an exact derivation, we

have to assume a definite ratio between rock volume and screes volume.

$$\frac{\text{rock volume}}{\text{screes volume}} = \frac{1-c}{1}$$

in which c is constant.

Our starting points thus being fixed, we get a differential equation with the aid of fig. 3 in which we choose F as zeropoint of our co-ordination system.

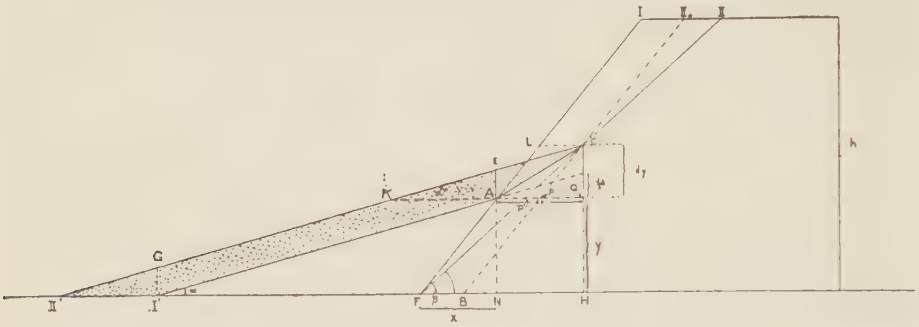


Fig. 3. Diagram from which our differentialequation may be obtained. AC represents an infinitely small increase in the profile of the convex nucleus.

Anticipating the result of our derivations, it will appear that underneath the screes we shall find a nucleus with convex profile of which AC in fig. 3 is an infinitely small part.

$$(1-c) \times II'I'AK = LCII_0I + II_0CII$$

$$II'I'AK = GI'AE = ay \times \left(dy - \frac{dx}{a}\right)$$

for cot $\alpha = a$; cot $\beta = b$

$$(1-c)\ y\ (a\ dy - dx) = (dx - b\ dy)\ (h-y) + II_0CII$$

$$II_0CII : FCB = (h-y)^2 : y^2$$

$$II_0CII : \tfrac{1}{2}\,(dx - b\,dy)\ y = (h-y)^2 : y^2$$

$$II_0CII = \frac{(h-y)^2}{2y}\,(dx - b\,dy)$$

$$(1-c)\,y\,(a\,dy - dx) = (dx - b\,dy)\,(h-y) + \frac{(h-y)^2}{2y}(dx - b\,dy) \quad . \quad . \quad (9)\ ^4)$$

$$(1-c)\,y\,(a\,dy - dx) = (dx - b\,dy)\,(h-y)\left\{1 + \frac{(h-y)}{2y}\right\}$$

Now putting cot $\beta = b = \dfrac{x}{y}$

$$(1-c)\,y\left(a - \frac{dx}{dy}\right) = \left(\frac{dx}{dy} - \frac{x}{y}\right)\left(\frac{h^2-y^2}{2y}\right)$$

$$\frac{dx}{dy}\left\{\frac{h^2+(1-2c)\,y^2}{2y}\right\} = \frac{h^2-y^2}{2y^2}\,x + ay\,(1-c). \quad . \quad . \quad (10)$$

---

4) For the equations 1—8, see literature (10).

When $y = h$, we have, according to the equation (10)

$$\frac{dx}{dy} = a. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

This is the tangent of the maximum slope-angle of the screes in Nature.

Formula (11) is an analogue to LEHMANN's formula on page 93 of his theory [our formula (8), 10, p. 538].

When the steeper part of the mountain has completely disappeared and the screes have consequently reached the level of the top-plateau, their rectilinear cross profile forms the tangent on our curve (formula 14) at the intersecting point with the plateau. (See fig. 7 and formula (11a) in the second part of our theory.)

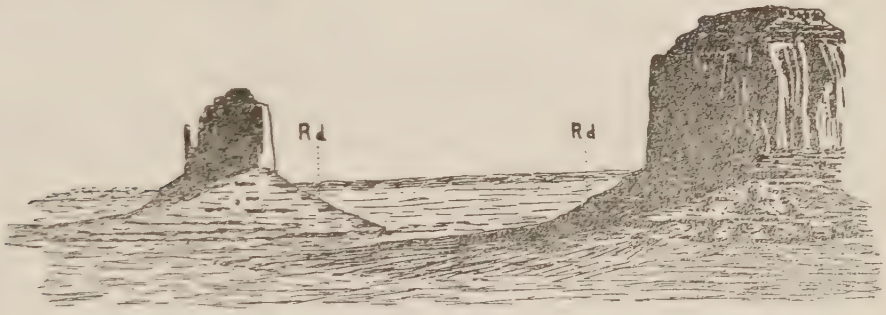


Fig. 4. Outliers in the Monument Valley, Utah (after VON ENGELN).
Rd = RICHTER's denudationslope.

It follows from formula (10), that

$$\frac{dx}{dy} - \frac{h^2 - y^2}{[h^2 + (1-2c)\,y^2]\,y}\,x = \frac{2a\,(1-c)\,y^2}{h^2 + (1-2c)\,y^2} \quad . \quad . \quad . \quad (12)$$

a common linear (first degree) differential equation

$$\left[\frac{dx}{dy} + P(y)\,x = Q(y)\right]$$

The solution of this equation is well known

$$\left[x = e^{-\int P(y)\,dy}\left\{\int Q(y)\,e^{\int P(y)\,dy}\,dy + C\right\}\right]$$

Substituting the functions $P(y)$, $Q(y)$ and C we get [5])

$$x = ay - (a-b)\,y\left[\frac{h^2 + (1-2c)\,y^2}{h^2}\right]^{\frac{c-1}{1-2c}} \quad . \quad . \quad . \quad . \quad (14)$$

---

[5]) Indeed we have

$$\int P(y)\,dy = -\int \frac{h^2 - y^2}{[h^2 + (1-2c)\,y^2]\,y}\,dy = -\frac{1}{2}\int\left[\frac{1}{y^2} + \frac{2(c-1)}{h^2 + (1-2c)\,y^2}\right]dy^2 = \left.\begin{array}{l} \\ \\ \end{array}\right\} \quad (13a)$$

$$-{}^e\log y - \frac{c-1}{1-2c}\,{}^e\log\,[h^2 + (1-2c)\,y^2] + {}^e\log K$$

K is constant

Measuring length in h as a unit

$$x = ay - (a-b)\, y\, [1+(1-2c)\, y^2]^{\frac{c-1}{1-2c}} \quad . \quad . \quad . \quad . \quad (14a)$$

In the case of slow regular central rectilinear recession of slopes (initial slope angle $\beta > \alpha$; height h) in a homogeneous rock, the screes with slope angle α, accumulated on a horizontal terrace at the foot, protects a nucleus, of which the shape of the cross profile is dependent upon the cotangent of the initial slope angle β, the cotangent of the screes angle α and upon the ratio between the solid rock removed and the debris deposited per unit of time, and is represented by the formula:

$$x = ay - (a-b)\, y \left[\frac{h^2+(1-2c)\, y^2}{h^2}\right]^{\frac{c-1}{1-2c}} \quad . \quad . \quad . \quad . \quad (14)$$

Formula 14 and 14a can be derived too in other ways.

First, we shall examine the two cases, which strike one in LEHMANN's theory (10, p. 537—538). In LEHMANN's theory we obtain FISHER's parabola for $\beta = 90°$ ($b = 0$) and $c = 0$.

In our theory we get in these conditions the following formula of a cubic curve

$$x = ay - \frac{ay}{1+y^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

The premiss $\beta = 90°$, however, is rather unlikely at valley slopes and faultscarps, which have the tendency to recede rectilinearly by continual decrease of the slope angle value. That is why the more general formula for $c = 0$

$$x = ay - \frac{(a-b)\, y}{1+y^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15a)$$

is to be preferred.

RICHTER's straight-line denudationslope with the slope angle α is found in LEHMANN's theory for $c = -\infty$. In LEHMANN's case the denudation

---

and

$$\left.\begin{aligned} \int Q(y)\, e^{\int P(y)\, dy}\, dy &= \int \frac{2ay^2(1-c)}{h^2+(1-2c)\, y^2} \cdot \frac{K\,[h^2+(1-2c)\, y^2]^{-\frac{c-1}{1-2c}}}{y}\, dy = \\ &= a\, K\,(1-c) \int [h^2+(1-2c)\, y^2]^{-\frac{c-1}{1-2c}-1}\, dy^2 \\ &= a\, K\, [h^2+(1-2c)\, y^2]^{-\frac{c-1}{1-2c}} + C' \\ &\qquad\qquad C' \text{ is constant} \end{aligned}\right\} \quad (13b)$$

$$x = ay + Cy\, [h^2+(1-2c)\, y^2]^{\frac{c-1}{1-2c}} \qquad C = \frac{C'}{K} = \text{constant}$$

For $y = 0, \frac{x}{y} = b$ $\qquad C = \frac{-(a-b)}{h^{\frac{2(c-1)}{1-2c}}}$.

slope is formed at the base of the receding upstanding mountain part with slope angle β. As the basic point of the steep front, we started from, remains intact, this possibility is excluded in our theory. Nevertheless for high negative values of c our formula 14a may approximate

$$x = ay. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)\ ^{6})$$

This equation refers likewise to a straight line having the same slope angle α as the screes in Nature of the rock formation in question. It implies that RICHTER's denudation slope can be formed in two ways.

In this respect we must distinguish between *direct* and *indirect transformation of mountain slopes:*

1. Our fig. 1 is an example of *direct slope transformation.* The slope becomes gentler, but apart from that it remains as a unity preserved. So RICHTER's denudation slope can be formed only because β gradually diminishes to value α.

2. We speak of *indirect slope transformation,* when at the foot of the initial mountain wall, a *new* slope, which grows at the cost of the first one, is formed.

Indirect slope transformation will take place for all values of c, as soon as the basic



Fig. 5. The totem poles, pillars of red sandstone, Monument Valley, Utah (after A. HOLMES). Rd = RICHTER's denudationslope.

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6) For $c = -\infty$, we may ignore h^2, if y does not approximate zero too closely

$$x = ay + C^* y\,[(1-2c)\,y^2]^{\frac{c-1}{1-2c}}$$

$$x = ay + C^* y\,[Gy^2]^{-\frac{1}{2}}$$

If G is practically ∞ and C^* practically zero ($b \to a$) we may write

$$x = ay + C^* G$$

C^*G is a constant, dependent upon the way, in which C^* approximates zero and b approximates a. See formula (16a) in the second part of our theory.

point of the mountain slope, we started from, moves at not too small distances, apart from the fact that the slope recession proceeds either in a parallel or non-parallel manner. In our theory indirect slope transformation takes place for positive and small negative values of c.

In fig. 4, reproduced from VON ENGELN's Geomorphology (11, p. 292) we see RICHTER's denudation slope on outliers in the Monument Valley in Arizona. Fig. 5, taken from HOLMES' Physical Geology (12, plate 64 *B*) shows the same phenomenon among the well-known totem-poles. Both the permeability of the horizontal stratified sandstones and the proximity of canyons with nearly perpendicular slopes and also the fact that, in spite of a considerable difference in breadth of the buttes in fig. 4 and 5, the perpendicularity of the cliffs has approximately been preserved, are an indication, that in this case we have to do with parallel recession. It means that here RICHTER's denudation slopes are therefore examples of *in*direct slope transformation.

In order to gain a fuller understanding of the properties of our formula (14a) the way in which the shape of the curves change when c, α and β alter, respectively, and the other factors remain constant, will be investigated in the second part of our theory.

Crystallography. — *Crystal description of 1-methylamino-2 : 4-dinitronaphthalene,* $C_{10}H_5(NO_2)_2NH \cdot CH_3$. By L. P. G. KONING. (Communicated by Prof. H. A. BROUWER.)

(Communicated at the meeting of September 27, 1947.)

The organic compound 1-methylamino-2 : 4-dinitronaphthalene, $C_{10}H_5(NO_2)_2NH \cdot CH_3$, has been prepared by H. W. TALEN [1]). Prof. Dr P. TERPSTRA of the Crystallographic Institute of the University of Groningen has been so kind to put the crystals at my disposal for crystallographic investigation with the two-circle goniometer. The results of this investigation will be given here.

The orange-coloured crystals with a size of 0.1—1.0 mm are short prismatic parallel to the c-axis.

Triclinic pinacoidal.

Forms: c(001), b(010), a(100), m(110), r(101), q(011), o(111), t(121), $i(\bar{1}01)$, $v(1\bar{2}1)$, $\sigma(\bar{4}34)$, $\tau(\bar{4}31)$, $\pi(\bar{5}41)$.

Face	Symbol	φ	ϱ
1 c	(001)	86° 08′	46° 23′
2 b	(010)	0	90
3 a	(100)	98 51	90
4 m	(110)	57 25	90
5 r	(101)	91 55	61 30
6 q	(011)	55 08	51 39
7 o	(111)	72 28	62 40
8 t	(121)	56 21	65 39
9 i	$(\bar{1}01)$	47 41	16 55
10 v	$(1\bar{2}1)$	126 12	66 16
11 σ	$(\bar{4}34)$	18 29	35 19
12 τ	$(\bar{4}31)$	238 21	68 42
13 π	$(\bar{5}41)$	238 06	74 36

[1]) H. W. TALEN, Replacement of the halogen atom or the alkyl group in 1-chloro-, 1-methoxy-, or 1-ethoxy-2 : 4-dinitro- and 2 : 4 : 5-trinitronaphthalenes by various other groups, Rec. d. Trav. Chim. d. Pays-Bas, T. **47**, no. 2, 346—362 (1928).

$X'_0 = 1.0515$	$\varrho_0 = 46° 30'$	$p'_0 = 0.7955$
$Y'_0 = 0.3461$	$\varphi_0 = 86° 14'$	$q'_0 = 0.6388$
$\alpha = 100° 11'$	$\lambda = 87° 16'$	$a = 1.130$
$\beta = 136° 26'$	$\mu = 44° 58'$	$b = 1$
$\gamma = 105° 37'$	$\nu = 99° 03'$	$c = 0.6206$
	$d = 48° 47'$	
	$f = 28° 23'$	

Optical properties:

Biaxial positive, 2V large; birefringence very large, dispersion $v > r$ strong. Optical ax.plane about parallel a(100).

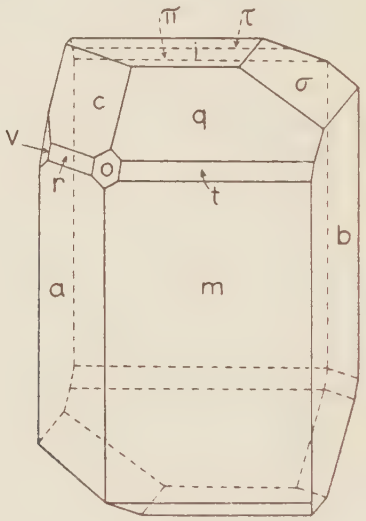


Fig. 1.

Pleochroism strong: n_α = brownish yellow, $n_\beta = n_\gamma$ = greenish yellow.

According to BARKER's [2]) systematic classification of crystals the following characteristic angles have been calculated:

$cr = 15° 46'$	$am = 41° 26'$	$bq = 63° 38'$
$ra = 29° 12'$	$mb = 57° 25'$	$qc = 23° 37'$

Geological Institute of the University of Amsterdam.

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[2]) T. V. BARKER, Systematic crystallography, an essay on crystal description, classification and identification, London (1930). — P. TERPSTRA, Kristallometrie, Groningen (1946).

Crystallography. — *Crystal description of 1-n-propylamino-2 : 4-dinitronaphthalene* $C_{10}H_5(NO_2)_2NH \cdot C_3H_7(n)$. By L. P. G. KONING. (Communicated by Prof. H. A. BROUWER.)

(Communicated at the meeting of September 27, 1947.)

The organic compound 1-n-propylamino-2 : 4-dinitronaphthalene, $C_{10}H_5(NO_2)_2NH \cdot C_3H_7(n)$, has been prepared by H. W. TALEN [1]). The crystals kindly put at my disposal by Prof. Dr P. TERPSTRA of the Crystallographic Institute of the University of Groningen have been subjected to crystallographic measurements with the two-circle goniometer, the results of which will be given here.

The orange-coloured crystals with a size of 0.1—1.0 mm are prismatic parallel to the c-axis and flattened parallel to b(010).

Cleavage: c(001) and a(100) good.

Triclinic pinacoidal.

Forms: c(001), b(010), a(100), m(110), p(310), n($\bar{3}$10), q(011), t(0$\bar{1}$1).

Face	Symbol	φ	ϱ
1 c	(001)	108° 12′	14° 40′
2 b	(010)	0	90
3 a	(100)	80 48	90
4 m	(110)	25 38	90
5 p	(310)	51 18	90
6 n	($\bar{3}$10)	295 32	90
7 q	(011)	16 11	41 28
8 t	(0$\bar{1}$1)	166 08	46 06

$X'_0 = 0.2481$ $\quad \varrho_0 = 14° 37'$ $\quad p'_0 = 0.4904$

$Y'_0 = 0.0805$ $\quad \varphi_0 = 107° 59'$ $\quad q'_0 = 0.9290$

$\alpha = 83° 14'$ $\quad \lambda = 94° 28'$ $\quad a = 1.9354$

$\beta = 103° 56'$ $\quad \mu = 77° 02'$ $\quad b = 1$

$\gamma = 100° 32'$ $\quad \nu = 80° 48'$ $\quad c = 0.9103$

$d = 51° 25'$

$f = 53° 46'$

As is shown in the figures this compound occurs also twinned.

Optical properties:

Biaxial positive, 2V moderate. Birefringence very strong; Dispersion strong: v > r. Optical ax.plane nearly $\perp$ a(100). $n_\beta \wedge c = 7°$.

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[1]) H. W. TALEN, Replacement of the halogen atom or the alkyl group in 1-chloro-, 1-methoxy- or 1-ethoxy-2 : 4-dinitro- and 2 : 4 : 5-trinitronaphthalenes by various other groups, Rec. d. Trav. Chim. d. Pays-Bas, T. 47, no. 2, 346—362 (1928).

Pleochroism strong: $n_\alpha = n_\beta$ = brownish yellow, n_γ = greenish yellow.

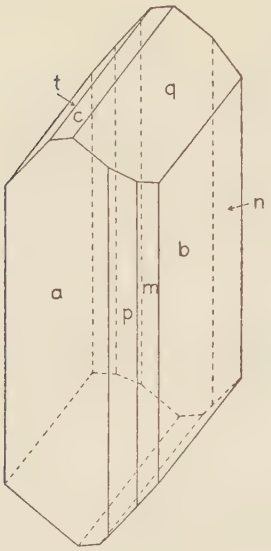


Fig. 1.

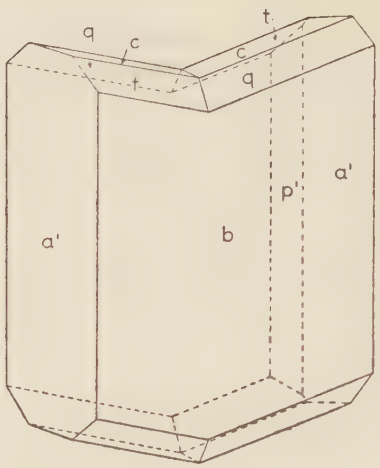


Fig. 2.

According to BARKER's [2]) classification of crystals the following characteristic angles have been calculated:

cr = 30° 36′	**am = 44° 02′**	bq = 51° 18′
ra = 46° 26′	mb = 50° 26′	qc = 29° 30′

Geological Institute of the University of Amsterdam.

[2]) T. V. BARKER, Systematic crystallography, an essay on crystal description, classification and identification, London (1930). — P. TERPSTRA, Kristallometrie, Groningen (1946).

Crystallography. — *Crystal description of 1-n-butylamino-2 : 4-dinitronaphthalene,* $C_{10}H_5(NO_2)_2NH \cdot C_4H_9(n)$. By L. P. G. KONING. (Communicated by Prof. H. A. BROUWER.)

(Communicated at the meeting of September 27, 1947.)

The organic compound 1-n-butylamino-2 : 4-dinitronaphthalene, $C_{10}H_5(NO_2)_2NH \cdot C_4H_9(n)$, prepared by H. W. TALEN [1]) has been kindly put at my disposal by Prof. Dr P. TERPSTRA of the Crystallographic Institute

[1]) H. W. TALEN, Replacement of the halogen atom or the alkyl group in 1-chloro-, 1-methoxy-, or 1-ethoxy-2 : 4-dinitro- and 2 : 4 : 5-trinitronaphthalenes by various other groups, Rec. d. Trav. Chim. d. Pays-Bas, T. 47, no. 2, 346—362 (1928).

of the University of Groningen for crystallographic investigation with the two-circle goniometer, the results of which will be given here.

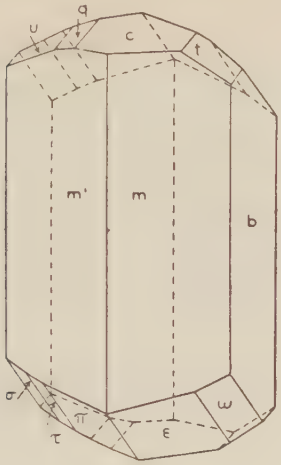


Fig. 1.

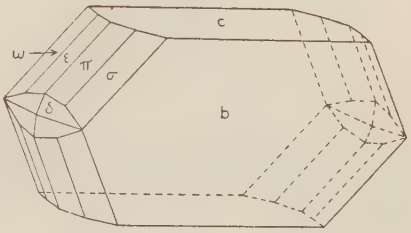


Fig. 2.

The orange-coloured crystal with a size of 0.1—1.0 mm are prismatic parallel to the c-axis and flattened parallel to b(010).

Cleavage b(010) perfect.

Monoclinic prismatic.

Forms: c(001), b(010), m(110), q(0$\bar{1}$1), t(031), u(0$\bar{2}$1), ω($\bar{1}$31), ε($\bar{1}$21), π($\bar{1}\bar{2}$1), τ($\bar{2}\bar{5}$2), σ($\bar{1}\bar{3}$1), δ($\bar{3}$31).

Face	Symbol	φ	ϱ
1 c	(001)	90°	12° 02′
2 b	(010)	0	90
3 m	(110)	73 30′	90
4 q	(0$\bar{1}$1)	138 13	17 51
5 t	(031)	16 31	36 51
6 u	(0$\bar{2}$1)	155 55	27 24
7 ω	($\bar{1}$31)	320 14	42 44
8 ε	($\bar{1}$21)	308 42	37 25
9 π	($\bar{1}\bar{2}$1)	231 18	37 25
10 τ	($\bar{2}\bar{5}$2)	224 51	40 08
11 σ	($\bar{1}\bar{3}$1)	219 47	42 47

$X'_0 = 0.2132$ $\quad \varrho_0 = 12° 02'$ $\quad p'_0 = 0.8074$

$Y'_0 = 0$ $\quad \varphi_0 = 90°$ $\quad q'_0 = 0.2384$

$\beta = 102° 02'$ $\quad a = 0.3019$

$\mu = 77° 58'$ $\quad b = 1$

$c = 0.2384$

$d = 76° 35'$

$f = 44° 25'$

As is shown in the figures this compound occurs also twinned as bipyramidal flakes.

Optical properties:

Biaxial negative, 2V large. Birefringence very strong, and dispersion $r > v$ very strong. Optical ax.plane nearly $\perp$ b(010), $n\beta \wedge c = 22\frac{1}{2}°$.

According to BARKER's [2]) systematic classification of crystals the following characteristic angles have been calculated:

cr = 33° 33′	am = 41° 32′
ra = 44° 25′	bq = 54° 54′

Geological Institute of the University of Amsterdam.

[2]) T. V. BARKER, Systematic crystallography, an essay on crystal description, classification and identification, London (1930). — P. TERPSTRA, Kristallometrie, Groningen (1946).

Crystallography. — *Crystal description of 1-n-heptylamino-2 : 4 : 5-trinitronaphthalene, $C_{10}H_4(NO_2)_3NH \cdot C_7H_{15}(n)$.* By L. P. G. KONING. (Communicated by Prof. H. A. BROUWER.)

(Communicated at the meeting of September 27, 1947.)

The organic compound 1-n-heptylamino-2 : 4 : 5-trinitronaphthalene, $C_{10}H_4(NO_2)_3NH \cdot C_7H_{15}(n)$, has been prepared by H. W. TALEN [1]). Prof. Dr P. TERPSTRA of the Crystallographic Institute of the University of Groningen has been so kind to put the crystals at my disposal for crystallographic investigation with the two-circle goniometer. The results of this investigation will be given here.

The yellow-brown coloured crystals with a size of 1—2 mm are elongated parallel to the a-axis. Cleavage c (001) perfect.

Orthorhombic dipyramidal.

Forms: c(001), b(010), r(101), t(102), o(111), s(201), n(121).

Face	Symbol	ϱ	φ
1 c	(001)	0°	
2 b	(010)	90	0°
3 r	(101)	46 52′	90
4 t	(102)	28 21	90
5 o	(111)	51 50	56 43′
6 s	(201)	64 42	90
7 n	(121)	60 41	36 49

[1]) H. W. TALEN, Replacement of the halogen atom or the alkyl group in 1-chloro-, 1-methoxy-, or 1-ethoxy-2 : 4-dinitro- and 2 : 4 : 5-trinitronaphthalenes by various other groups, Rec. d. Trav. Chim. d. Pays-Bas, T. 47, no. 2, 346—362 (1928).

$p'_0 = 1.0676$ $\quad d = 54° 48'$
$q'_0 = 0.7055$ $\quad f = 43° 08'$

$a = 0.6608$
$b = 1$
$c = 0.7055$

Optical data:

Birefringence very strong; optical biaxial positive;
2V small; dispersion $v > r$.
Opt. ax. plane = b(010), $n_\gamma = c$.
Pleochroism: $n_\alpha = n_\gamma$ = yellow-brown, n_β = yellow-greenish.

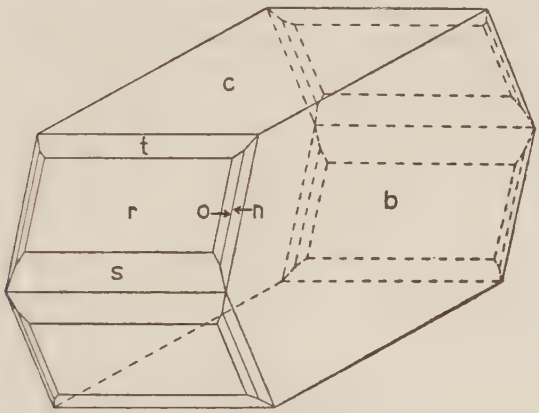


Fig. 1.

According to BARKER's [2]) classification of crystals the following characteristic angles have been calculated:

$cr = 36° 49'$
$am = 43° 08'$
$bq = 54° 48'$

Geological Institute of the University of Amsterdam.

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2) T. V. BARKER, Systematic crystallography, an essay on crystal description, classification and identification, London (1930). — P. TERPSTRA, Kristallometrie, Groningen (1946).

Crystallography. — *Crystal description of benzotribromide, $C_6H_5CBr_3$.* By L. P. G. KONING. (Communicated by Prof. H. A. BROUWER.)

(Communicated at the meeting of September 27, 1947.)

The organic compound benzotribromide = toluenetribromide = $C_6H_5CBr_3$, of which the preparation has been mentioned by W. H. HUNTER and D. E. EDGAR in 1932 and L. S. HEBLE, D. R. NADKARNI and

T. S. WHEELER [1]) in 1938, crystallized from a saturated solution in petroleum-ether during chemical experiments performed by Mr. B. JIBBEN in the Laboratory for Organic Chemistry of the University of Amsterdam.

In the saturated solution the ideally developed prismatic crystals seem highly lustreous, but out of the mother liquor the faces appear dull.

Several crystals have been subjected to crystallographic and optical investigation, the results of which will be given here.

The colourless crystals with a size of 5—10 mm are elongated parallel to the c-axis.

Orthorhombic dipyramidal.

Forms: b(010), a(100), m(110), c(001) and o(111).

Face	Symbol	ϱ	φ
1 b	(010)	90°	0°
2 a	(100)	90	90
3 m	(110)	90	51 46′
4 c	(001)	0	—
5 o	(111)	70 13′	51 46

$p'_0 = 2.1838$ $\quad$ $a = 0.7879$

$b = 1$

$q'_0 = 1.7206$ $\quad$ $c = 1.7206$

$d = 30° 10'$

$f = 24° 36'$

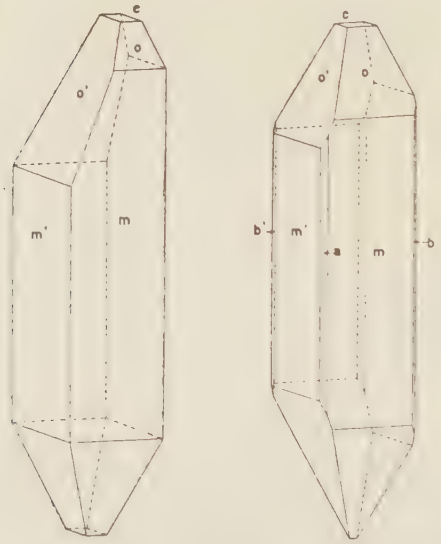


Fig. 1.

[1]) W. H. HUNTER and D. E. EDGAR, Carbon tetrabromide as a brominating agent, Am. Chem. Soc., Vol. 54 II, 2025—2028 (1932). — L. S. HEBLE, D. R. NADKARNI and T. S. WHEELER, Mixed laterally halogenated toluenes. Journ. Chem. Soc. London, 1322—1323 (1938).

No cleavage has been observed. Fracture conchoidal, uneven.

Optical data:

Biaxial positive, 2 V large; n_α slightly higher than 1.74.
Birefringence and dispersion are very strong.
Optical orientation: $n_\alpha = c$, $n_\beta = b$, $n_\gamma = a$.

According to BARKER's [2]) systematic classification of crystals the following characteristic angles have been calculated:

$$cr = 65° 24'$$
$$am = 38° 14'$$
$$bq = 30° 10'$$

Geological Institute of the
University of Amsterdam.

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[2]) T. V. BARKER, Systematic crystallography, an essay on crystal description, classification and identification, London (1930). — P. TERPSTRA, Kristallometrie, Groningen (1946).

Microbiology. — *An electron-microscopical study of Leptospira biflexa.* By P. H. VAN THIEL and WOUTERA VAN ITERSON. (From the Parasitological Laboratory of the University at Leiden and the Netherlands Institute for Electron Microscopy at Delft.) (Communicated by Prof. A. J. KLUYVER.)

(Communicated at the meeting of June 28, 1947.)

Notwithstanding the fact that the morphology of leptospiras has frequently been studied, there is still much diversion of opinion regarding several structural details. This uncertainty has increased since researches in *Treponema pallidum* with the electron microscope have shown that their structure is probably quite different than has been supposed before.

NOGUCHI (1917, 1918) created the genus *Leptospira*, on account of the following characteristics:

1. the presence of fine and minute elementary windings; 2. the absence of a terminal filament and of minute flagellum-like projections; 3. the absence of an axial filament and of a membrane; 4. when in a free space, one or both ends may be semicircularly hooked; 5. the typical movement; 6. the resistance to 10 per cent saponin.

By staining the leptospiras first with Giemsa's solution and then partly decolourizing them, ZUELZER (1931), and also TIMMERMAN (1927) stated that the spirals become decolourized and that a straight stretched axial filament remains, which was before entwined by the now decolourized plasmatic spiral. After having added specific immune serum or sodium taurocholic acid (1 : 10) to the living leptospiras only the axial filament remains. In contradiction to ZUELZER, TIMMERMAN observed that the more resistant axial filament also finally disappears in sodium taurocholate. HERREWEGHE (1943) also thinks to have observed this filament.

Objections have been raised concerning the alleged resistance against 10 per cent saponin only in 1938, when MOLLARET and SIMONS found a few strains of *Leptospira icterohaemorrhagiae* extremely sensitive to saponin. The presence of an axial filament, however, has not been questioned for the following reasons. In the first place this structure is supposed to have been clearly observed in larger non-parasitic spirochaetes so that it has been thought to be present in all sorts of spirochaetes. Further the typical movement of the spirochaetes could be explained by its presence, as the axial filament was thought to function as an internal skeleton.

The correctness of this conception has been questioned since MARTON (1942), UDO WILE, PICARD and KEARNY (1942), MUDD, POLEVITZKY and ANDERSON (1943) observed with the electron microscope that no axial filament is present in *Treponema pallidum* and that relatively long lopho-

trichous flagella are found on different places of their cell body. The investigation by MORTON and ANDERSON (1943) carried out with the aid of the R.C.A. electron microscope showed indeed that an axial filament is lacking in *Leptospira icterohaemorrhagiae* and *canicola*.

It seemed desirable to us to make some additional observations on the morphology of the leptospiras by means of the electron microscope in order to arrive at an independent opinion.

Methods.

Three days old cultures of *Leptospira biflexa* Wolbach and Binger were submitted to investigation. In order to avoid disturbance of any flagella-like or other protrusions if present, and to avoid a contact of the leptospiras with distilled water longer than necessary, the organisms were mounted on the specimen-film in a drop of the original diluted serum. After a short moment the serum drop was taken away with a pipette and substituted by a drop of distilled water; this operation was repeated two or three times.

Object-holders used in the Delft electron microscope are small metal cones with a hole of 0.2—0.3 mm diameter in the top to allow for the passage of the electron beam. Over the hole a film of "Geisselthallack" is brought on which the specimens are mounted.

After the washing off the object-holders are placed on a little tray, quickly submerged in liquid air, and brought into a vacuum chamber where they are dried and shadowed with gold according to the method devised by WILLIAMS and WYCKOFF (1946). By this technique a piece of gold, placed in an oblique position to the preparation holders, is evaporated by electrical heating as soon as the vacuum has reached 5×10^{-5} mm Hg. In consequence hereof the specimens are coated with a thin layer of metal molecules, which raises their contrast, whilst a "shadow" is left where in the shelter of protruding structures no gold particles could settle. Printed in "negative" the micrographs of specimens treated in this way give an impression of their 3-dimensional structure.

In the background of fig. 1, 2, 3 one notices a rough granulation, which can be attributed to the combined effect of the shadowed primary structure of the Geisselthallackfilm, protein molecules from the original serum which have remained on the film after the washing procedure, and perhaps partly to a too fierce illumination with the electron beam which causes a coagulation of the gold particles. This influences the image unfavourably.

The micrographs were taken at 75 kv with the Delft electron microscope as built by J. B. LE POOLE (1947).

Discussion.

The micrographs give us the following information regarding the morphology of the species of *Leptospira* under examination.

1. *Cell wall.* No distinct ectoplasmatic envelope is present. In unshadowed air-dried leptospiras we sometimes distinguished a clear sheath-like zone as observed by MORTON and ANDERSON (1947). This was certainly not reminiscent of a protoplasmatic contraction from a cell wall as so often encountered in bacterial micrographs, but — as was confirmed by a shadowed specimen — could only be explained as a groove in the layer of protein molecules originating from the remaining serum caused by the convulsions of the leptospira during the drying on the supporting film.

Although the outer layer of the protoplasm is very probably somewhat more rigid than the central part, its appearance is markedly different from that of the cell wall of bacteria as seen with the same microscope. This makes us suppose that there can only be question of the presence of a periplast. If this is true leptospiras must be called naked organisms. KNAYSI (1944) wrongly considers the membrane of the spirochaetes as homologous to the cell wall of the bacterial cell.

The absence of a rigid structure comparable to a bacterial cell wall is clearly shown by the irregular contours of the protoplasm in the degenerated cell of fig. 3.

2. *Axial filament.* We have not been able to detect the slightest indication of such a structure. Neither were MUDD, POLEVITZKY and ANDERSON (1943) able to detect a differentiated axial filament within the protoplasm of *Treponema pallidum.* In several cases motile structures consisting of an axial filament surrounded by a sheath were clearly revealed by the electron microscope (BRETSCHNEIDER and VAN ITERSON (1947) in sperm cells, BROWN (1945) and SCHMITT (1942) in protozoa and sperm cells), but in these instances they always made the impression of being highly specialized structures forming a part of a relatively complicated object. Therefore it seems highly improbable to the authors that in the primitive spirochaetal cell such a complex differentiation as in the above mentioned objects should occur. The problem remains to be reinvestigated for other spirochaetes.

3. *Flagella.* Flagella such as have been found to exist in *Treponema pallidum* were never observed, so that in this point we fully agree with MORTON and ANDERSON.

Micrograph 1 clearly shows that NOGUCHI rightly described in *Leptospira* the absence of a terminal filament and of peritrichal flagella. The end of the body is typically hooked. In old or badly grown cultures frequently end-buds, one on each pole, are found. Fig. 3 shows that such a bud originates from a rolling up of the last windings, whereas evidence that this should result from the curling up of a terminal filament is lacking. It therefore seems beyond doubt that flagella play no rôle in the movement of leptospiras. The statement of STANIER and VAN NIEL (1941) that spirochaetes are motile either by means of an elastic axial filament or of a modified fibrillar membrane certainly does not apply to leptospiras.



Fig. 1.
(18.000 ×)



Fig. 2.
(21,000 ×)



Fig. 3.
(16.000 ×)

Neither can KNAYSI's opinion that "the cell should be provided with an additional anatomical feature in the form of an axial filament which may project beyond one or both cell ends forming flagellum-like extensions" be confirmed.

Fig. 2 shows, however, that irregular threadlike protrusions can be found situated terminally or sidelong on leptospiras. These threads are quite different from the very distinct bacterial flagella with their constant width. Probably in the leptospiras they are of a slimy nature, and perhaps they are set free after the organism's contact with distilled water during the washing procedure, so that they should not be considered as true appendices.

4. *Nuclei.* Neither nuclei nor any other granular bodies, nor any other internal structure could be observed in young cultures of leptospiras.

With the above description of morphological characteristics no opinion is expressed on the existence of the "granules leptospirogènes", described by GASTINEL and MOLLINEDO (1942).

Conclusion.

The absence of flagella in leptospiras removes this organism from the spirochaetal subgenera *Borrelia* and *Treponema*. Yet it is to be recommended to maintain *Leptospira* as a subgenus of the genus *Spirochaeta,* because the conception that a leptospira would no longer be a spirochaete would cause great confusion, the more so because they have several characteristics in common.

Summary.

A discussion is given of the morphology of *Leptospira biflexa* Wolbach and Binger on the basis of electron photomicrographs.

LITERATURE.

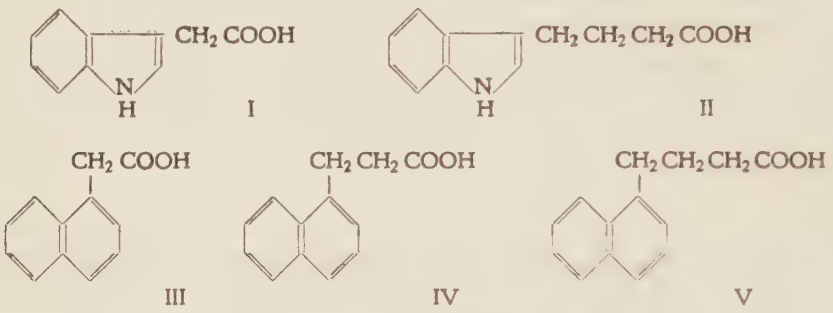
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Botany. — *Researches on plant growth regulators.* XII. *Comparative investigation of a number of homologous and isomeric synthetic growth substances in the rooting of cuttings.* By W. KRUYT and H. VELDSTRA. (Communicated by Prof. V. J. KONINGSBERGER.)

(Communicated at the meeting of September 27, 1947.)

Introduction.

If one compares the effect in root formation between plant growth substances derived from indole and from naphthalene respectively, it may be noticed that e.g. indole acetic acid (I) and indole butyric acid (II) are on the whole equally active (in certain cases indole butyric acid is even more active). When, however, the side-chain is lengthened, starting from α-naphthalene acetic acid (III) the activity clearly decreases and thus α-naphthalene propionic acid (IV) and α-naphthalene butyric acid (V) are practically never used.



The difference between derivatives of indole and naphthalene is all the more outstanding in the primary growth reaction as determined in the pea-test, in which at a similar activity of indole acetic acid, indole butyric acid and α-naphthalene acetic acid, the activity of α-naphthalene butyric acid is but very small.

In the studying of the relation between structure and activity (VELDSTRA (1)) as yet no explanation has been proved for this remarkable fact. None of the comparative methods used shows a differentiation running parallel to the mentioned differences. As we suspected an important detail of the connection between structure and activity to be hidden in these relations, we thought it advisable to compare once more with certain cuttings all the named acids, especially the naphthalene derivatives, as to their activity in root formation, in order that we might verify the results obtained by other investigators partly on heterogeneous material (compare GRACE (2)).

Beside the already mentioned homologous series, isomeric β-substituted naphthalene-derivatives were also tested.

Material and methods.

Our experiments on cuttings were carried out in the hothouses of "De Proeftuin" at Boskoop [1]). The material was collected in the morning in fair weather and used as soon as possible. As testing subject we used × *Viburnum Burkwoodii* Burkw. From the branches measuring 17—28 cm the part directly under the soft top was taken. The cuttings all had 4 leaves and consisted of two internodes, cut off directly above the node. Finally the length was 11—18 cm and the diameter = 2½ mm. The cuttings from the various sources were always distributed equally among the groups.

In the course of the afternoon the bundled cuttings were put 2 to 3 cm deep in glass dishes with fresh growth substance solutions (prepared with distilled water). Beside an entirely untreated group another lot was always placed in pure water only as another control. A dish filled with water and without cuttings was present during the treatment to allow a determination of the evaporation in the end so that the uptake could be more or less accurately calculated and thus the dose of growth substance for each cutting.

During the experiment maximum and minimum temperature were registered.

After 24 hours of soaking the lower ends of the cuttings were washed in rain-water and placed in so-called Rhododendron-pots (inside measurements: top diam. 9 cm, bottom diam. 5 cm and depth 8 cm). In so doing 5 cuttings came to be placed along the rim.

The rooting medium (peat sand = 1 : 2) had been prepared beforehand in sufficient quantity to eliminate possible differences in composition between the various lots. The amounts of fresh, sifted peat and washed coarse sand were measured with a large flower-pot so that the proportion peat/sand was known by volume. An investigation by KRUYT (3—6) in sequence of the work done by CHADWICK (7); ESPER and ROOF (8); HITCHCOCK and ZIMMERMAN (9—10); HUBERT, RAPPAPORT and BEKE (11); LAURIE (12); LAURIE and CHADWICK (13); LONG (14); SMITH (15) etc., had proved that by a right choice in the composition of the medium one can exert an important influence on the root formation. In this case probably the p_H plays the most important part. The p_H of a mixture peat/sand = 1 : 2 shortly after mixing is 5.3 whereas this value increases to 5.7 after a full month.

After carefully mixing the peat and the sand the material was watered now and then and mixed again until the right humidity had been obtained. The pots which had previously sucked up plenty of water were then filled with the mixture. After placing the cuttings in previously prepared holes in the medium it was pressed firmly.

The pots were placed in a hothouse under double glass and dug in into humid peat. We have strived to distribute the influence of the position as much as possible among the various groups. Therefore the groups were not kept separately but the pots were placed in the breadth of the hotbed or hothouse, behind each other, in sequence of the groups of which the rooting experiment consists. When the row is full a new one is placed next to it till one pot of very group has been dug in. Then one starts again with the first group and so on. Of course one must see to it that the number of pots in one row does not happen to correspond with the number of groups because then there would be no shifting. In that case an empty pot is occasionally placed in between the filled ones (so as to get a shifting of position).

As soon as everything was dug in, it was watered thoroughly with rain-water after which the windows were closed. The tending took place in the usual way (5). Soil and air temperatures were registered nearly daily during the root formation.

At the end of the experiment the most important result for us, viz. the number of roots on each cutting, was determined and the manner in which they were attached to the cuttings was also noted. The measuring of all the roots was practically impossible because of the large number of cuttings and the extensive root formation (in some cases more than an average of 40 per cutting !). It would have damaged too much the material which had

[1]) We owe many thanks to the Board and Director of "De Proeftuin" for the hospitality granted.

to serve again in further experiments. Therefore measuring was confined to the longest root of each cutting.

We owe many thanks to Messrs. H. BOSCH and P. DE VOGEL, chief assistants of the "Tuinbouwvoorlichtingsdienst" at Boskoop for their assistance in the collecting and preparing of the cuttings and for the careful attending of the experiments.

Discussion of the results.

The first experiment which lasted from July 26 till August 24 1944 consisted of 21 groups containing 30 cuttings each. Beside the two control groups (untreated and water) we examined the reaction of α- and β-naphthalene acetic acid, α- and β-naphthalene propionic acid, α- and β-naphthalene butyric acid (in the form of their potassium salts), each in three concentrations (6, 3 and 1.5×10^{-4} mol./1 respectively) and besides these β-indole acetic acid (as the free acid) in a concentration of 25 mg/1 [2]).

During the treatment the minimal temperature was 17.4° C, the maximal temperature 31.3° C, the average 24.4° C; temperature of the soil during the process of root formation: minimal 18.8°, maximal 28.8°, average 23.4° C; air temperature minimal 17.1°, maximal 31.7°, average 23.4° C. No soil heat was supplied.

Table I shows the situation at the ending of the experiment.

As a result of the uptake of water we already note a rise in the percentage of root formation; the average number of roots for each rooted cutting does not change however. The concentration of n.a.a., n.p.a. and n.b.a. was always 6×10^{-4} mol./1 at the start; corresponding with 111.6, 120 and 128.4 mg of acid per litre. From the data obtained we may deduce the following.

α-N.a.a. is without doubt the most active substance. Of its homologues α-n.b.a. still has a moderate effect, but α-n.p.a. has only a very weak effect. So the oscillation in the homologous series, already noticed by GRACE (2), evidently also occurs in this case. The order α-n.a.a. $>$ α-n.b.a. $>$ α-n.p.a. is also to be seen in the type of the root formation. With the first-mentioned acid no basal rooting takes place, α-n.b.a. at 3×10^{-4} mol./1 gives two specimens with basal roots, while in the groups of α-n.p.a. a total of 25 such cases occur.

The β-isomers are weaker than the α-substituted naphthalene-derivatives, which shows up strongly especially with n.a.a. For the remainder the order β-n.a.a. $>$ β-n.b.a. $>$ β-n.p.a. also holds in this case. β-N.a.a. and β-n.b.a. have only a slightly stronger effect than water.

The effect of β-i.a.a. (free acid) here lies between that of α-n.a.a. and α-n.b.a.

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[2]) We use the following abbreviations:
n.a.a. = naphthalene acetic acid
n.p.a. = naphthalene propionic acid
n.b.a. = naphthalene butyric acid
i.a.a. = indole acetic acid
i.b.a. = indole butyric acid

TABLE I.

Group	Number of cuttings	Treatment (Solution 24 hrs; concentrations in mg of acid/l)	ml soaked per cutting	µg of acid per cutting	Dead	Number of cuttings without roots				Total number without roots	Number of cuttings rooted									Total number rooted	Percentage rooted cuttings	Total number of roots	Average number of roots per rooted cutting	Average length of longest roots in mm
						with basal decay	no callus	some callus	callus-knob		only higher up			not only basal			at the basal cut							
											basal decay	no callus	some callus	no callus	some callus	callus-knob	no callus	some callus	callus-knob					
1	30	untreated	—	—	—	—	5	7	1	13	—	—	—	—	5	1	—	9	2	17	57	88	5	30
2	30	water	0.55	—	—	—	7	—	—	7	—	2	1	5	1	—	8	6	—	23	77	111	5	26
3	30	α-n.a.a. 111.6 mg/l	0.75	83.7	9	5	—	—	—	5	16	—	—	—	—	—	—	—	—	16	53	191	12	42
4	30	„ 55.8 „	0.58	32.4	2	—	—	—	—	0	16	—	—	12	—	—	—	—	—	28	93	795	28	45
5	30	„ 27.9 „	0.53	14.8	1	1	—	—	—	1	6	—	—	20	2	—	—	—	—	28	93	736	26	47
6	30	β-n.a.a. 111.6 mg/l	0.75	83.7	2	—	4	—	—	4	3	3	1	16	—	—	1	—	—	24	80	163	7	39
7	30	„ 55.8 „	0.80	44.6	1	—	4	—	—	4	1	5	—	15	—	—	4	—	—	25	83	195	8	28
8	30	„ 27.9 „	0.57	15.9	—	2	6	—	—	8	—	—	—	17	—	—	5	—	—	22	73	152	7	34
9	30	α-n.p.a. 120 mg/l	0.68	81.6	—	6	2	—	—	8	10	1	—	6	—	—	5	—	—	22	73	120	5	32
10	30	„ 60 „	0.62	37.2	—	1	5	—	—	6	3	3	—	4	—	—	14	—	—	24	80	111	5	19
11	30	„ 30 „	0.73	21.9	—	1	5	—	—	6	—	4	—	14	—	—	6	—	—	24	80	147	6	32
12	30	β-n.p.a. 120 mg/l	0.72	86.4	—	3	8	—	—	11	3	1	—	3	—	—	11	1	—	19	63	118	6	35
13	30	„ 60 „	0.85	51.0	—	1	3	—	—	4	3	3	—	7	—	—	11	1	1	26	87	152	6	26
14	30	„ 30 „	0.88	26.4	1	2	5	—	—	7	—	—	—	7	1	—	13	1	—	22	23	121	6	25
15	30	α-n.b.a. 128.4 mg/l	1.12	143.8	3	8	2	—	—	10	8	2	—	7	—	—	—	—	—	17	57	205	12	35
16	30	„ 64.2 „	0.78	50.1	1	—	6	—	—	6	2	2	—	17	—	—	2	—	—	23	77	221	10	35
17	30	„ 32.1 „	0.72	23.1	3	—	6	—	—	6	1	5	—	14	1	—	—	—	—	21	70	151	7	24
18	30	β-n.b.a. 128.4 mg/l	0.93	119.4	—	5	4	—	—	9	2	9	—	9	—	—	1	—	—	21	70	174	8	28
19	30	„ 64.2 „	0.82	52.6	—	3	2	—	—	5	3	2	—	17	—	—	2	1	—	25	83	184	7	25
20	30	„ 32.1 „	0.68	21.8	—	—	1	2	—	3	2	5	—	10	2	—	7	1	—	27	90	172	6	24
21	30	i.a.a. 25 mg/l	0.62	15.5	—	—	4	—	—	4	2	—	—	21	—	—	2	1	—	26	87	393	15	49

On account of the number of dead specimens and the occurrence of basal decay with the rooted as well as with the non-rooted cuttings, the concentration of 6×10^{-4} mol./1 is in most cases too high. It is remarkable that the most active compound (α-n.a.a.) evidently gives an increase in root length (in the next experiment this phenomenon is not so prominent). β-I.a.a. also has a favourable influence on the root length. This conclusion, however, could only have been drawn definitely if the length of all the roots had been determined and thus the average root length. For this the measuring of only the longest root of every cutting is very likely insufficient.

On the whole the rooting was normal; the roots were coloured white and had a tip tinged with yellow. In the groups 3 and 15 (with the highest concentrations of α-n.a.a. and α-n.b.a. respectively) the roots were slightly thicker than normal.

The second, more complete, experiment in which also the potassium salts of β-i.a.a. and β-i.b.a. in three different concentrations were included, followed on September 20 1944.

During the treatment with the growth substance solutions the minimal temperature was 12.4° C, the maximal temperature 21.7° C, the average therefore 17.1° C. The number of ml sucked up by each cutting varied between 0.53 and 0.80; the dose calculated in μg per cutting may be found in the fifth column of table II.

During the proces of rooting from Sept. 27 '44 up to and including Sept. 28 '44; from Oct. 3 '44 up to and including Oct. 8 '44 and from Oct. 16 '44 onwards soll heating was applied. The temperature of the bed was not read at the place of experiment itself in consequence of lack of thermometers, but in another place in the hothouse. The soil temperature was minimal 10.4° C and maximal 19.9° C (with an average of 15.3°) whereas these values for the air temperature were 9.7° C and 24.3° C respectively (with an average of 16.5°). In the place at which these temperatures were taken there was no soil heating, so that very possibly during the period that the hothouse was heated the soil and air temperatures around the cuttings were higher than is stated above.

The results of this experiment are collected in table II and also recorded in photographs made on Oct. 30 and Oct. 31 1944, that is some days after the ending of the experiment. During this time the cuttings were dug in in boxes with the cutting-medium and so placed in the hothouse bed. By doing this the roots had the opportunity to increase in length during these few days and even the number of roots had increased. Evidently the transferring into loose medium stimulates the elongation of the roots. Therefore the photographs of the second experiment do not show an exact picture of the situation at the end on October 26 and 27, but can only be used for mutual comparison. We may suppose that the growth in length was approximately of the same proportion in all groups. The figures in table II give a reliable summary.

The starting concentrations of n.a.a., n.p.a., n.b.a., i.a.a. and i.b.a. were again 6×10^{-4} mol./1, that is 111.6, 120, 128.4, 105 and 121.8 mg of

TABLE II.

Group	Number of cuttings	Treatment (Solution 24 hrs; concentrations in mg of acid/l)	ml soaked per cutting	µg of acid per cutting	Dead	Number of cuttings without roots					Total number without roots	Number of cuttings rooted												Total number rooted cuttings	Percentage rooted cuttings	Total number of roots	Average number of roots per rooted cutting	Total length of longest roots in mm	Average length of longest roots in mm
												only higher up				not only basal				at the basal cut								
						with basal decay	no callus	some callus	callus	callus-knob		basal decay	no callus	some callus	callus-knob	no callus	some callus	callus	callus-knob	no callus	some callus	callus	callus-knob						
1	30	untreated	—	—	—	—	—	4	1	—	5	—	—	—	1	—	9	6	2	2	2	2	1	25	83	181	7	727	29
2	30	water	0.75	—	—	—	2	7	5	—	14	—	—	—	—	1	4	2	—	2	6	1	—	16	53	92	6	256	16
3	30	α-n.a.a. 111.6 mg/l	0.65	72.5	3	8	2	—	—	—	10	13	—	—	—	4	—	—	—	—	—	—	—	17	57	384	23	508	30
4	30	,, 55.8 ,,	0.73	40.7	1	2	2	—	—	—	4	11	1	—	—	9	2	1	—	1	—	—	—	25	83	435	17	697	28
5	30	,, 27.9 ,,	0.70	19.5	—	2	5	—	—	—	7	2	1	—	—	15	3	2	—	—	—	—	—	23	77	354	15	770	33
6	30	β-n.a.a. 111.6 mg/l	0.72	80.4	—	2	8	1	—	—	11	1	1	—	—	9	8	—	—	—	—	—	—	19	63	271	14	454	24
7	30	,, 55.8 ,,	0.77	43.0	—	—	5	6	1	—	12	1	—	—	—	6	3	—	—	6	1	1	—	18	60	173	10	380	21
8	30	,, 27.9 ,,	0.80	22.3	—	—	5	2	—	—	7	2	1	—	—	4	9	1	—	3	1	1	1	23	77	176	8	446	19
9	30	α-n.p.a. 120 mg/l	0.58	69.6	—	1	7	—	1	—	9	—	1	—	—	8	—	1	—	8	3	—	—	21	70	143	7	327	16
10	30	,, 60 ,,	0.63	33.8	—	—	2	—	—	—	2	1	—	—	—	12	3	1	1	5	5	—	—	28	93	267	10	690	25
11	30	,, 30 ,,	0.57	17.1	—	—	3	1	—	—	4	—	1	—	—	6	10	2	—	1	5	1	—	26	87	184	7	510	20
12	30	β-n.p.a. 120 mg/l	0.55	66.0	—	5	13	—	—	—	18	3	—	—	—	2	—	—	—	6	1	—	—	12	40	80	7	215	18
13	30	,, 60 ,,	0.65	39.0	1	—	6	6	—	—	12	—	—	—	—	2	1	1	—	7	6	—	—	17	57	89	5	195	12
14	30	,, 30 ,,	0.62	18.6	—	—	2	4	4	4	14	—	—	—	—	—	1	1	—	—	12	1	1	16	53	61	4	287	18
15	30	α-n.b.a. 128.4 mg/l	0.70	89.9	2	2	10	—	—	—	12	—	2	—	—	6	2	—	—	5	1	—	—	16	53	85	5	300	19
16	30	,, 64.2 ,,	0.73	46.9	—	1	14	1	—	—	16	—	1	—	—	7	2	1	—	1	2	—	—	14	47	93	7	262	19
17	30	,, 32.1 ,,	0.68	21.8	—	4	16	2	—	—	22	—	—	—	—	5	1	—	—	2	—	—	—	8	27	42	5	104	13
18	30	β-n.b.a. 128.4 mg/l	0.73	93.7	—	4	5	—	—	—	9	7	3	—	—	8	—	—	—	3	—	—	—	21	70	130	6	334	16
19	30	,, 64.2 ,,	0.68	43.7	—	2	14	1	—	—	17	—	2	—	—	6	2	—	—	2	1	—	—	13	43	95	7	240	18
20	30	,, 32.1 ,,	0.67	21.5	1	—	2	4	—	—	6	—	—	—	—	6	5	1	3	3	5	—	—	23	77	208	9	615	27
21	30	β-i.a.a. 150 mg/l	0.65	68.3	1	6	1	—	—	—	7	6	—	—	—	14	2	—	—	—	—	—	—	22	73	946	43	840	38
22	30	,, 52.5 ,,	0.53	27.8	1	—	—	—	—	—	0	2	—	—	—	15	11	1	—	—	—	—	—	29	97	916	32	1250	43
23	30	,, 26.25 ,,	0.62	16.3	—	1	1	—	—	—	2	1	—	—	—	15	8	—	—	—	4	—	—	28	93	549	20	1221	43
24	30	β-i.b.a. 121.8 mg/l	0.58	70.6	—	1	—	1	1	—	3	6	—	—	—	20	1	—	—	—	—	—	—	27	90	803	30	732	27
25	30	,, 60.9 ,,	0.58	35.3	—	1	3	2	—	—	6	1	—	—	—	14	7	—	—	1	1	—	—	24	77	570	24	602	25
26	30	,, 30.45 ,,	0.68	20.7	—	—	3	—	—	—	3	—	—	—	—	15	7	2	—	2	—	1	—	27	90	534	20	843	31

acid per litre respectively. On account of the number of dead specimens and the occurrence of basal decay this concentration here also proves to be too high in most cases for the rooted as well as for the non-rooted cuttings. This was even more noticeable in the first experiment, in which case the material was slightly less ripened and therefore softer.

It is remarkable that in this case the water-control had a smaller percentage of rooting than the totally untreated cuttings.

From the table and from the photographs it may be concluded that β-i.a.a. is the most active, so that with the highest concentration of 6×10^{-4} mol./l even an average of 43 roots per cutting may be counted. It is followed by β-i.b.a. and α-n.a.a.; β-n.a.a. is clearly weaker than the α-isomer, whereas α-n.p.a. and α-n.b.a. only differ slightly (probably n.p.a. is slightly more active, see photographs) and find their place at the end of the series. β-N.p.a. and β-n.b.a. are practically inactive. (It may be noticed that here β-n.a.a. has much less influence than in the previous experiment; possibly the hardyness of the cuttings plays a part in this). Of course the sequence of activity of the compounds examined here need not be exclusively the result of their structure and activity but may perhaps also be due to a specific sensibility of the object tested in these experiments. Therefore it might be advisable to make such trials on one or more other plants too.

Just as in the first experiment we again notice that by β-i.a.a. the longitudinal growth of the roots is considerably stimulated. The effect of α-n.a.a. in this respect is weaker than was noted in the first experiment.

The following remarks can be made about the rooting process in the following groups:

group 1: normal rooting, roots white with yellowish tips.
,, 2: weaker rooting, as a whole shorter than in group 1; roots normal.
,, 3: thick, short roots; more heavier rooting than group 1.
,, 4: also rather thick roots; somewhat longer than in group 3.
,, 5: roots normal; slightly longer than the roots of group 4, approximately as in group 1 (though of course more abundant).
,, 6: roots normal; somewhat heavier rooting than in group 1.
,, 7: roots normal; amount of rooting slightly less than in group 1.
,, 8: roots normal; rooting as in group 7.
,, 9: roots normal; rooting as in group 2.
,, 10: roots normal, amount of rooting as in group 1; probably slightly shorter.
,, 11: roots normal, approximately as in group 9; roots shorter than in group 1.
,, 12: roots normal; weaker rooting than in group 2.
,, 13: roots normal; still weaker than in group 12.
,, 14: roots normal; as in group 13.
,, 15: roots slightly thickened; amount of rooting sometimes as in group 1; as a whole less than group 2.
,, 16: roots slightly thickened; rooting as in group 15.
,, 17: roots normal; very weak rooting.
,, 18: roots normal; rooting as in group 2.
,, 19: roots normal; weaker rooting than in nr 18.

group 20: roots normal; stronger rooting than groups 18 and 19; approximately as in group 1.
„ 21: roots thinner than in group 1; heavy rooting, length of roots as in group 1. Too high a concentration shows itself very occasionally by the appearance of aerial roots at the top of the cutting (see fig. 1).
„ 22: roots still thin; heavy rooting, roots longer than in group 21.
„ 23: roots normal; heavy rooting, slightly longer than in group 22.
„ 24: roots normal; heavy rooting, shorter than in group 1.
„ 25: roots normal; heavy rooting, as in group 24.
„ 26: roots normal; heavy rooting; roots longer than in group 25, approximately as in group 1.



Fig. 1. Formation of aerial roots at the top of a cutting due to treatment with indole acetic acid (105 mg/l).

Summary.

On cuttings of × *Viburnum Burkwoodii* Burkw. the activity of the potassium salts of α- and β-naphthalene acetic acid, α- and β-naphthalene propionic acid and α- and β-naphthalene butyric acid has been compared with that of β-indole acetic acid and β-indole butyric acid. The starting concentration was always 6×10^{-4} mol./1 whilst beside this two dilutions (3 and 1.5×10^{-4} mol./1) have been examined.

The highest concentration proved to be too high in most cases, resulting in dead specimens and basal decay. β-I.a.a. is the most active substance after which come β-i.b.a. and α-n.a.a.; β-n.a.a. has a definitely weaker activity than the α-isomer; α-n.p.a. and α-n.b.a. do not differ greatly, their activity is very weak and weak respectively. Their β-isomers are practically inactive.

Thus a typical difference between indole- and naphthalene-derivatives could be established. In the former case lengthening of the side-chain did not greatly decrease the activity; however, in the latter case it did. We could establish, as did the other investigators that the β-substituted naphthalene-derivatives are less active than the α-isomers.

Of the most active compounds β-i.a.a. and α-n.a.a. also stimulate the longitudinal growth of the roots.

One should take into account a possible specific sensibility of a testing subject in comparing the activity of synthetic growth substances so that a repetition with cuttings from other plants would be advisable.

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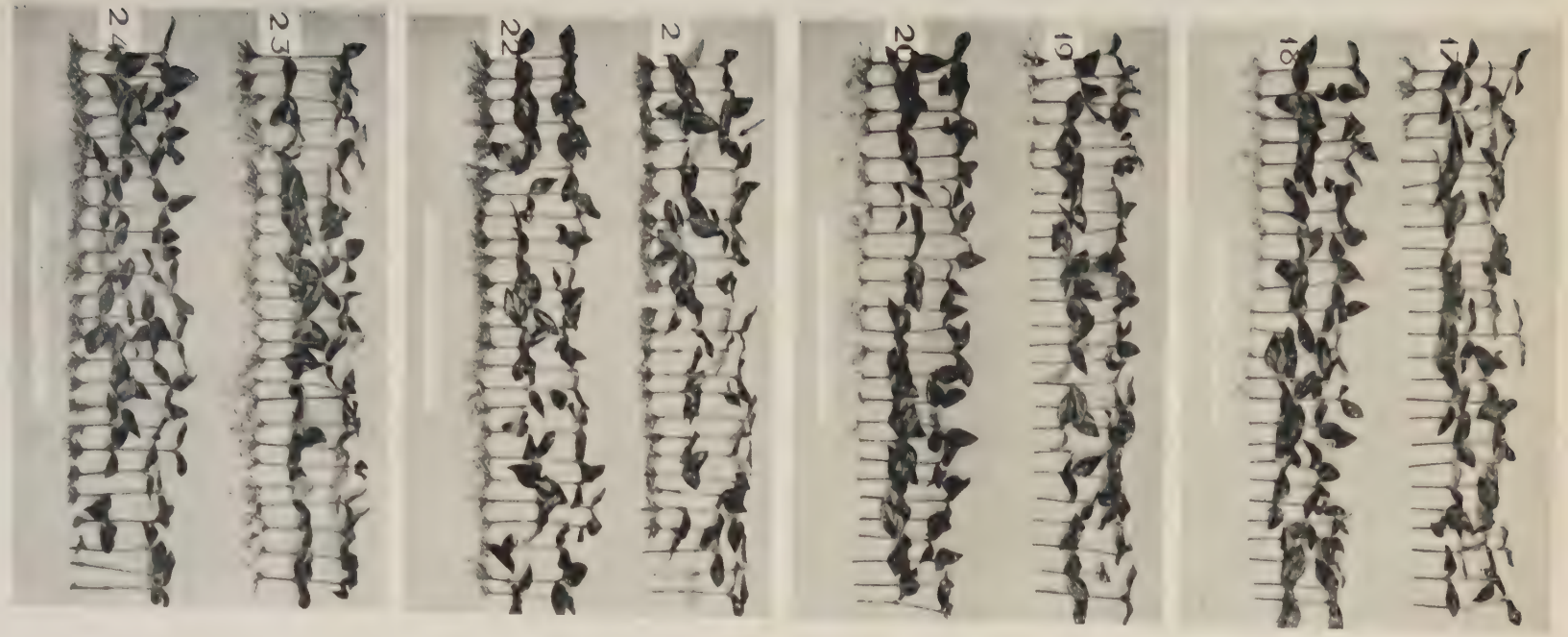

1
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11
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15
16



1 = untreated
2 = water
3, 4, 5 = α-n.a.a., resp. 111.6, 55.8, 27.9 mg/l
6, 7, 8 = β-n.a.a., resp. 111.6, 55.8, 27.9 mg/l
9, 10, 11 = α-n p.a., resp. 120 , 60 , 30 mg/l
12, 13, 14 = β-n.p.a., resp. 120 , 60 , 30 mg/l
15, 16, 17 = α-n.b.a., resp. 128.4, 64.2, 32.1 mg/l
18, 19, 20 = β-n.b.a., resp. 128.4, 64.2, 32.1 mg/l
21, 22, 23 = β-i.a.a., resp. 105 , 52.5, 26.25 mg/l
24, 25, 26 = β-i.b.a., resp. 121.8, 60.9, 30.45 mg/l



Chemistry. — *Intermolecular forces in soap crystals.* By V. VAND and J. H. DE BOER.

(Communicated at the meeting of October 25, 1947.)

§ 1. Soap molecules consist of long hydrocarbon chains terminating in polar carboxyl ions carrying single negative charges. The negative charges are neutralised by positive charges of metal ions. Soaps of univalent metals are therefore binary compounds of the type AB, like the alkali halides. The long hydrocarbon tails of the negative ions, however, make it impossible for them to form three-dimensional ionic lattices. A two-dimensional lattice of the metal- and carboxyl-ions results. There may even be a double ionic layer formed, but the presence of the long hydrocarbon chains attached to the negative ions make formation of more than a double ionic layer impossible. The chains stick out on both sides of the ionic lamella of one or two ions thick and they shield it on both sides from the addition of more ions. A bimolecular lamella is thus formed. In the real crystal more such lamellae may be on top of each other but it is easily seen that the forces between successive bimolecular lamellae must be of a far weaker nature than the forces prevailing in the lamellae.

The arrangement is similar to that in other so-called layer lattices (boric acid, talcum, cadmium iodide, graphite) where layers of molecular dimensions are found held together in two directions by strong forces — ionic or homopolar — whereas in the third direction perpendicular to the layers only relatively weak VAN DER WAALS' forces are operating. Such crystals split very easily along planes parallel to the layers, a property so prominent that the splitting takes place when the crystals are rubbed between the fingers, giving rise to a "fatty" touch. The same distribution of forces favours the two-dimensional growth of these crystal lattices to relatively large thin plates.

Turning to the orientation of the hydrocarbon chains which are on both sides of an ionic layer, it is easily seen that there is hardly room for them unless they stick out, forming parallel rods pointing away from the central ionic layer at right angles or anyhow forming steep angles with it.

The thickness of the lamellae in monoclinic soap crystals, as given by the "long spacing", d, is in accordance with this picture, the "long spacing" being:

$$d = c \sin \beta = (2N_1 l_1 + l_0) \sin \tau,$$

where c is the dimension of the edge of the lattice, β is the monoclinic angle, N_1 is the number of $-CH_2-CH_2$-groups in the molecules, l_1 the increase in length of the molecule per $-CH_2-CH_2$-group, l_0 an additional length arising from the end groups on both sides of the molecules and τ

the angle of tilt of the axis of the molecules, measured from the ionic plane. It should be noted that τ can be quite different from β.

§ 2. We will now direct attention to the so-called "short spacings" in the crystals, arising from X-ray reflections from planes determined by the side distances of the molecules within the lamellae. If the middle ionic layer were not present, the hydrocarbon chains would nevertheless, due to their mutual VAN DER WAALS' forces, be orientated, forming parallel rods. Experimentally such an arrangement is known to be present in the crystals of saturated hydrocarbons (paraffins). We may expect that owing to the increase of the total VAN DER WAALS' forces between neighbouring chains, the short spacings will decrease with increasing length of the chains [1]). This can easily be shown in the following way. Suppose we have only two single $-CH_2$-groups (of different molecules) and we study their mutual forces. The potential curve giving the mutual energy of such a pair as a function of the distance is given by the equation:

$$E = -\frac{C_1}{d^6} - \frac{C_2}{d^8} + \frac{B}{d^n}.$$

The first term stands for the dipole-dipole effect of the VAN DER WAALS' forces, the second for the dipole-quadrupole effect, whilst the third term stands for the repulsive action caused by the interpenetration of electronic clouds. We have neglected the quadrupole-quadrupole term: this term is small and does not affect our result in any way. It is a known empirical fact, that the first term $\frac{C_1}{d^6}$, when taken separately and when evaluated either with the approximation formula of London [2]) or that of SLATER and KIRKWOOD [3]) gives the right value of D (the dissociation energy) for all practical purposes [4]), which means that the second and third term balance each other almost completely at the point were $d = d_m$ (the distance between the centres at equilibrium). The numerical value of D for a CH_2 group in a chain is of the order 0.25 Kcal/mole (see footnote 7).

Making use of this, we write:

$$C_1 = D d_m^6,$$
$$B = C_2 d_m^{n-8}.$$

Moreover we have the following condition for the equilibrium:

$$\left(\frac{dE}{dd}\right)_{d=d_m} = \frac{6C_1}{d_m^7} + \frac{8C_2}{d_m^9} - \frac{nB}{d_m^{n+1}} = 0,$$

[1]) A similar effect is observed with respect to the distance between the two-dimensional hexagonal layers in graphite. This distance, which is governed by VAN DER WAALS' forces increases from 3.345 Å in normal graphite to 3.6 Å for extremely small crystals (see J. H. DE BOER, Rec. Trav. Chim. **59**, 826 (1940).

[2]) F. LONDON, Z. Physik, **63**, 245 (1930); Z. physikal Chem. **11** B, 222 (1931).

[3]) J. C. SLATER and J. G. KIRKWOOD, Phys. Rev. **37**, 682 (1931).

[4]) J. H. DE BOER, Trans. Far. Soc. **32**, 10 (1936).

which leads to

$$C_2 = \frac{6}{n-8} C_1 d_m^2 = \frac{6}{n-8} D d_m^8.$$

In this way we can express all the constants C_1, C_2 and B in terms of d_m and D, and the resulting form of the potential curve is therefore:

$$E = -D\left[\left(\frac{d_m}{d}\right)^6 + \frac{6}{n-8}\left(\frac{d_m}{d}\right)^8 - \frac{6}{n-8}\left(\frac{d_m}{d}\right)^n\right].$$

E here is therefore a function of d_m/d and the whole form of the curve is only dependent on the choice of n. From a comparison of the equilibrium distances of molecules, bound by VAN DER WAALS' forces, and their collision diameters, we may derive that in many practical cases n may be taken as 20. In Figure 1 the curve for $n = 20$ is given.

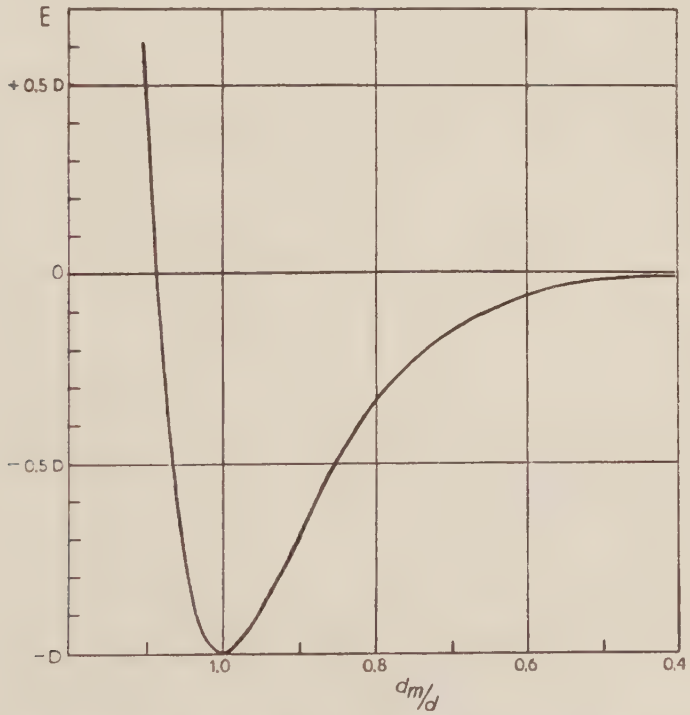


Fig. 1. Curve for mutual VAN DER WAALS' potential energy of a pair of CH_2 groups.

We now take two bi-atomic molecules AB and CD in a position parallel to each other, and we assume that the distance between the atoms A and B in the molecule AB and between C and D in the molecule CD is a third of the distance d_m which we should find if the atoms A and C could attract each other separately. With the aid of the curve in Figure 1 it is now a relatively easy matter to asses the mutual energy of the two molecules as a function of their variable distance d. In such a way we may obtain a potential curve for this pair of bi-atomic molecules, and similarly

we can construct potential curves for straight molecular chains consisting of more than two atoms. In Figure 2 the curves for molecules consisting of 1, 2 and 3 atoms are given, and in Figure 3 those for 1, 2, 4, 6 and 8.

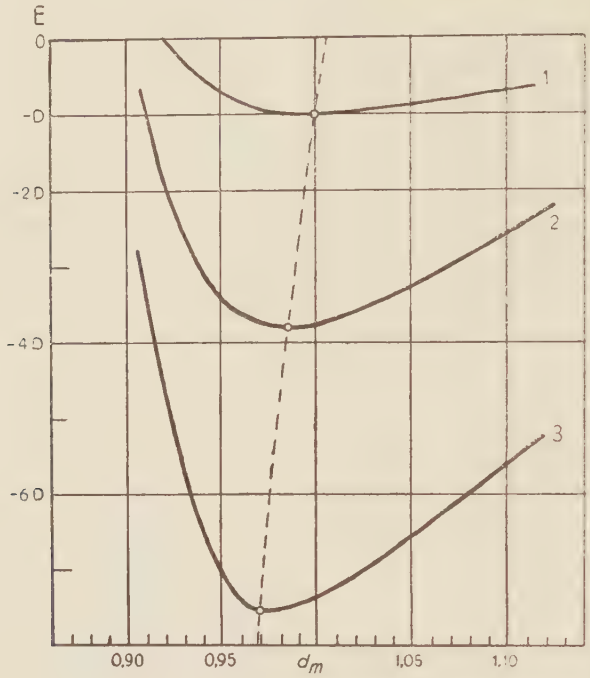


Fig. 2. Potential energy curves for pairs of molecules, each consisting of 1, 2 and 3 atoms or CH_2 groups.

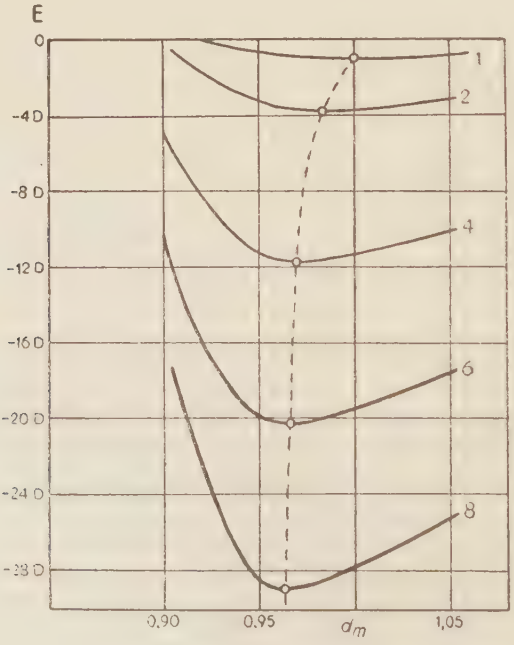


Fig. 3. Potential energy curves for pairs of molecules, each consisting of 1, 2, 4, 6 and 8 atoms or CH_2 groups.

The distance is expressed in fractions of the distance d_m, which is the minimum for the two single atoms, and the energy is expressed in D as a unit. We see that as a result of the additivity of the VAN DER WAALS' forces the equilibrium distance of these curves decreases with increasing chain length. This is equally true for hydrocarbon chains, composed of CH_2 groups, and the top curve in Figure 5 gives the decrease of the equilibrium distance as a function of $1/N$, where N is the number of CH_2 groups in the molecule.

§ 3. Experimentally such a decrease with increasing chain length is indeed found. Even when one takes the earliest figures of MÜLLER and SAVILLE [5]) of 1925 for the dimensions of the unit cells of paraffin hydrocarbons this decrease is clearly demonstrated. We have taken available figures from the literature and calculated the cross-sections per molecule and their probable errors by the GAUSSIAN method of least squares assuming that the cross-section S may be expressed by the equation $S = S_\infty + \frac{Q}{N}$, where Q is a constant and S_∞ is the cross-section of an infinite chain. In doing this we treated the hydrocarbons with odd numbers of carbon atoms separately from those with even numbers, for it is possible that their crystal structures differ.

If the cell edges a, b, of a monoclinic crystal are known, then we can calculate S from the following relation:

$$S = \frac{ab \sin \tau}{k}$$

where k is a multiplicity factor, which is equal to the number of molecules occupying one cell cross-section ab, and τ is the angle of tilt of the molecules. Paraffins crystallise in two forms; one is orthorhombic, the other is monoclinic. Only the orthorhombic form is considered here, for which $\tau = 90°$, $k = 2$. The following results were obtained:

Odd series (N odd):

$$S_\infty = (18.12 \pm 0.10),\ Q = 9.68 \pm 2.66,\ \text{in } kX^2.\ ^6)$$

Even series (N even):

$$S_\infty = (18.00 \pm 0.14),\ Q = 10.00 \pm 4.00,\ \text{in } kX^2.$$

The literature data and calculated values are given in Tables I and II.

It can be seen that the value of Q is positive and significantly different from zero, and so the contraction seems to be real, and both S_∞ and Q are the same within experimental error for odd and even series. This allows us to combine the results, and we obtain for the paraffins finally:

$$S_\infty = (18.08 \pm 0.08),\ Q = 9.78 \pm 2.40,\ \text{in } kX^2.$$

[5]) A. MÜLLER and W. B. SAVILLE, J. Chem. Soc. **127**, 599 (1925).

[6]) According to new values, 1 Å = 1.00202 kX. The older values of lattice constants were as a rule measured in kX, although denoted as Å.

It should be noted that BUNN's (1939) value of $S = 18.24$ for hydrocarbons having N between 130 and 1000, is a value which according to our relation would have S for $N = 62$, but it certainly lies within the experimental error of BUNN's measurements.

The effect can be expressed by the equation:

$$S = 18.08\,(1 + \frac{0.54}{N}) \text{ in } kX^2,$$

and is even greater than would follow from our potential curves of Figures 2 and 3, or from the upper curve of Figure 5. Our calculation, however, was only for pairs of molecules; the effect will be increased in forming a lattice from a great number of molecules.

§ 4. In the soaps, however, we have in the first place the two-dimensional ionic sheet of the carboxyl and metal ions. If we take the sodium or silver soaps we might expect that the strong forces between such small cations and the anions in this ionic sheet would enfore a smaller cross-section on to the molecules, thus compressing them. The energy of combination for the ionic sheet would, moreover, be very much higher than that for the hydrocarbon chains. Consequently the potential curve of such an ionic pair would show a minimum at a far lower energy content and at a smaller distance. Let us assume the equilibrium distance to be 0.9 of that of our potential curve of VAN DER WAALS' attraction of two centres (Figure 1). The energy content of such

TABLE I. Paraffins N Odd.

N	a	b	c	$S = \frac{1}{2} ab$	S calc.	Reference
19	7.55	5.01		18.91	18.63	MÜLLER (1932)
23	7.435	4.97		18.48	18.53	„
25	7.41	4.96		18.38	18.50	„
27	7.40	4.93		18.24	18.48	„
29	7.42	4.94	77.2	18.33	18.45	MÜLLER (1932)
29	7.45	4.97		18.51	18.45	MÜLLER (1928)
31	7.49	4.97	41.11	18.61	18.43	SCHOON (1938)
31	7.42	4.95		18.36	18.43	THIESSEN (1937)
31	7.45	4.94		18.40	18.43	RIGAMONTI (1936)
31	7.40	4.93		18.24	18.43	MÜLLER (1932)
35	7.43	4.97		18.46	18.39	HENGSTENBERG (1928)
> 130	7.40	4.93		18.24	18.12	BUNN (1939)

MÜLLER, A., Proc. Roy. Soc., A **138**, 514 (1932).
MÜLLER, A., Proc. Roy. Soc., A **120**, 437 (1928).
SCHOON, T., Z. phys., chem., B **39**, 385 (1938).
THIESSEN, T. A. and T. SCHOON, Z. phys. chem., B **36**, 216 (1937).
RIGAMONTI, R., Gazz. chim. ital., **66**, 174 (1936).
HENGSTENBERG, J., Zeits. Krist., **67**, 583 (1928).
BUNN, C. W., Trans. Far. Soc., **35**, 482 (1939).

TABLE II.

Paraffins N Even.

N	a	b	$S = \frac{1}{2} ab$	S calc.	Reference
18	7.56	5.00	18.90	18.55	MÜLLER (1927)
24	7.41	4.94	18.30	18.42	MÜLLER (1932)
26	7.415	4.94	18.31	18.38	„
30	7.452	4.965	18.50	18.33	KOHLHAAS (1938)
30	7.51	4.955	18.23	18.33	SCHOON (1938)
30	7.33	4.92	18.03	18.33	MÜLLER (1932)
34	7.40	4.95	18.32	18.29	„
44	7.33	4.93	18.07	18.23	„
> 130	7.40	4.93	18.24	18.00	BUNN (1939)

MÜLLER, A., Proc. Roy. Soc., A **114**, 542 (1927).
MÜLLER, A., Proc. Roy. Soc., A **138**, 514 (1932).
KOHLHAAS, R., and K. SOREMBA, Z. Krist, A **100**, 47 (1938); Z. anorg. Chem., **51**, 483 (1938).
SCHOON, T., Z. phys. Chem., B **39**, 385 (1938).
BUNN, C. W., Trans. Far. Soc., **35**, 482 (1939).

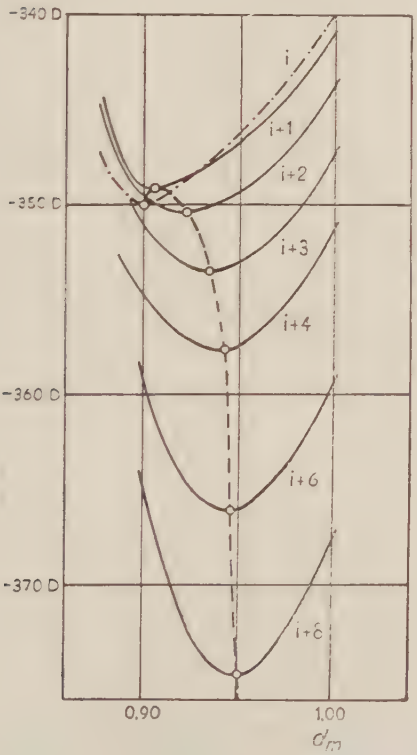


Fig. 4. Potential energy curve for an ionic pair (i) and for an ionic pair in which each ion is combined with a chain containing 1, 2, 3, 4, 6 and 8 atoms or CH_2 groups.

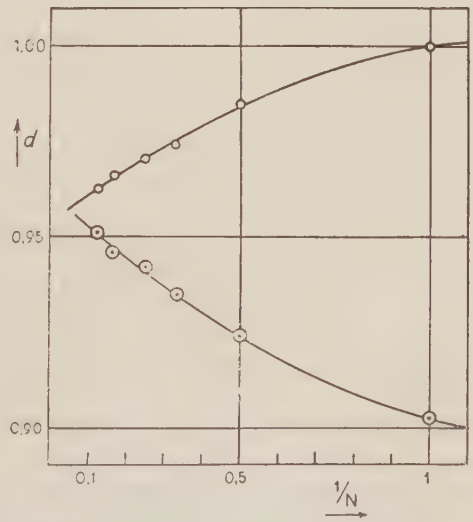


Fig. 5. Calculated effect of the chain length on the equilibrium distance of between the chains. Top curve: VAN DER WAALS' forces alone. Bottom curve: Ionic forces combined with VAN DER WAALS' forces.

an ionic pair will be about 350 times [7]) the value of that of D in Figure 1. With these two figures for the fixation of the minimum of the potential curve of the ionic pair, namely distance is 0.9 d_m, and energy is 350 D, the potential curve for the ionic pair can be easily constructed; it is the curve indicated with the letter i in Figure 4.

In order to see what happens in soap crystals, we have now to combine this curve with the set of curves of Figures 2 and 3. The result is shown in Figure 4. We obtain curves indicated with $i = 1$ for 1 atom in the chain to $i = 8$ for 8 atoms in the chain. Although the VAN DER WAALS' forces are smaller than the ionic forces and are even of another order of magnitude, the influence of the VAN DER WAALS' curves on the equilibrium distance is paramount. Consequently we must expect here a dilation of the lattices when we proceed towards longer length of the hydrocarbon chains, as is indicated in the lower curve of Figure 5 where the distance of these minima are plotted against $1/N$.

§ 5. The experimental figures for the cross-sections of silver soaps and γ-sodium soaps were calculated as follows:

The volume V_1 occupied by one molecule is obtained from the density ϱ by means of the following relation:

$$V_1 = \frac{1.6502\, M}{\varrho},$$

where M is the molecular weight and V_1 is measured in cubic kX units. There are 2 molecules along the c-cell edge in these soaps.

$$\text{The expression } S' = \frac{2\, V_1}{d},$$

where d is the value of the long spacing, which can be directly measured, gives the surface areas per molecule measured in a plane parallel to (001). The surface area S of a molecule measured in a plane perpendicular to its axis is thus:

$$S = \frac{2\, V_1 \sin \tau,}{d},$$

---

[7]) The VAN DER WAALS' interaction energy of two $-CH_2-$ groups at an equilibrium distance of 4.0 Å is 0.25 Kcal/mol, as calculated with the formula of SLATER and KIRKWOOD, (loc. cit 3)

$$E = \frac{163 \times 10^{-12}\, n^{\frac{1}{2}}\, \alpha^{\frac{3}{2}}}{a^6} \text{ Kcal/mol.},$$

where n is the number of outer electrons and α is the polarizability; we took $n = 6$ and $\alpha = 1.9 \times 10^{-24}$ cm^3. An ionic pair at a distance 3.6 Å has an interaction energy of 84 Kcal/mol, as calculated with

$$E = \frac{\varepsilon^2}{d}\left(1 - \frac{1}{n}\right),$$

where $n = 12$, the exponent of the repulsion term.

where τ is the angle of tilt of the chains measured from the (001) plane.

The angle of tilt τ can be calculated from the increment of the long spacing on increasing the chain length, assuming tetrahedral angle and carbon to carbon bond length. It should be noted that the calculation assumes (*a*) constancy of the parameters of the unit cell not directly affected by the chain increase, including constancy of the angle of tilt, (*b*) constancy of the tetrahedral angle and carbon to carbon distance in various compounds. For the carbon to carbon distance, a value of $1.54kX$ is usually assumed, whilst for the angle between the C—C bonds the tetrahedron valency angle of 109° 28′ is taken. This angle, however, may be strained to various degrees, so that the angle of tilt cannot be calculated with any degree of accuracy. However, this does not affect our results much.

Table III contains the data for γ-sodium soaps and Table IV for silver

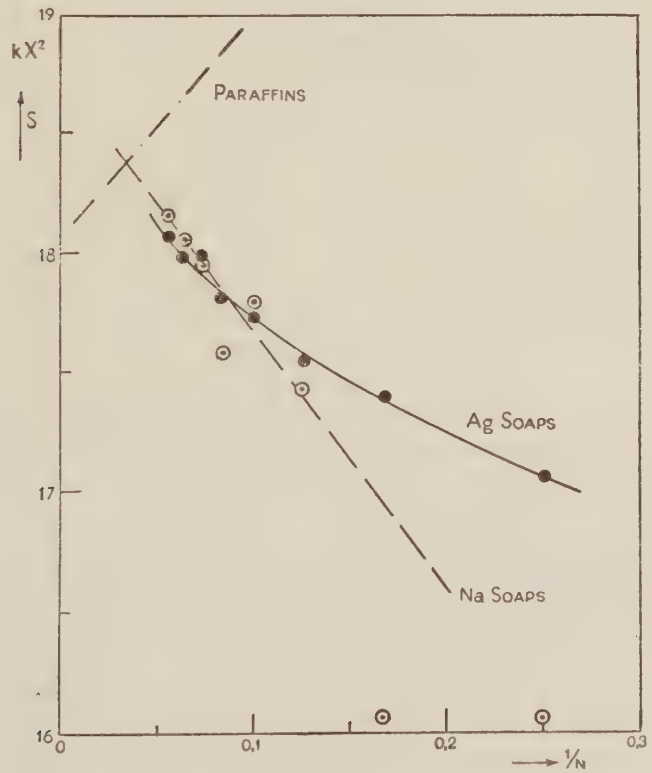


Fig. 6. Cross-section S of the chains in paraffins, in silver soaps and in sodium soaps as a function of the chain length.

soaps. The values for S for both these soaps are plotted in Figure 6. We see, indeed, that the cross-section increases with increasing chain length as predicted from the co-operation of ionic and VAN DER WAALS' forces.

§ 6. Let us direct our attention for a few moments to the mutual orientation of the molecules in the layers. There are several possibilities of

TABLE III.

Sodium soaps.

Form γ

$\sin \tau = 0.8461, \quad \tau = 57° 48'$

Soap	N	M	ϱ	V_1	d	S
Sodium butyrate	4	110.10	1.340	135.6	14.30	16.04
,, caproate	6	138.16	1.290	176.7	18.65	16.03
,, caprylate	8	166.21	1.170	234.4	23.15	17.42
,, caprate	10	194.24	1.135	282.4	27.30	17.80
,, laurate	12	222.31	1.110	330.5	31.80	17.58
,, myristate	14	250.36	1.090	379.0	35.70	17.96
,, palmitate	16	278.42	1.075	427.4	40.00	18.08
,, stearate	18	306.47	1.060	477.1	44.40	18.18

TABLE IV.

Silver soaps.

$\sin \tau = 0.9537, \quad \tau = 72° 30'$

Soap	N	M	ϱ	V_1	d	S
Silver butyrate	4	195.0	2.403	133.9	14.95	17.08
,, caproate	6	223.0	2.043	180.1	19.74	17.40
,, caprylate	8	251.1	1.838	225.4	24.48	17.56
,, caprate	10	279.1	1.686	273.2	29.36	17.74
,, laurate	12	307.2	1.588	319.2	34.18	17.81
,, myristate	14	335.2	1.506	367.3	38.93	17.99
,, palmitate	16	363.3	1.452	412.9	43.82	17.97
,, stearate	18	391.3	1.403	460.2	48.56	18.07

orientation, in all of which the axes of the molecules are parallel. In a parallel row, for instance, the *planes* through the zig-zag configurations of the hydrocarbon chains may be parallel (see case *a* on Figure 7) or in alignment (*b*), or they may alternate (*c*). They may also be in general positions (*d*), or finally, they may even rotate (*e*) at higher temperatures. If they are in alignment it may easily be seen that such a mutual orientation of the hydrocarbon chains is not unfavourable even from the point of view of the anisotropy of the polarisability. There are hardly any cases of parallel axes of greatest polarisability then. When, however, the planes of the zig-zags are parallel, the attraction due to the VAN DER WAALS' forces at the same distance will be smaller because of the unfavourable position with regard to the anisotropy of polarisability. We see that there are distinct possibilities that even in the bimolecular lamellae the forces are much stronger in one direction than in another. In such cases the plates can easily be split to needles or fibres. Many soap lattices grow easily as needles or fibres. In all such cases we find that the molecules are perpendicular or nearly prependicular to the needle or fibre axis, an orientation quite different therefore from that found in fibres of high polymers where the molecules are stretched in the direction of the

fibre axis. An orientation of the molecules perpendicular or nearly perpendicular to the needle or fibre axis may always be expected from such

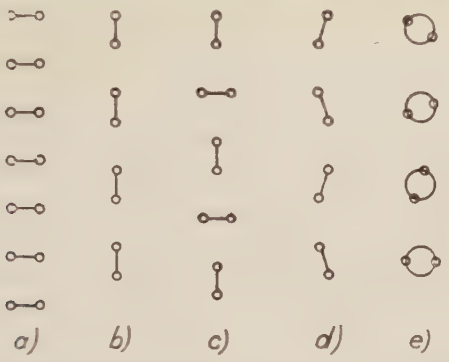


Fig. 7. Possible arrangements of the hydrocarbon chains in a parallel row. Cross-sections perpendicular to the axes of the chains are shown for chains (*a*) parallel, (*b*) in alignment, (*c*) alternating, (*d*) in general positions, (*e*) rotating.

lattices which, because of the forces operating in them, might not only be called layer lattices but even rod lattices. Many long shaped molecules of dyestuffs crystallise in this way.

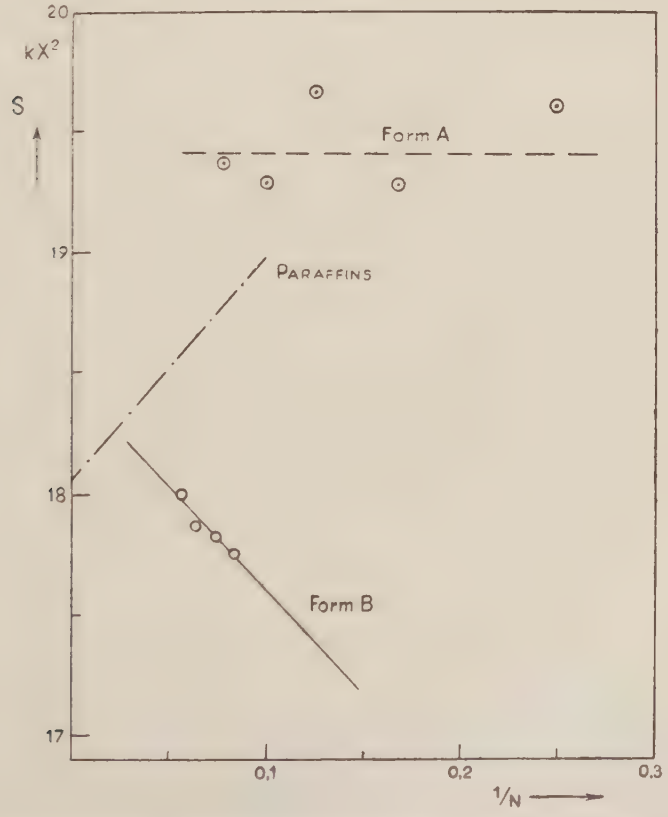


Fig. 8. Cross-section *S* of the chains in paraffins and in two forms A and B of potassium soaps as a function of the chain length.

§ 7. Usually it is accepted, though without sufficient proof, that long chain molecules such as soap molecules are all parallel to each other in the crystalline state. This, however, need not be so. Recently VAND, LOMER and LANG [8]) have found that in the so-called *A* form of potassium caprate rows of molecules with parallel axes alternate with other rows of molecules, the directions of the axes of which, however, are such that they cross the directions of the axes in the first set of rows. A row of hydrocarbon chains which stick out from the ionic layer with an angle of tilt of 58° is bordered on both sides by another row of hydrocarbon chains sticking out from the ionic layer with an angle of tilt of 122°. The two directions of chains, therefore, cross each other with an angle of 66°.

It is remarkable that this *A*-form, found in potassium soaps of up to 12 C-atoms does not show the increase of cross-section with increasing chain length. The so-called *B*-form, found with potassium soaps of 12 C-atoms and higher, however, shows the same effect as the sodium- and silver-soaps, as is shown in Figure 8.

Acknowledgments.

The authors wish to thank the Directors of Lever Brothers & Unilever Limited for permission to publish these results.

Summary.

It is shown from theory and by experiment that in a series of homologous long chain compounds, such as paraffins or soaps, cross-sections of the hydrocarbon chains vary with the chain length, so that the side spacings and the shorter edges of the unit cells do not remain strictly constant.

These effects are due to the interplay of the VAN DER WAALS' forces between the chains and ionic forces within the ionic parts of the structures.

With paraffins, when only VAN DER WAALS' forces are operating, the cross-sections per molecule decrease with increasing chain-length. In the case of sodium- and silver-soaps, however, a smaller cross-section is caused by the ionic forces, which is increased with increasing chain-length.

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[8]) V. VAND, T. R. LOMER and A. LANG, Nature **159**, 507 (1947).

Chemistry. — *On the Graphical Representation of Chemical processes.* By L. G. M. Baas Becking D. Sc., Ph. D., Buitenzorg.

(Communicated at the meeting of September 27, 1947.)

1. G. G. Stokes, in 1891, proposed a graphical representation of the properties of three components by means of an equilateral triangle that has been in general use ever since.

The method is based upon a property of the equilateral triangle; that the sum of the distances of any point within this triangle to the sides is constant. In this paper the method will be used to represent by points on an equilateral triangle compounds of C—H—O, N—H—O, S—H—O, respectively.

2. In order to take into account the valencies of the atoms, the corners of the triangles represent C—4H—2O (figure 1), 2N—6H—5O (figure 2),

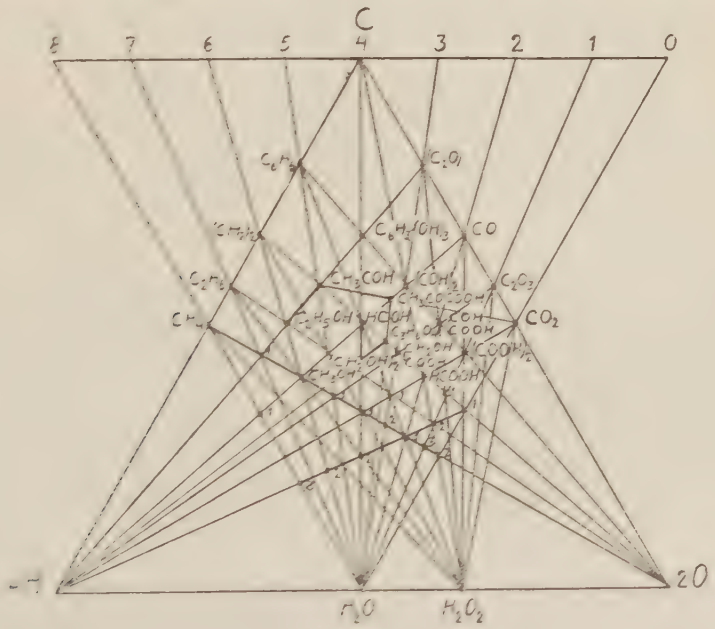


Fig. 1.

S—2H—3O (figure 3) respectively. To construct the point H_2O a line parallel and equal to the base is run through the apex (C, N or S) of the triangle. This line is devided into eight equal parts, in recognition of the fact that there are 8 H-atoms required to reduce the highest oxidation-stage (CO_2, N_2O_5, SO_3) of the element to the highest stage of reduction (CH_4, NH_3, H_2S).

For carbon, a tetravalent atom, the carbon-point is situated in the middle of the line through the apex, for nitrogen, the N-point is situated at $\frac{3}{8}$ of the distance, for sulphur the S-point is situated at $\frac{1}{4}$ of this distance. This represents, for Carbon, the valencies 4 and 4, for Nitrogen, the valencies 3 and 5 and for Sulphur, the valencies 2 and 6.

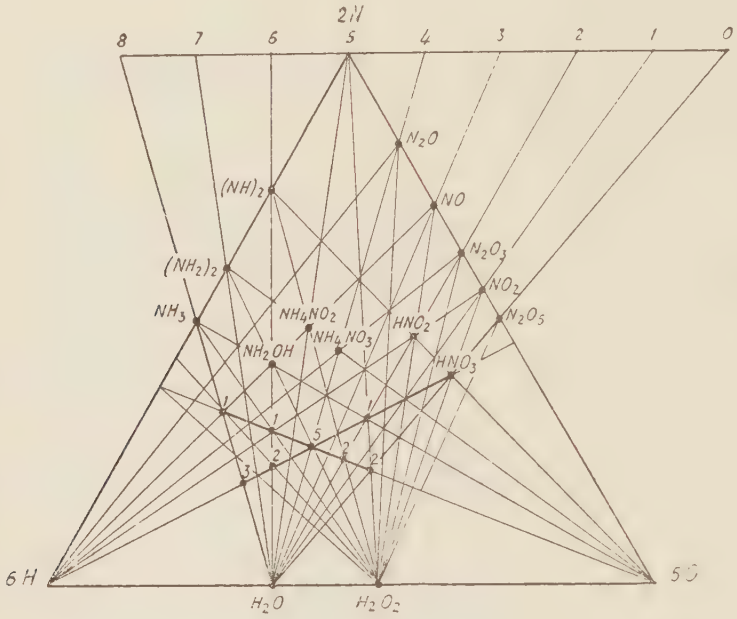


Fig. 2.

By drawing lines from the ends of the upper graduated line through the middle of the adjoining sides of the triangle, these two lines meet at a point on the base line, which point represents H_2O. The geometrical proof will be omitted here.

From the point H_2O lines are drawn to the subsequent marks on the graduated line, from 0 to 8. The line H_2O—8 represents the highest reduction-stage of the element, the line H_2O-zero the highest oxidation-stage.

3. In order to represent a compound by a point on the triangle we could calculate the percentage composition. It is, however, more accurate, and at the same time more illuminating, to construct this point.

To construct the point CO, for instance we, first of all, determine its reduction-stage (see fig. 1). We consider a C-atom equivalent to 4H and an oxygen atom to —2H.

The reduction stage of CO is, therefore, $4 - 2 = 2$. Now we join point 2 on the top-horizontal line with the point H_2O. As point CO is, obviously, situated on the line C—2O, we find point CO at the intersection of both lines.

For SO_2 (see fig. 3) the reduction stage is $6 - 2 \times 2 = 2$, meaning that it takes 2H atoms to change it from zero-level (SO_3). Joining point 2 with point H_2O we obtain point SO_2. In a similar fashion we find the reduction-stages for

$S_2O_5 = 2 \times 6 - 5 \times 2 = 2$ or, per S-atom, 1.
$S_2O_3 = 2 \times 6 - 3 \times 2 = 6$ or, per S-atom, 3.
$SO = 6 - 2 = 4$.
$S = 6$.
$H_2S = 2 + 6 = 8$.
$(HS)_2 = (2 + 12)/2 = 7$.

After joining these points with the H- and O-corners, we obtain a grid which enables us to construct the position of any compound containing only S—H and O.

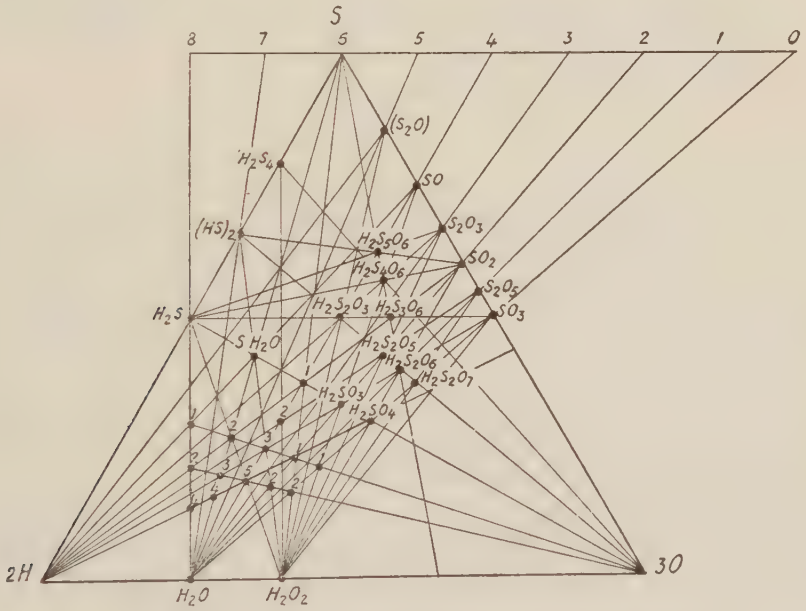


Fig. 3.

Point H_2SO_4 lies obviously on the line SO_3 and H_2O, and also on the line H_2S and O. The point is found at the intersection of both lines. Sulphoxylic acid, H_2SO, may be split into S and H_2O, but also into H_2S and O. H_2SO_3 is found at the intersection of H—SO_3 and H_2S—O. The other points are found in a similar way. H_2O_2 is constructed by substracting SO_2 from H_2SO_4, and as these points are widely apart on the triangle, this insures sufficient accuracy. Any point may be, of course, constructed in many different ways.

4. Considering the Carbon-diagram, one may note that points moving on the bundle of lines through the corner-point 4H represent hydro-

genations and dehydrogenations; points moving on the lines emanating from corner 20 represent oxidations and desoxidations; and it is clear that hydratations and dehydratations are represented on a bundle through the point H_2O.

5. Isomers are represented by the same point, this is the greatest drawback of the method, and also substances like lactic acid and glucose cannot be told apart. In the latter case the hydrates, however, are represented by different points.

6. Homologous series are represented by points on the same line. The monobasic acids may all be found on the line running from corner 20 to ethylene, the simple alcohols on the line connecting water and ethylene, a.s.o.

7. In physiology the respiratory quotient is defined by the molecular ratio CO_2 produced and oxygen consumed in complete combustion. For a compound $C_nH_mO_p$ this quotient is equal to

$$RQ = \frac{4n}{4n + m - 2p}$$

Now the stage of reduction of a compound is indicated in the diagram by lines passing through the point H_2O. All points on the line H_2O—C, for example, have the reduction-stage 4. Now this reduction-stage equals $(4n + m - 2p)/n$ and, therefore, $RQ = 4/E$. All substances, therefore, represented by points on the same line running through the point H_2O have the same respiratory quotient.

8. The heat of combustion of a carbon compound is, at first approximation, proportional to its stage of reduction E, neglecting the effects of cyclic structures and of double- and triple-bonds.

$\triangle H = kE$. If Hydrogen is oxidized to water $H_2 + 1/2O_2 = H_2O$. $\triangle H = k\,(4n + m - 2p)$, in which n and p equal zero. Therefore $\triangle H = 2k$ and the value of k should be approx. 26.2 K. Cal.

When we write, therfore

$$\triangle H = 26.2\,(4n + m - 2p)\text{ k. cal.}$$

we are able to determine, in first approximation, the heat of combustion of compounds containing only C, H and O.

9. Of five hundred compounds, taken from LANDOLT's tables, the heat of combustion was determined by means of the simple expression developed above. In 55 % of the cases, the error was $\pm$ 1 % or less, in 24 % of the cases, the error was between 1—5 %, in 11 % of the cases the error was between 5—10 %.

Deviations occurred especially in small molecules, in ethers and alcohols and in small molecules containing double and triple bonds.

But the diagram given in figure 1 may be considered as a first approximation of an energy diagram, in which the energy-levels are represented by lines running through the H_2O point.

The numbers at the top of the figure also correspond to energy-levels and this level is roughly inversely proportional to the respiratory quotient of the compound.

Figure 4 shows the relation graphically.

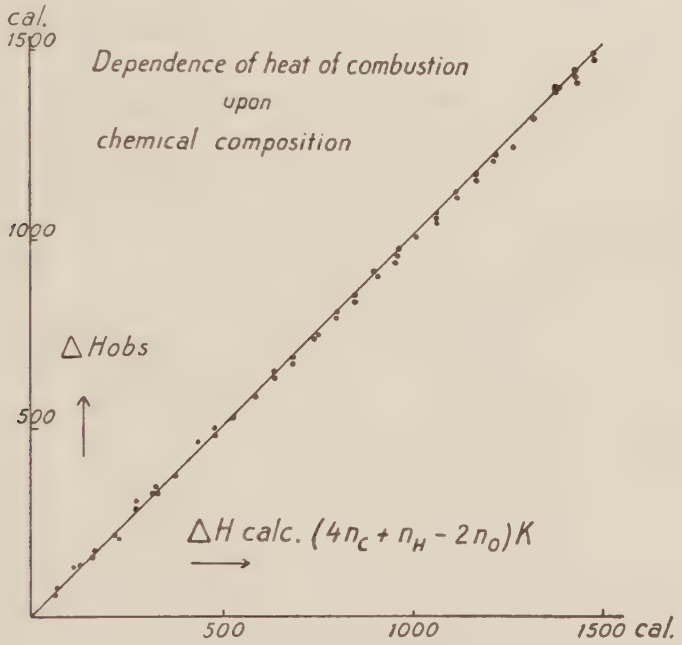


Fig. 4.

10. The relation established allows us, at a glance, to see whether a biochemical reaction proceeds with appreciable energy-exchange For the equation of alcoholic fermentation of glucose

$$C_6H_{12}O_6 - 2\,C_2H_5OH + 2\,CO_2$$

we get, for E

$$6 \times 4 + 12 - 2 \times 6 - 2\,(2 \times 4 + 6 - 2) + 2 \times 0.$$

Both sides give 24, therefore the reaction proceeds with little energy-production (less than 26 K. cal).

The equation for photosynthesis

$$6\,CO_2 + 12\,H_2O + \Delta H = C_6H_{12}O_6 + 6\,H_2O + 6\,O_2$$

yields:

$$0 + 2 \times 0 + \Delta H = 24 + 0 + 6 \times 0.$$

The reaction is endothermic, demanding energy uptake of $24 \times 6.5 = 643$ K. cal. (found 674 K. cal.).

The fundamental equation for CO_2-assimilation however,

$$6\,CO_2 + 24\,H = C_6H_{12}O_6 + 6\,H_2O$$

gives

$$6 \times 0 + 24 = 24 + 6 \times 0.$$

Therefore it may proceed with very little energy-exchange.

When we consider the alcoholic fermentation more in detail we start with glucose (symbol HCOH in fig. 1).

The triose-hydrate is situated vertically below the glucose point.

The first dehydrogenation yields glyceric acid (line through H-point).

Pyruvic acid is found by dehydratation (line through H_2O-point). Pyruvic acid is decarboxylated and yields aldehyde. Aldehyde is hydrogenated to alcohol.

$$E = \frac{C_3H_6O_3}{12} + \frac{H_2O}{0} = \frac{C_3H_6O_4}{10} \quad \frac{+2H}{2} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$E = \frac{C_3H_6O_4}{10} - \frac{H_2O}{0} = \frac{C_3H_4O_3}{10} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$E = \frac{C_3H_4O_3}{10} = \frac{CO_2}{0} + \frac{C_2H_4O}{10} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$E = \frac{C_2H_4O}{10} \quad \frac{+2H}{2} = \frac{C_2H_6O}{12} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

None of these reactions imply important energy transfers, as the hydrogen, liberated in (1) is used in reaction (4).

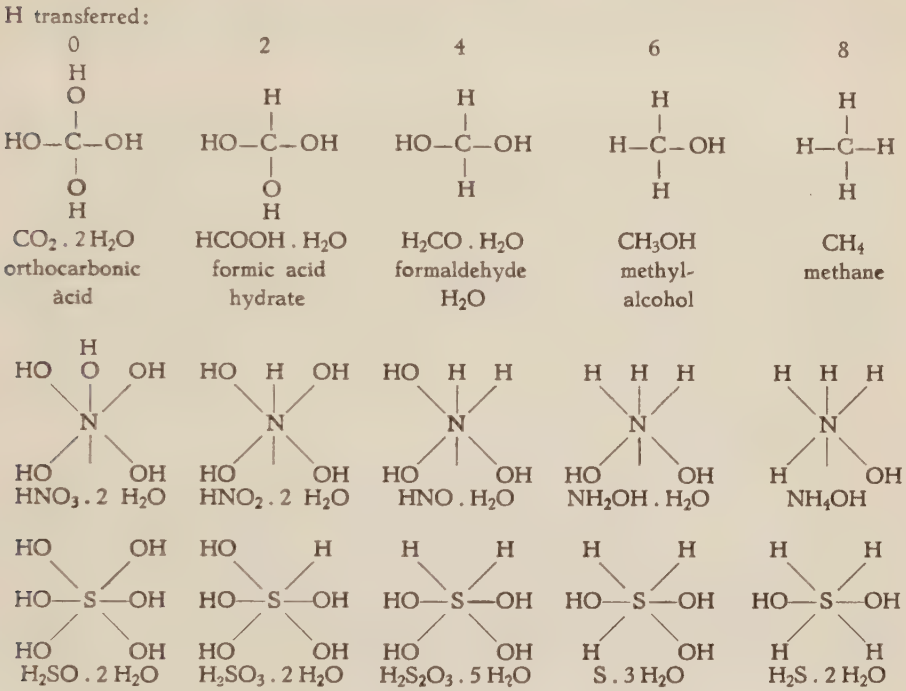
11. In order to homogenize the representation of oxido-reductions we have to represent the compounds by points on the same line. Figure 1 will make this clear.

Let us consider the intermediate stages of the reduction $CO_2 - CH_4$. It is clear that this represents the transfer of 8 hydrogen atoms per C-atom. On the line $CO_2 - H_2O$ we find point 1 (monohydrate of CO_2) and point 2 (dihydrate, orthocarbonic acid). About the possible role of this compound in CO_2-reduction see BAAS BECKING and HANSON 1937.

From orthocarbonic acid to methane we have a desoxidation, passing a trihydrate (3) of oxalic acid, the monohydrate of formic acid, the dihydrate of glucolic acid, the monohydrate of formaldehyde, the monohydrate of glycol, methanol, monohydrate of ethane, before arriving at point CH_4 (see figure 5).

We also may represent the process by direct hydrogenation. In that case we start with H_2CO_3 (point 1 on the $CO_2 - H_2O$ line, figure 1) and proceed towards the hydrogen point.

We pass the dihydrate of oxalic acid, monohydrate of formic acid, dihydrate of formaldehyde, dihydrate of methanol, finally arriving at the dissolved CH_4.



12. Figure 2 represents the relations of nitrogen, hydrogen and oxygen. Transfer of 8 hydrogen atoms is again necessary to reduce the highest oxide to ammonia. The products found in the bacterial reduction of nitrate, or the oxidation of ammonia, are the following: nitrite (MUNTZ), N_2O (BEYERINCK and MINKMAN), nitrogen (MUNTZ), hydrazin (VIRTANEN) and hydroxylamin (VIRTANEN).

The relation between the various oxides will be seen at a glance and also the "inevitability" of the occurrence of the various reduction stages. The desoxidation of nitrate clear to ammonia starts with point 2 on the line $NHO_3 — H_2O$ and proceeds away from point 5 over the dihydrate of nitrous acid, pentahydrate of nitrous oxide, monohydrate of hydroxylamine, monohydrate of ammonia.

The hydrogenation (see also table) starts at point HNO_3 and proceeds over nitrogen, pentahydrate of nitrous oxide, dihydrate of hydroxylamin to $NH_3\ 3\ H_2O$.

13. Figure 3 represents the relations of sulphur, hydrogen and oxygen. As in the nitrogen cycle most reactions stated in MILLOR's large handbook may be read from the diagram at a glance.

Complete sulphate reduction, as represented by desoxidation is represented by dihydrate of sulphuric acid, dihydrate of sulphurous acid, penta-

hydrate of thiosulphuric acid, trihydrate of sulphur up to tetrahydrate of hydrogen sulphide. The hydrogenation runs from H_2SO_4 over sulphur monohydrate, pentahydrate of thiosulphuric acid, tetrahydrate of sulphur to tetrahydrate of hydrogen sulphide.

All compounds play a role in the microbiological sulphur cycle (BUNKER 1938).

14. The above considerations merely want to call attention to possibilities of a graphic representation of chemical reactions which is primarily of didactic interest, as much that is left to memory may be reproduced now at will.

The diagrams make us realize the significance of hydrogenations, hydratations and oxidations. It enables us, moreover, in case of carbon compounds, to realize, in first approximation, energy-exchange in various reactions.

Leiden-Buitenzorg 1945—1946.

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Biochemistry. — *Oleate systems containing potassium chloride in which the KCl concentration is still too low for coacervation.* II. *Influence of polar organic non-electrolytes.* By H. G. BUNGENBERG DE JONG and G. W. H. M. VAN ALPHEN.

(Communicated at the meeting of March 29, 1947.)

1. *Influence of polar organic non-electrolytes on the shape of the KCl curve.*

In the first communication [1]) we have investigated the influence of KCl on the viscosity of diluted oleate sols and in this communication now follows the influence of non-electrolytes, particularly of normal primary alcohols on this KCl curve.

40 gr. Na-oleate (MERCK) is dissolved in 2000 cc dest. water by rolling it in a large glass stoppered bottle. Afterwards is added 200 cc KOH 2 N. This oleate stocksolution may be preserved in the frigidair.

The mixtures contain: 5 cc oleatestocksol. + a cc KCl 3.8 N + b cc alcoholsolution + $[15 - (a + b)]$ cc dest. water.

For organic compounds which dissolve difficultly in water another method was used according to which a known quantity of organic compound was dissolved in a definite quantity of stocksol., while by diluting this latter sol. with the stocksol. a series of sols was prepared with various concentrations of organic compound.

It appears now that we may devide the organic compounds into two groups, concerning their influence on the KCl curve.

In the first group — e.g. methylhexyl keton or n. octanol [2]), see lower parts of fig. 1 and 2, the KCl curve moves as a whole to smaller concentrations, while the maximum of the curve becomes lower and the inflexion point in the ascending curve branch becomes less pronounced or even disappears.

In the second group — e.g. aceton or ethanol, see upperparts of fig. 1 and 2 — the KCl curve just moves to higher concentrations, while here too the maximum becomes lower and the inflexion in the ascending curve branch becomes less clear or disappears. In contrast to the first group, the maximum flattens more and more when the curve moves on.

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[1]) H. G. BUNGENBERG DE JONG and G. W. H. M. VAN ALPHEN, Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, 8, 849—858 (1947).

[2]) In the experimental series with methyl-hexyl-keton is not used the usual Na-oleate serving for the investigation of the three other compounds, but a Na-oleate prepared from oleic acid and recrystallised from alcohol. Though the blank curve has the same character as the other blank curves (compare Communication No. I, 4), there still are some differences: the KCl concentrations where *a*) the system is pulling threads, *b*) the viscosity maximum is reached and *c*) coacervation occurs, are obviously smaller.

As to the KCl concentration at which the system is pulling threads for the first time (indicated by arrows ↓) this is lowered by methylhexyl-keton and n. octanol and raised by aceton and ethanol.

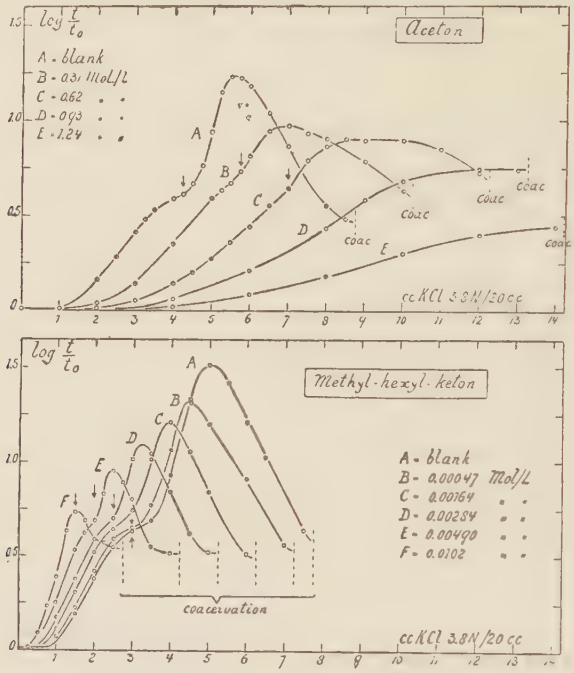


Fig. 1.

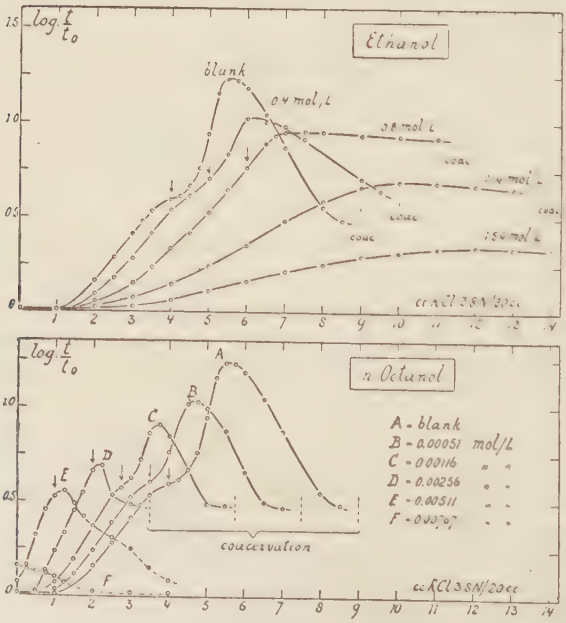


Fig. 2.

Further all curves agree in principle with that which holds for the blank curve: at the arrow the property of pulling threads appears for the first time, it is becoming more distinct with increasing KCl concentrations, to decrease again at approaching the coacervation limit. These properties gradually grow weaker however.

This is particular clear for aceton and ethanol at which the system even fails to pull threads at the two largest constant concentrations.

With n. octanol the systems of curve F are not pulling threads at all, but this is probably based on another cause: these systems are not clear, but blue opalescent.

Along the curve E a similar opalescence appears already along the right descending curve branch and though no coacervation occurs anymore (the same along curve F), the property of pulling threads stops at KCl = 3.

Finally we want to discuss the coacervation limit. We notice that compounds of the first group move on these limit to smaller KCl concentrations, compounds of the second group to higher concentrations.

Compare fig. 3 in which the coacervate volumes (in % of the total volume) are plotted as functions of the KCl concentrations (the

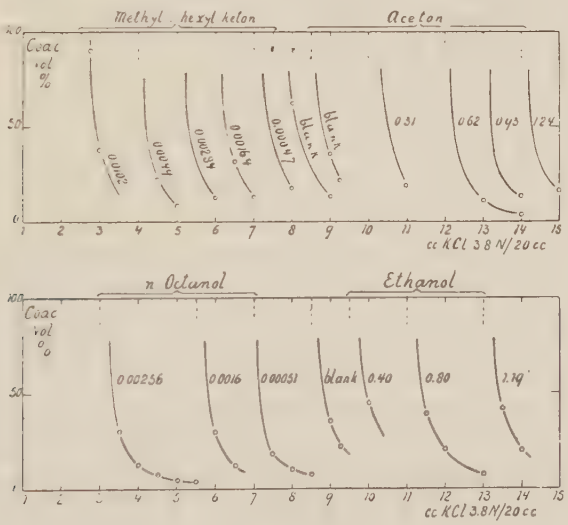


Fig. 3.

coacervation limit being the KCl concentration at which the coacervate volume = 100 %).

Thus we meet in several properties a similarity with previous experiences concerning the influence of organic non-electrolytes on oleate coacervates; (at constant KCl concentrations) [3]). There we could also discern two groups of non-electrolytes.

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[3]) H. G. BUNGENBERG DE JONG, G. G. P. SAUBERT and H. L. BOOY, Protoplasma **30**, 1 (1938).

The higher terms of a homologous series (e.g. methylhexylketon, octanol) act "condensing" while the first terms of a homologous series act "opening" (e.g. aceton, ethyl alcohol).

Thus we may suppose, that in the oleate systems, which we are studying here (at KCl concentrations smaller than the coacervation limit) similar rules concerning the relation between effect and constitution will exist.

2. *Influence of increasing concentrations of n. primary alcohol at three different but constant KCl concentrations.*

The first six terms of the n. primary alcohols have been further investigated.

They will be indicated in the following by a figure representing the number of C-atoms : methyl- (1); ethyl- (2); n. propyl- (3); n. butyl- (4); n. amyl- (5) and n. hexyl- (6) alcohol.

In this investigation a Na-oleate preparation was used, of which the viscosity-KCl curve is already given in communication I (fig. 2) showing the following characteristic KCl concentrations:

just pulling threads 4,5 cc KCl 3,8 N/20 cc;
viscosity maximum 5,75 cc KCl 3,8, N/20 cc;
coacervation limit 9,25 cc KCl 3,8 N/20 cc.

We now made measurements at three KCl concentrations which were kept constant, viz. at

3,5 cc KCl 3,8 N/20 cc, at which the oleate system thus still pulls no threads (fig. 4 A);
at 5,75 cc KCl 3,8 N/20 cc, at the viscosity maximum (fig. 4 B) and
at 8,0 cc KCl 3,8 N/20 cc, just before the coacervation limit (fig. 4 C).

The final oleate concentration was always 0,455 % and the KOH concentration was also kept constant viz. 0,0455 N. (The latter to exclude auto-sensibilisation, see communication I).

In fig. 4 the values of log. t/t_0 are plotted as functions of the logarithms of the alcohol concentrations (in molls per L.).

The values t_0 are here equal to the times of flow of similarly composed mixtures (KCl, KOH and alcohol) without oleate.

In the figures A, B and C the dotted horizontal line presents the value of log. t/t_0 at an alcohol concentration $= 0$. The figure added to a curve indicates the concerning alcohol (e.g. 5 = amyl-alcohol). The curves are drawn continuously when the oleate system is homogeneous, dotted when they are heterogenous (coacervated) or blue opalescent.

A first view on fig. 4 shows already that the influence of an alcohol is the stronger the longer its carbon chain.

At the viscosity-maximum of the KCl curve (fig. 4 B), all six alcohols appear to act in the same direction. They very strongly decrease the viscosity.

The opposition between compounds which move the KCl curve to smaller or higher KCl concentrations resp., already discussed in 1) is not indicated here.

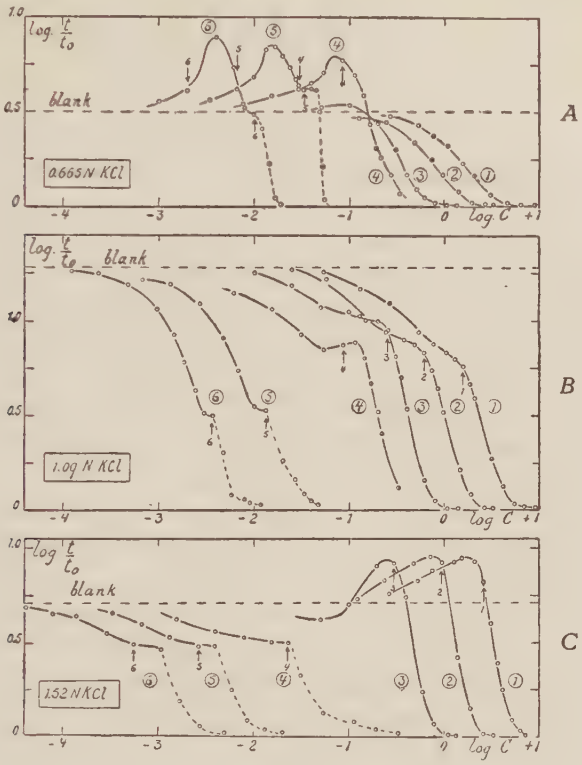


Fig. 4.

Compare fig. 1 and 2, from which we may predict this already: notice the intersection of the different curves with a vertical line drawn through the maximum of the blank curves.

At the other chosen KCl concentrations however, an entirely different behaviour occurs as the six alcohols do not alter the viscosity in *one* direction only.

In fig. 4 A 4, 5 and 6 first increase the viscosity to a maximum, but after that the viscosity decreases. 3, 2 and 1 on the contrary just decrease the viscosity.

This is also to predict from the fig. 1 and 2, if we notice the intersection of the different curves with a vertical line at a KCl concentration smaller than that at which the blank system gets pulling threads.

Considering in the same way the intersection points with a vertical line at KCl = 8 c.c. 3,8 N/20 cc, it becomes clear that the action in fig. 4 C will be the reverse of fig. 4 A 6, 5 and 4 are decreasing already immediately, while 3, 2 and 1 first increase to a maximum, only to decrease later on.

We thus conclude from fig. 4, that in the series of the primary alcohols

a reversal occurs in the way on which its individual terms influence the KCl curve.

Methyl-, ethyl- and n. propyl alcohol move the KCl curve to higher KCl concentrations, while n. butyl-, n. amyl- and n. hexylalcohol (and still higher terms also e.g. n. octylalcohol, see fig. 2) to lower concentrations.

A more close inspection of fig. 4 A and C shows propylalcohol as a transition term.

In small concentrations its action corresponds just for a while to butyl-, amyl- and hexylalcohol, in higher concentrations however it corresponds entirely to ethyl- and methylalcohol.

This means an obvious similarity with the previous investigated influence of alcohols on oleate coacervates, in which the same characteristic points have been found: *a*) an alcohol is acting in a lower concentration the longer the carbonchain, *b*) there occurs a change of action in the homologous series concerning the direction in which an alcohol is acting *c*) n. propyl alcohol comes out as transition term.

One would expect now that butyl-, amyl-, and hexylalcohol in a sufficient concentration always will produce coacervation.

This is the case at the highest KCl concentration anyway (fig. 4 C); at lower KCl concentrations there is a removal to higher terms in the homologous series with respect to the coacervation.

In this manner butylalcohol does not perform coacervation at 5.75 cc KCl (fig. 4 B), while amylalcohol and hexylalcohol still do.

At 3.5 cc KCl (fig. 4 A) the systems with n. amylalcohol do not even typically coacervate anymore (blue opalescent systems appear), though n. hexylalcohol still does.

Finally we have to discuss the influence of the alcohols on the elastic properties (thread-pulling capacity).

The capacity of pulling threads is strongly pronounced at the maximum of the KCl curve.

It appears now that all alcohols act on this phenomenon in one direction, namely annulating (see arrows ↑ in fig. 4 B).

The same holds for the blank at KCl = 8 cc 3.8 N/20 cc, where the capacity of pulling threads is weaker (see arrows ↑ in fig. 4 C).

Concerning alcohols producing coacervation the thread-pulling stops just before the coacervation limit, as for the alcohols producing no coacervation (4, 3, 2, 1 in fig. 4 B and 3, 2, 1, in fig. 4 C) the thread-pulling stops near to a distinctive change of course in the viscosity curve.

In fig. 5 the concentration trajectories in which threads are pulled are indicated by continuous lines.

The alcohol concentration for annulating these properties appears to be the lower, the longer carbon-chain it contains.

In the cases B and C nothing indicates a division of the alcohols into two groups, as they all annulate the thread-pulling.

In case A, the KCl concentration (3.5 cc) is chosen in such manner that the blank system is still not pulling threads.

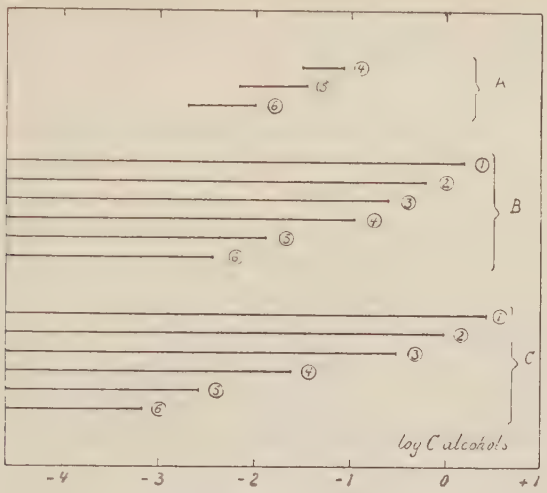


Fig. 5.

The alcohols 6, 5 and 4 may here originate the threadpulling (arrows ↓ in fig. 4 A).

The alcohols 3, 2 and 1 on the contrary do not proceed any thread-pulling.

The alcohols 6, 5 and 4 apparently annulate the thread-pulling in higher concentrations, which they originated in lower concentrations (arrows ↑ in fig. 4 A).

See A in fig. 5, with indicated trajectories in which threads are pulled.

Nevertheless we will still see in communication III that at the maximum of the viscosity KCl curve (5.75 cc KCl) a division may be shown into two groups, with n. propylalcohol as transition term, from measurements of these elastic systems.

If we review the data from this and the first communication the question arises, if the high viscous and elastic oleate systems appearing at KCl concentrations still too low for coacervation may be considered as stadia preparing the coacervation or as systems sui generis.

We are inclined to the latter.

Indeed it was established already over and over again that factors, strongly attacking the typical high viscosity or the thread-pulling, do not prevent the coacervation anyway (even sometimes stimulate the coacervation, e.g. influence of temperature in Communication I).

Then it may be mentioned that the change of action in the homologous series of the alcohols concerning the influence on the viscous and elastic behaviour is *independent* of the KCl concentration; as for the influence on the coacervation however, the change of action shifts to higher terms when the KCl concentration decreases.

Summary.

1. A constant concentration of a polar non-electrolyte (e.g. an alcohol or keton) moves the viscosity KCl curve either to higher concentrations (the first terms of a homologous series) or to lower concentrations (the higher terms of a homologous series).

A middle term may show a transition character: in smaller concentrations it corresponds with the higher terms of a homologous series, in larger concentrations with the lower terms.

2. While the viscosity curve is moving as mentioned in 1), the maximum of viscosity decreases, the inflexion point in the left ascending curve branch gets less pronounced or disappears, while the phenomenon of thread-pulling declines or stops altogether.

3. The influence at 3 different, but constant KCl concentrations of methyl- (1), ethyl- (2), n. propyl- (3), n. butyl- (4), n. amyl- (5) and n. hexyl-alcohol (6) on the viscosity and the thread-pulling was investigated more in detail.

4. The influence on the viscosity still depends on the selection of the KCl concentration, being evident from 1) and 2).

At a KCl concentration before the inflection point. 6, 5 and 4 do increase the viscosity until a maximum is reached and decrease afterwards. 2 and 1 only decrease the viscosity, while 3 acts as transition term.

At the maximum of the KCl viscosity curve all terms of the homologous series just decrease the viscosity.

At still higher KCl concentrations now 2 and 1 increase the viscosity to a maximum and 6, 5 and 4 decrease, while 3 acts as transition term again.

5. The concentrations at which an alcohol acts on the viscosity (either increasing or decreasing) or on the phenomenon of threadpulling resp. (either stimulating or suppressing) is the smaller the more carbon atoms the alcohol contains.

Botany. — *Uptake and transport of chlorine by parenchymatic tissue of leaves of Vallisneria spiralis. I. The active uptake of chlorine.* By W. H. ARISZ.

(Communicated at the meeting of October 25, 1947.)

§ 1. *Introduction.*

The phenomenon of the active uptake of substances in plant cells has been extensively investigated on various objects. Many investigators have done this on disks of storage organs. Besides various advantages these objects have this disadvantage that the uptake is not a normal phenomenon in their case, but takes place as a result of wounding and regeneration. Therefore the uptake which is found here and which is inherent with the formation of new protoplasm and the protein metabolism departs in some respects from the normal uptake. STEWARD's generalization that only very active cells having the capacity for further growth show an active uptake seems premature.

Others have investigated the process of uptake in unicellular seaweeds which are much less apt to take up substances actively, such as Valonia and various Characeae as Nitella.

The greater number of researches, however, has been made with the roots of higher plants, in which an uptake occurs which in some respects resembles the active uptake in storage tissues and unicellular sea weeds. It has appeared, however, that this uptake is of a complicated nature because in their case we have not only to deal with the uptake by the root cells, but also with the transport of salts to the xylem. Therefore it is important that there is a fourth group of objects, the leaves of water plants, in which the problems are in some respects a little simpler.

Especially Elodea has served as an object for investigations.

In our own researches Elodea gave no distinct result, while with Vallisneria the results on uptake of salts proved to be much more suitable for analytical purposes. Besides this object has the advantage that not as in the case of Elodea, leaves and stalk were examined at the same time, but only the very homogeneous leaf tissue. For these long leaves it is moreover possible to trace the transport of the absorbed substances in the leaf. As far as I know, these leaves have not been used in absorption- and transport experiments, since BIERBERG worked with them in 1909 in a research on the influence of protoplasmic streaming on the transport.

The last ten years a series of researches have been made in Groningen on Vallisneria, in which the details of these processes have been extensively examined. As early as 1932 Miss A. KOK of Groningen had repeated BIERBERG's investigation on the uptake of Lithium salts and she had also examined the transport of caffeine. Next followed extensive quantitative

researches by ARISZ and OUDMAN in 1937 and 1938 into the uptake and the transport of asparagine and caffeine, in which it was found that absorption and transport of asparagine are active processes.

By ARISZ and VAN DIJK (1939) and ARISZ (1943) the details of the process of asparagine absorption were further determined. After that a research on the uptake of cat-and an-ions of anorganic salts was started, of which the results of the uptake of chlorine are summed up below.

Vallisneria spiralis can be cultivated in a hothouse all the year round. The leaves are so long that pieces of 30 cms. or longer can easily be cut off and can be divided into smaller pieces. The leaves grow only in their basal parts. Those parts which are used in the experiments do not display any further growth. A disadvantage of Vallisneria is that growing it requires much more care than the growing of Elodea and that only material cultivated in favourable circumstances in a suitable exposure to light can be used for these experiments. In discussing the method in § 2 the most important items which have to be observed in making the experiments, will be treated extensively. Another disadvantage of Vallisneria is the presence of big intercellular ducts which though they don't constitute an open communication, because of the presence of septa cf. A. KOK, may give rise to difficulties, when as a result of inefficient technics external liquid enters them. This namely takes place in the dark when the surrounding cells withdraw oxygen from the intercellulars, which are for a great part, filled with it. As a result water is being sucked up in the intercellular spaces. The infiltration is injurious to the leaf and causes a quicker dying off, but in addition, if the external solution is concentrated, the infiltration of this solution into the intercellular spaces may cause important errors, because of its giving the impression that the substances have been absorbed by the tissue.

If however, diluted solutions are used as is the case for this publication, this phenomenon need not be considered, provided the leaves continue to stay in a good condition. This obtains for all solutions more dilute than 1/50 to 1/100 mol. Vallisneria leaves have a number of vascular bundles running parallel in the length of the leaf. They consist almost entirely of parenchymatic cells. Vessels do not occur. This renders these leaves particularly suitable for experiments to investigate into the transport in parenchymatic tissue.

For the experiments and the analyses I received the valuable aid of the analysts Miss J. VAN DER SCHANS, Miss E. BOSMA and Miss H. MEESTER, for which I wish to express my indebtedness in this place.

§ 2. *Method.*

In this publication the uptake of Cl ions is exclusively discussed. These were determined by VOLHARD's method in about 200 mgrs fresh weight, while the analysis figures always refer to a certain size of leaf surface, mostly 19.2 or 20 cms with a width of 4 mms.

Some investigators have preferred examining Br ions, therefore something has to be said here about the choice of chlorine ions. Since HOAGLAND and STEWARD showed in a convincing manner that the uptake of Br ions is more complicated than that of Cl ions, because simultaneously with the uptake of Br a loss of Cl occurs, the use of Cl ions has to be preferred, if one wants to study the process of active uptake, as in that case the interchange of Cl and Br would only give an undesirable complication.

The Vallisneria plants were grown in concrete tanks measuring 200 to 58 cms. with a depth of 50 cms. Heating took place with hot water pipes and electric heating with an ozurite cable in the ground. The temperature was kept constant by a thermoregulator at 23° C.

Part of the year extra exposure was administered with a couple of 200 watt lamps during some hours in the evening. Of great importance is the water used for the experiments. On account of the well known fact that destilled water is injurious, various investigators have used tap-water or springwater. This surely gives much better results for keeping the cut leaves alive, but it should be remembered that the leaves might absorb salts from these highly diluted salt solutions and as a result the osmotic value of the cell-sap may rise considerably, as was first shown in this laboratory by Miss VAN SCHREVEN (ARISZ 1943).

For the experiments made water destilled over glass was used. All the experiments were made in a room of a constant temperature usually at 25° C.

We must refer to a previous publication (ARISZ and OUDMAN 1937) in which the division of material into smaller pieces and the composing of series from these was communicated. Owing to the fact that they contain pieces of different leaves, taken from different parts of the leaves, these series are very homogeneous with each other, so that the results of the chemical analysis show a very slight variability.

Before the experiments the leaves were well washed and from their sides so much was cut off that the width everywhere amounted to 4 mms. Next they were cut into smaller pieces and put in destilled water in series. The pieces of one series are sewn into a bit of tulle, so that they don't overlap. The tulle is weighted so much on one side that it floats in the solution in a vertical plane. All leaf-lengths can then be exposed to the sideways entering light of an electric lamp. Both during the preliminary treatment and during the uptake the leaf lengths were put in tumblers or in vessels with bottoms of sintered glass, a good aeration being seen to.

Regarding the rather great differences in the amount of the uptake in identical experiments it must be emphasized that conclusions must be based on the comparison of different treatments in one and the same experiment.

§ 3. *Influence of wounding.*

In experiments on uptake and transport of asparagine by ARISZ and

OUDMAN 1937, it had already appeared, that leaf lengths of which a small part had been brought in touch with asparagine, while the rest of the leaf had to get it from this small leaf length, took up more asparagine according as the free part was longer. This phenomenon was met again in the experiments on the uptake of chlorine and will be discussed when we deal with the transport. It was of essential interest to know what this influence of the free leaflength could be based upon. Presumably this could at least partly be due to the wounding, owing to which a longer leaf length would be less impeded in the uptake by the wound-stimulus, as the wounds inflicted lie farther apart. This phenomenon could better be investigated in this way that the uptake of leaf lengths of a different size which could absorb salt over their whole length, were compared and it was traced, whether the uptake calculated for an equal leaf surface, was always equally great. If the wounding should affect the uptake, it would have to appear from a slighter relative uptake by short leaf lengths.

To ascertain this the following experiment was made. Leaf lengths of 7.5 cms, 2.5 cms and 0.83 cm were brought into a solution of 1/1000 mol. KCl with $CaSO_4$. After 24 hours it was determined how much Cl had been absorbed. For this experiment 8 leaves of a length of 30 cms were used. These were cut to a uniform width of 4 mms and divided into 4 pieces of 7.5 cms. By combining in a series 8 leaf lengths taken from the 8 different leaves but from a different place, 4 comparable series were obtained. In two of these series the leaf lengths were further divided into 3 lengths of 2.5 cms, so that the experiment was made with:

A. two series, each of 8 leaf lengths of 2.5 cms = 20 cms, which were analyzed at the beginning of the experiment.

B. two series of 8 leaf lengths of 7.5 cms. After the uptake these leaf lengths were cut into 3 lengths of 2.5 cms and the Cl was separately determined in the 3 zones. Here too 20 cms of leaf is available for each analysis.

C. two series, each of 8 leaf lengths of 2.5 cms = 20 cms.

D. two series each consisting of 24 lenghts of 0.83 cm, i.e. likewise 20 cms per series.

The table gives the amount of chlorine in each of the series B, C and D after a 24 hours'uptake. The difference with the series sub A gives the increase of chlorine per 20 cms leaf length.

Short leaf lengths take up considerably less than longer ones. This is due to a wound influence. This follows from the analysis of the 3 different zones of the leaf lengths of 7.5 cms, of the series *B 1* and *B 2*. The marginal zones take up considerably less than the central part. As this wound influence could not be prevented, it was desirable, to discover if the wound influence grows weaker in the course of time. For this purpose the cut leaf lengths were put in destilled water for a different length of time. The

TABLE I. Absorption of chlorine by leaves of different length. Uptake during 24 hours from a solution of 0.001 mol KCl with $CaSO_4$, exposed to light (200 Watt at 50 cm). Cl content per 20 cms leaf length.
B_1 B_2 leaf length 7.5 cms C_1 C_2 2.5 cms D_1 D_2 0.83 cm

	γ Cl	uptake γ Cl	average uptake per 20 cms leaf length γ Cl
A_1 control series 2.5 cms	432		
A_2 " " " "	437		
B_1 first zone 2.5 cms	568	134	
second zone 2.5 cms	611	177	146
third zone 2.5 cms	561	127	
B_2 first zone 2.5 cms	571	137	
second zone 2.5 cms	614	180	148
third zone 2.5 cms	561	127	
C_1 leaf length 2.5 cms	532	98	
C_2 leaf length 2.5 cms	536	102	100
D_1 leaf length 0.83 cm	504	70	
D_2 leaf length 0.83 cm	512	78	74

following experiment gives the result of a preliminary treatment of 2, 4, 7 and 24 hours on the uptake of 1 1000 mol KCl with $CaSO_4$ in the light at 25° C. In this case the series consisted of 8 leaf lengths of 2.5 cms each. We only mention the increase of chlorine per series of 20 cms total leaf length.

TABLE II. Influence of duration of pretreatment in destilled water on the uptake of chlorine.

Preliminary treatment no.	uptake of Cl in γ.
	98
2 hours in destilled water	120
4 hours in destilled water	131
7 hours in destilled water	142
24 hours in destilled water	246

From this it appears that preliminary treatment in destilled water has the desired effect and that after 24 hours the inhibition due to wound-influence has disappeared for the greater part.

After this the experiment with bits of leaf of various lengths was repeated, now after a 24 hours' preliminary treatment in destilled water.

Table 3 gives the results of this experiment. It appears that lengths of 2.5 cms and 7.5 cms give the same results. Besides the difference between

TABLE III. Absorption of chlorine by leaves of different length after pretreatment 24 hours in destilled water. Uptake during 24 hours from a solution of 0.001 mol KCl with $CaSO_4$, exposed to light (200 Watt at 50 cms' distance). Cl content per 20 cms leaf length.

B_1 B_2 leaf length 7.5 cms C_1 C_2 2.5 cms D_1 D_2 0.83 cm

	γ Cl	uptake γ Cl	avarage uptake per 20 cm leaf length γ Cl
A_1 2.5 cms no uptake	404		
A_2 „ „ „ „	401		
B_1 first zone 2.5 cms	644	241	
second zone 2.5 cms	640	237	238
third zone 2.5 cms	640	237	
B_2 first zone 2.5 cms	644	241	
second zone 2.5 cms	648	245	242
third zone 2.5 cms	644	241	
C_1 leaf length 2.5 cms	640	237	234
C_2 leaf length 2.5 cms	634	231	
D_1 leaf length 0.83 cm	564	161	154
D_2 leaf length 0.83 cm	551	148	

marginal and central zones has disappeared with the 7.5 cm leaf lengths. In the 0.83 cm leaf lenghts the uptake after a 24 hours'preliminary treatment is still considerably slighter. From these and other experiments not mentioned it may be concluded that a 24 hours'preliminary treatment in destilled water eliminates the wound effect sufficiently for leaf-lengths of 2.5 cms and longer. From the result of a great number of comparative experiments it has appeared that the wound effect not only influences the strength of the uptake, but also influences the transport. To this we shall revert on discussing the transport.

§ 4. *Experiments on the uptake of chlorine.*

A. *from solutions.*

The experiments were partly made in 1944, when the wound effect had not yet been investigated, partly in 1946 and 1947, when this could be entirely taken into account. The results of the two series are in perfect agreement. This is partly due to the fact that the wound effect is not always so great as in the above experiments, but also to the fact that the wound effect only impedes the processes of uptake without altering them in

essence. Initially experiments were made on the uptake of chlorine from aqueous solutions of various chlorides. Various chlorides, especially LiCl and Mg Cl_2 but in some experiments NaCl as well were not absorbed at all in a concentration of 1/320 mol. On the contrary a loss of chlorine from the tissue was often stated which indicated the toxicity of these salts. In more resistant leaves the loss of chlorine changed into an uptake. $CaCl_2$ was taken up most easily, KCl considerably less. Since this indicated a toxicity of the salt solution and as in lower concentrations such as 1/1000 mol toxicity was somewhat less than in higher concentrations, though the results remained variable, we proceeded to operating with a mixture of potassium and calcium ions to balance the toxicity of the cations.

Seeing the Vallisneria plants were grown in well-water and in the tanks a continual evaporation of water occurred the quantity of salts and particularly of calcium in the water increased during the growing. It is known that under these circumstances Vallisneria leaves depose $CaCO_3$ on the surface exposed to the light. Older leaves were indeed often covered with $CaCO_3$. Though usually younger leaves were used for the experiments, they can also be covered more or less with $CaCO_3$, so that on our making experiments in water containing carbonic acid, $Ca(HCO_3)_2$ will also have been present. This may account for the fact that in some experiments an equal uptake from KCl as from a mixture of KCl and $CaSO_4$ was found.

Besides the toxicity of the cations the destilled water used may have its influence. It has repeatedly appeared that in solutions with ordinary destilled water no uptake but exosmosis occurred. That is why, as already stated above, in all experiments water destilled over glass has been used. The solutions of KCl and $CaCl_2$ did give better results yet no satisfactory ones. By adding $CaSO_4$ instead of $CaCl_2$ a solution was obtained which always gave favourable results. As a rule a mixture of KCl and $CaSO_4$ (747 mgs KCl + 368 mgs $CaSO_4$ 2 aq in 1 L water) was used and this mixture was diluted if required. This gave a ratio of potassium and calcium ions of 62 : 18 and of chlorine and SO_4 ions as 63.5 : 36.5. Seeing in this case the issue is the absorption of chlorine, the strength of the concentration is indicated in mol KCl. With this combination of salts a great number of experiments have been made, the results of which were very regular, which makes them seem sufficiently reliable.

Influence of deprivation of oxygen.

Some experiments have been made to trace whether in an anaerobic medium chlorine was taken up. Of course these experiments had to be made in the dark, as in the light oxygen is set free during assimilation. Never was any absorption found in these experiments as a rule loss of chlorine occurred, because the leaves started showing dying off phenomena.

Influence of salt concentration.

Table 4 contains a number of data on experiments in which the in-

TABLE IV. Influence of concentration on the uptake of chlorine by leaves of Vallisneria from KCl solutions containing $CaSO_4$. Experiments in the light 200 Watt at 50 cms, 25° C, 24 hours. The figures give the increase in Cl concentration in milli mol of 8 leaf pieces of 2.5 cms length and 4 mm width, A and B with a short pretreatment, C and D with pretreatment during 24 hours in destilled water.

Chlorine conc. outer solution in milli mol	uptake of Cl in milli mol				accumulation factor	
	A	B	C	D	C	D
4	30		35		9	
2	29		30		15	
1	29		33		33	
1/2	29	24	32	31	64	62
1/4	28	22	27	28	108	112
1/8		15		25		200
1/16		9		17		272
1/32		2		11		352

fluence of the concentration of chlorine ions in the solution on the uptake has been traced. All the experiments were made in the light, those of 1947 after a 24 hours'preliminary treatment in destilled water in the light. The data are in perfect agreement: from 1/32 to 1/2 millimol there is a rather strong increase in the uptake, about proportional to the logarithm of the concentration. When, however, results of various experiments are compared, rather great differences may appear. This is due to the fact that the condition of the plants at the beginning of the experiment greatly influences the uptake. This also appears from experiments made with a different preliminary treatment. An influence on the uptake is exercised by the strength of the exposure, the supply of oxygen, the composition of the air used for aeration and the concentration of hydrogen ions of the medium.

Accumulationfactor.

From the data of table IV an accumulation factor can be evaluated. When we suppose that the chlorine taken up is solved in the water present in the tissue we get an approximative value of the inner concentration. The accumulationfactor that is the relation between the concentration of the chlorine in the tissue and in the outer solution increases from 9 at a concentration of the outer solution of 4 millimol Cl to 352 at a concentration of 1/32 millimol Cl.

Influence of exposure during the uptake.

After a 24 hours'preliminary treatment in the light in destilled water, the uptake from 1/1000 mol KCl + $CaSO_4$ in the light was 157 γ Cl, whereas in the dark it was 16 γ Cl. As a source of light during the preliminary treatment a 100 watt lamp was used at a 50 cms'distance (6450 lux), and during the uptake a 200 Watt lamp at 50 cms (12900 lux). Also the

intensity of the light has its influence. On this subject only some orientating experiments were made.

TABLE V. Influence of light-intensity on the uptake of chlorine from 1/1000 mol. KCl with $CaSO_4$. Time of uptake with A 28 hours, with B 26 hours. Light source 200 Watt at 50 cms' distance.

at a distance of	A	B
50 cms	208	215
100 „	154	190
200 „	90	115

From these data it is apparent that the intensity of the light during the uptake has a considerable influence.

Analysis of the influence of light on the uptake of chlorine.

In this series of experiments the effect of light on the uptake was further examined. Seeing light also gives photosynthesis, it was interesting to know whether light in an environment free from carbonic acid also effects the absorption of chlorine. The result of these experiments has been given in table 6. An environment free from carbonic acid was obtained by letting

TABLE VI. Analysis of the influence of light on Cl-accumulation. Absorption 24 hours of 0.001 mol KCl with addition of $CaSO_4$ at 25° C. A and B in the light, C and D in the dark. A and C in solution aerated with common air. B and D in an aerated solution deprived of CO_2. Experiment V pretreated 24 hours with destilled water, exposed to light and aerated with CO_2 free air.

	aeration	uptake in γ Cl				
		I	II	III	IV	V
A. light	with air	118	133	296	134	264
B. light	without CO_2	122	188	360	209	292
C. dark	with air	13	1	97	28	
D. dark	without CO_2			117	39	

air free from carbonic acid bubble through the solution before and during the experiment, so that for photosynthesis at most the small quantity of carbondioxyde originating from the respiration was available. In addition in some experiments the solution had been boiled beforehand and cooled after closing off with a tube containing soda-lime. It is evident that light without carbonic acid has the same effect as light with carbonic acid. This proves that the effect of light on the active uptake of chlorine is not connected with carbondioxide assimilation. Besides it appears from table 6 that the uptake in an environment free from carbonic acid is considerably greater than in a solution containing carbonic acid. This also holds good for experiments in the dark, only the level is much lower in this case. It seems probable that in case of withdrawal of carbonic acid the stronger uptake may be the result of the higher pH in the medium.

Influence of pH on the uptake of chlorine.

Therefore the influence of the concentration of *H* ions on the uptake was examined. The experiments were made in a continuous artificial light. In these experiments the pH of the unbuffered solutions decrease during the experiment. This may be due to an increase of carbonic acid in the medium. Aeration was brought about with common air. The change is somewhat slighter if during the experiment aeration takes place with air free from carbonic acid. In these unbuffered solutions the optimum has not always been found at the same level. It lies at about pH 6—7. Evidently pH 4.5 is detrimental, so that a loss of chlorine occurs due to exosmosis, likewise pH 9 is toxic. It is therefore not impossible that on aeration with air free from CO_2 the uptake is greater owing to the fact that the pH lies nearer to the neutral point. In the same way aeration during the preliminary treatment with air free from CO_2 may give a stronger uptake than aeration with air containing CO_2.

Preliminary treatment.

The preliminary treatment is of great importance, which appears from some experiments on the influence of light during this period on the strength of a later uptake. In one experiment e.g. the uptake from 0.001 mol $KCl + CaSO_4$ during 24 hours in the light, was 158 γ Cl, if during the 24 hours'preliminary treatment the leaflengths were exposed, but only 76 γ Cl if they were kept in the dark. Therefore in experiments with preliminary treatment the objects were always exposed to the light in this period. Also the intensity of the exposure during the preliminary treatment has its influence. Aeration during the pretreatment with carbondioxide free air and the withdrawal of carbondioxide formed by the tissue does not prevent this favourable influence on the following uptake. So this effect has nothing to do with photosynthesis of carbohydrates.

It seems interesting to know if an exposure to light during the pretreatment has the same effect as an exposure during the uptake. Both processes have in common that the influence of light is not through products of carbon dioxide assimilation. Still it appears that both effects are not identical because a leaf pretreated in light must be exposed during the absorption as well to get a normal uptake. It was ascertained wether the effect of light during the pretreatment influenced the uptake only in the next few hours following the pretreatment or that it lasted for a longer time. The results of two experiments are given in table 7. In each experiment three series with different pretreatments were exposed at the same time in a 0.001 mol KCl solution, and the uptake was determined after 6 and 24 hours. The uptake in the last 18 hours could be evaluated from these data. It is apparent that though the uptake in the first six hours depends on the pretreatment, the difference in strength is also present in the following hours. So the uptake in the second period was e.g. in experiment A 160 γ after exposure during the pretreatment to strong light, 105 γ to less strong

TABLE VII. Influence of exposure to light during pretreatment on the uptake of 0.001 mol KCl + $CaSO_4$ in the light, 25° C. Two experiments A and B. The uptake is determined after 6 hours and after 24 hours.

Pretreatment	uptake in γ Cl after 6 hours		uptake in γ Cl after 24 hours		uptake in γ Cl in the last 18 hours	
	A	B	A	B	A	B
24 hours exposed 12900 lux	144	146	304	250	160	104
24 hours exposed 6450 lux	122	108	227	193	105	85
24 hours in the dark	74	57	115	106	41	49

light and 41 γ when the pretreatment was in the dark. This means that the effect of the illumination during the pretreatment influenced the uptake all the time.

In another experiment the uptake took place in the dark after the same different pretreatments. In all cases the uptake was now only slight.

These results indicate that during the pretreatment in light a substance e.g. a sensitizer may be formed which favours the process of uptake in the light. The present data are not sufficient to give a complete analysis of this phenomenon.

B. *Uptake from agar.*

For all experiments in which it was necessary that only part of a leaf length should take up salt in which case it can be traced if transport takes place to the other zones of this leaf length, a 2 % agargel was used as a medium in which the salts were dissolved. For these experiments the agar was not specially purified, because in such a purification decomposition products of proteins are apt to be formed, of which a complicating action on the plasm is to be expected. So it appeared for instance, that if so called purified agar was used, the uptake in the dark was sometimes much stronger. As unpurified agar contains some chlorine, a leaf could already absorb some Cl from the blank agargel in 24 hours. This amounted to about 5—10 γ Cl in 24 hours per 20 cm leaf-length. It did not present difficulties however, that in these experiments, in which chlorides were added to the agar anyhow, a slight amount of chlorine was already found in the agar. The above experiments on the influence of different factors were repeated with lengths in agar and results were found in perfect agreement with those already stated. With regard to the wound stimulus as a result of cutting it was found that material which had had a 24 hours'preliminary treatment in destilled water in an exposure to 100 Watt at a 50 cms'distance, absorbs in 24 hours relatively more in leaf lengths of 7.5 cms than in leaf lengths of 2.5 cms, viz. 315 γ Cl against 286 γ Cl. This indicates that in this experiment with leaf lengths of 2.5 cms after 24 hours the wound

stimulus had not quite disappeared yet. This is in accordance with the experiments on the uptake from solutions.

With these experiments the difference between uptake in the light and in the dark is less pronounced than with experiments in solutions, because the uptake in the dark is slightly greater. This may very well be due to the presence of disintegration products in the agar. So in an experiment with leaf lengths of 7.5 cms for the uptake in the light in the first zone of 2.5 cms 229 γ was found, in the 2nd zone 233 γ and in the 3rd zone 218 γ Cl, whereas in the dark the amounts were 90, 83 and 79 γ respectively.

§ 5. *Exosmosis of absorbed chlorine.*

For the asparagine uptake there was formerly made an extensive study of the phenomenon of exosmosis of the first absorbed asparagine. Whereas on transport to an anaerobic medium no exosmosis was obtained, after transport into a fresh solution an exosmosis of part of the first absorbed asparagine took place for some hours. After that the leaf recovers itself and reabsorption takes place. This phenomenon is closely connected with the sensitiveness of the material. Resistant material, such as has been used for these experiments with salts, does not show exosmosis at all or in a much slighter degree. In the experiments on exosmosis of first absorbed salts no loss has been observed in undamaged tissue. Transport from light into dark or from the salt solution into destilled water had no influence at all. As already stated, an exosmosis was obtained with extreme pH. Neither does oxygen withdrawal in itself bring about exosmosis. If, however, the leaves continue under these unfavourable circumstances, in the dark for a long time, a loss is noticed right enough. We must therefore conclude that under normal circumstances salts which have once been absorbed are not returned to the environment again.

§ 6. *Osmotic value of the cell sap.*

It is essential to know whether the salts absorbed, of which here only the Cl ion was determined, get into the cytoplasm or are taken up partly or entirely in the vacuole. If they get into the vacuole they may be found there bound or absorbed or continue free. Only in the latter case they will cause an increase of osmotic value. If therefore after the uptake an increase of the osmotic value of the cell-sap is found, it may be concluded that the ions get into the vacuole and are present there in a free condition. The result of the experiments made to ascertain this fact are very convincing. The osmotic value of the vacuole at limiting plasmolysis has increased considerably after the uptake. The method that has to be applied to demonstrate the uptake of salt in the vacuole is simple and corresponds entirely with a similar research on the uptake of asparagine (ARISZ and VAN DIJK 1939). In a control series the osmotic value of a sucrose solution giving limiting plasmolysis is ascertained at the beginning of the experiment and in the experimental series which has absorbed Cl for 24 hours from a

solution of 1/1000 mol $KCl + CaSO_4$ this value is determined at the end of the experiment. As a plasmolyticum sucrose was used. In an experiment on uptake 217 γ Cl was found, while the osmotic value of the epidermal cells at limiting plasmolysis increased from 0.32 to 0.38 mol sucrose. In another experiment the uptake was 225 γ Cl and the increase from 0.30 to 0.38 mol. So the increase of the osmotic value is respectively 0.06 and 0.08 mol. If we consider the absorbed quantity of Cl dissolved in all the water present in the leaf length, this constitutes an increase of concentration of 0.031 mol and 0.032 mol. The quantity of water in the vacuole will of course be less than the total quantity of water present; moreover the salt will be dissociated. This renders it comprehensible that an even stronger increase of osmotic value was found (0,06 and 0,08 m instead of 0,031 and 0,032), but it indicates at the same time that the salt absorbed is found for the greater part free in solution in the cell sap. The chance that the very low concentration of the salt solution has caused metabolic processes, which should have brought about such an anatonosis of the cell-sap, may be considered out of the question.

§ 7. *Discussion.*

After that in some previous researches the active absorption of asparagine had been demonstrated, the uptake of salts, especially of chlorides, was further examined. As a result it was found that from a balanced salt solution the leaves of Vallisneria absorb chlorine. As the leaf lengths used in these experiments do not grow during the uptake we have here an active uptake which is independent of growth. The quantity absorbed depends on the concentration of chlorine in the external liquid, especially in concentrations lower than 1/4 millimol. The uptake is influenced by the pH of the solution, pH 4.5 and > 9 are detrimental and cause exosmosis of chlorine from the tissue. Exposure to light has a remarkable effect on the strength of the absorption. The stronger it is, the stronger is the absorption.

Light does not work photosynthetically here by forming carbohydrates, as it is also active when carbonic acid is not present in the medium. In literature various data are known on the fact that light affects the uptake of salts. In 1937 JÄRVENKYLA made an excellent summary of the extensive literature on the effect of light on the permeability of the protoplasm. In it he also treats the effect of light on the uptake in Nitella, Valonia and Elodea, found in 1923 and 1926 by HOAGLAND and collaborators, in 1934 by JACQUES and OSTERHOUT and in 1936 by INGOLD and points out that in these experiments light does possibly not affect permeability but processes of a different nature. Several of these investigators have connected the influence of light with photosynthesis. As it, however, appeared in our experiments with Vallisneria that in a medium free from carbonic acid, i.e. without photosynthesis light has the same influence, this conception cannot be accepted in the case of Vallisneria. PHILLIS and MASON (1937) found that cotton leaves only in the light and when supplied with oxygen,

absorb sugar from a sugar solution and form starch. This process also takes place in a medium free from carbonic acid.

From JÄRVENKYLA's publication and also from our own observations on Elodea cells it appears that the behaviour of various objects is different. Therefore it is advisable not to draw conclusions from the data given here for Vallisneria as to the behaviour of other objects. The problem will have to be carefully investigated for every ion and for every plant.

The process of absorption is an active process, as appears from the great sensitiveness of this process to outward circumstances. It is dependent on the presence of oxygen and on the temperature. Important is the great influence of the previous history. Wounding appears to have a rather strong effect due to which, especially in those parts of a leaf length which are nearest to the wound-edge, the processes of uptake are checked. This wound influence diminishes in course of time, so that in sufficiently great leaf lengths there is little to be observed of it after 24 hours. So to get a normal uptake it is necessary to apply a 24 hours'preliminary treatment in destilled water. Exposure, temperature, aeration during this preliminary treatment, they all have some influence on the strength of the ensuing uptake (Cf. p. 11).

It was possible to render it probable that the greater part of the substance absorbed gets into the vacuole. The increase in osmotic value at limiting plasmolysis tallies fairly well with the supposition that all chlorine absorbed gets into the vacuole. The cells do not return the absorbed chlorine to their environment. Neither through transport to destilled water nor through oxygen withdrawal from the medium, a loss was to be brought about. As soon, however, as injury was caused by toxic ions or by a too high or too low pH, exosmosis of chlorine could be shown. This proves that the protoplasm as a whole does not allow the chlorine ions to pass as long as it is in a perfectly normal condition.

Summary.

Leaves of Vallisneria spiralis take up Chlorine by an active process from balanced solutions containing KCl and $CaSO_4$.

The uptake depends on the presence of oxygen. It is influenced by exposure to light, by the pH of the solution, by temperature and by the pretreatment of the leaves. The influence of light is not indirect through the products of the process of photosynthesis but it is a direct effect on the processes occurring in the cytoplasm. From the cutting of the leaves ensues a wound-stimulus which checks the processes of active uptake. The normal condition returns after a prolonged stay in destilled water.

The increase of the osmotic value of the epidermal cells proves that the salts are accumulated into the vacuole.

Exosmosis does not take place from undamaged tissue. Under normal circumstances salts once absorbed remain in the vacuole. The protoplasm as a whole does not allow the chlorine ions to pass.

Zoology. — *The external shape as a specific character in Loxothylacus (Crustacea Rhizocephala)*. By H. BOSCHMA.

(Communicated at the meeting of October 25, 1947.)

In a previous paper (BOSCHMA, 1940. Temminckia, vol. 5), in which the specific characters of a number of species of the genus *Loxothylacus* were described in some detail, especially with regard to variation, no mention was made of the macroscopical external peculiarities of the specimens. As it proved that the shape of the animals described in the cited paper may be characteristic for the species, some remarks are given here concerning their external appearance.

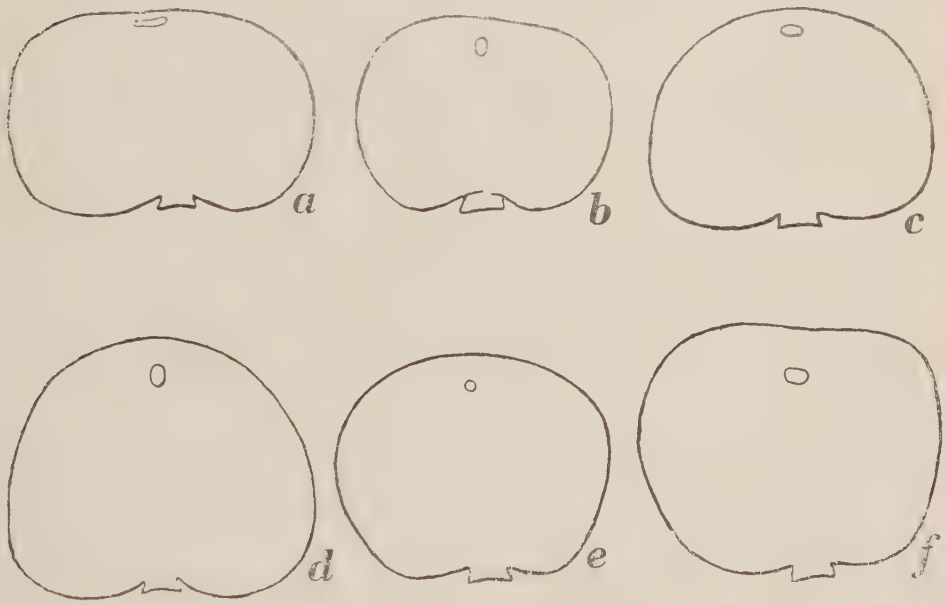


Fig. 1. Left side of six specimens of *Loxothylacus variabilis*. *a—d*, on *Chlorodiella nigra* from Koepang, Timor; *e*, on *Actaea rüppellii* from Mamoedjoe, Celebes; *f*, on *Actaea hirsutissima* from Taliaboe, Soela Islands. *a*, *c*, *d*, *e*, × 6; *b*, × 6½; *f*, × 5.

In *Loxothylacus variabilis* Boschma (fig. 1) the dorso-ventral diameter is slightly or appreciably larger than the antero-posterior diameter. The shape of the animals is elliptical or slightly panduriform or more or less reniform. The surface of the mantle does not possess any pronounced grooves or wrinkles. As a rule the mantle opening is rather narrow, it is found on the left side (the surface touching the thorax of the crab), not far from the anterior margin. This opening is very little conspicuous as it is

not surrounded by a ridge protruding above the surface. Specimens on *Chlorodiella nigra* (Forsk.) (fig. 1 *a–d*) do not differ in any important detail from those on *Actaea rüppellii* (Krauss) (fig. 1 *e*) or from those on *Actaea hirsutissima* (Rüpp.) (fig. 1 *f*).

In *Loxothylacus brachythrix* Boschma (fig. 2) as a rule the dorso-ventral diameter is distinctly larger than the antero-posterior diameter. The shape of the animals is elliptical or panduriform or more or less trapezoid. The surface of the mantle may be comparatively smooth or may show some grooves or wrinkles, which have a tendency especially to develop in the

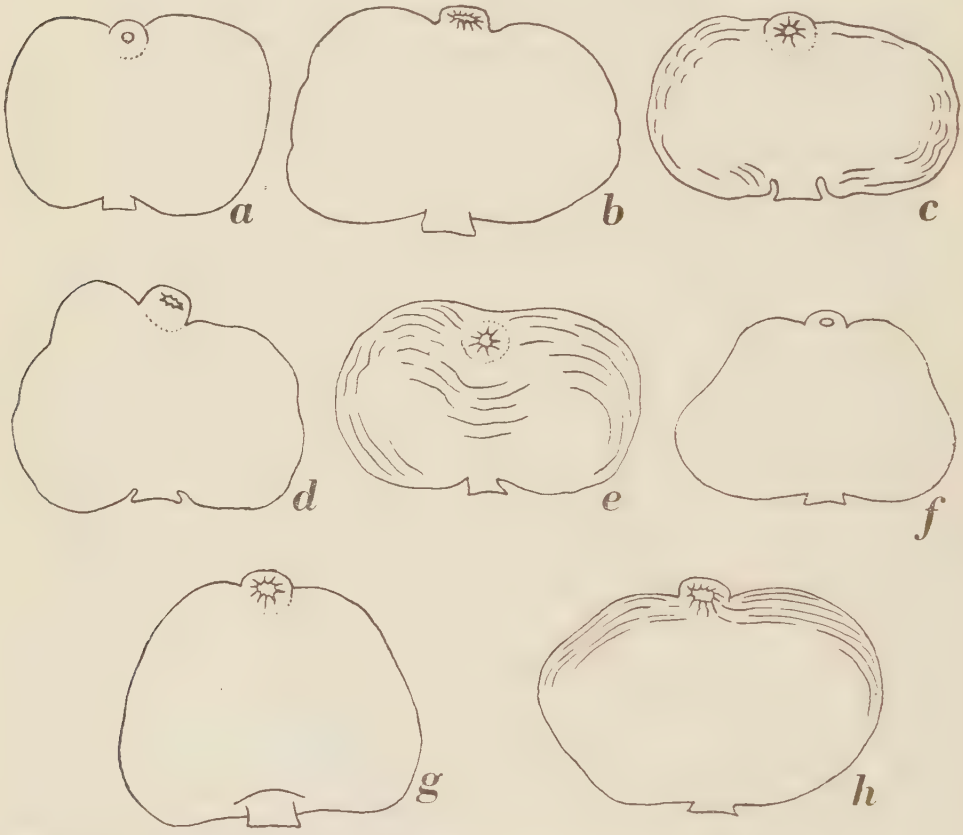


Fig. 2. Left side of eight specimens of *Loxothylacus brachythrix*. *a*—*c*, on *Xantho sanguineus* from Ternate; *f*, on *Lybia tessellata* from Koepang, Timor; *g*, on *Xantho exaratus* from Una Island, Mergui Archipelago; *h*, on *Phymodius ungulatus* from Tering Bay, Lombok Strait. *a*, × 3½; *b*, *c*, × 4; *d*, × 3; *e*, × 5; *f*, *g*, × 8; *h*, ×5.

peripheral regions. The mantle opening invariably lies at the extremity of a distinct tube or pronouncedly thickened protuberance. It occupies the centre of the anterior region or it occurs on the left side at some distance from the anterior region. In most specimens the internal margin of this tube is divided into a number of ridges separated by grooves. The specimens on *Xantho sanguineus* (H. M. E.) (fig. 2 *a-e*) show some individual

variation in their general appearance and in the shape and the position of the mantle opening. The specimen on *Lybia tessellata* (Latr.) (fig. 2 *f*) is more or less trapezoid, that on *Xantho exaratus* (H. M. E.) (fig. 2 *g*) has a similar shape, whilst the specimen on *Phymodius ungulatus* (H. M. E.) (fig. 2 *h*) bears a strong resemblance to some of the specimens on *Xantho sanguineus*.

In *Loxothylacus torridus* Boschma (fig. 3), which is known as a parasite on *Actaea tomentosa* (H. M. E.) only, the dorso-ventral diameter is about equal to the antero-posterior diameter. The specimens have a more or less quadrangular or trapezoidal shape. The region of the mantle opening does not project above its surroundings, this opening is found at the left side in the immediate vicinity of the anterior margin or in the anterior margin itself (in the specimen of fig. 3 *g, h*). In all specimens there is a pronounced convex region at the left side, extending from the stalk to the mantle opening, next to which there are distinct concavities in longitu-

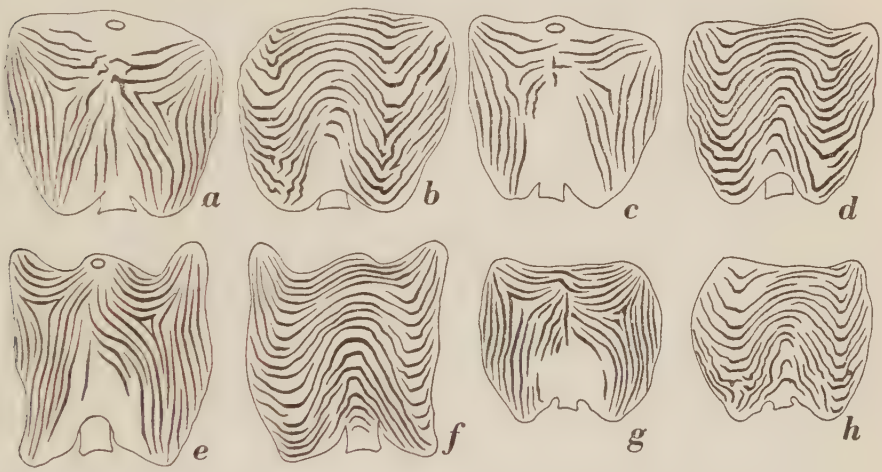


Fig. 3. *a, c, e, g,* left side of four specimens of *Loxothylacus torridus* on *Actaea tomentosa* from Koepang, Timor; *b, d, f, h,* right side of the same specimens. *a—f,* × 4; *g, h,* × 3.

dinal direction. At the right side the reverse is to be seen: here there is a distinct longitudinal concavity running from the stalk to the middle of the anterior region, whilst the remaining parts of the body form two conspicuous convex longitudinal ridges. The mantle is covered by a rather strong and hard external cuticle which shows numerous rather deep grooves, so that the greater part of the surface is noticeably wrinkled. At the left side there is a system of grooves running more or less parallel with the anterior region, forming an angle with another system of grooves which extend chiefly in the direction of the longitudinal convexity and concavities. At the right side the grooves are running in wavy lines from the dorsal to the ventral margin, their direction therefore is more or less at right angles to that of the longidutinal concavity and convexities.

In *Loxothylacus musivus* Boschma (fig. 4) the dorso-ventral diameter is noticeably larger than the antero-posterior diameter. In every other respect the specimens show a strong resemblance to those of the former species. The left side has a median longitudinal convex ridge whilst the dorsal and ventral regions are concave, the right side shows a median longitudinal concavity flanked by convex dorsal and ventral parts, and a

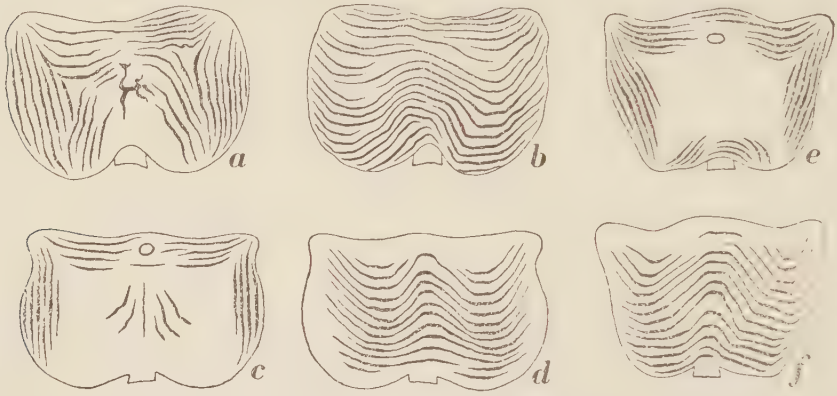


Fig. 4. *a, c, e,* left side of three specimens of *Loxothylacus musivus; b, d, f,* right side of the same specimens. *a—d,* on *Phymodius ungulatus* from Koepang, Timor; *e, f,* on *Cymo melanodactylus* from Koepang, Timor. *a, b,* × 4; *c, d,* × 4½; *e, f,* × 4¼.

similar system of grooves is present. The region of the mantle opening does not protrude above its surroundings, it is found at the anterior margin of the left side (fig. 4 *c, e*) or in the middle of the anterior margin (in the specimen of fig. 4 *a, b*). The shape of the figured specimens on *Phymodius ungulatus* (H. M. E.) (fig. 4 *a-d*) is more or less panduriform, whilst the specimen on *Cymo melanodactylus* de Haan is more or less trapezoidal.

Loxothylacus torridus and *Loxothylacus musivus* are very similar as far as concerns the shape of their internal organs and the structure of their external cuticle. This similarity manifests itself also in the external characters of the parasites.

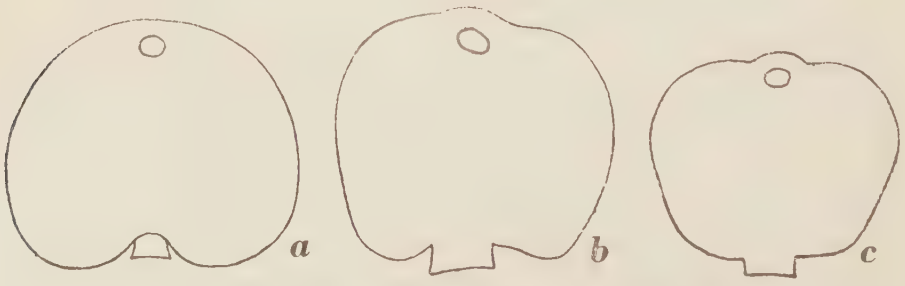


Fig. 5. *a, Loxothylacus echioides* on *Euxanthus exsculptus* from Mamoedjoe, Celebes, left side. *b, Loxothylacus sclerothrix* on *Actaea boletaria* from Amirante, Western Indian Ocean, left side. *c, Loxothylacus amoenus* on *Phymodius granulatus* from Koepang, Timor, left side. *a,* × 3; *b,* × 9; *c,* × 6½.

In fig. 5 the type specimens of three other species of the genus are represented, *Loxothylacus echioides* Boschma (fig. 5 *a*), a parasite of *Euxanthus exsculptus* (Herbst), from Mamoedjoe, Celebes, *Loxothylacus amoenus* Boschma (fig. 5 *c*), a parasite of *Phymodius granulatus* (Targ.-Torz.), from Koepang, Timor, and *Loxothylacus sclerothrix* Boschma (fig. 5 *b*), a parasite of *Actaea boletaria* Rathbun from Amirante, Western Indian Ocean.

The external characters of the only known specimen of *Loxothylacus echioides* bear a strong resemblance to those of some specimens of *Loxothylacus variabilis*. In both species the external cuticle bears two kinds of excrescences: numerous small hairs and a smaller number of large excrescences. The latter, however, are entirely different in the two species.

The type specimens of *Loxothylacus sclerothrix* and of *Loxothylacus amoenus* are strikingly alike in their external shape. In both specimens the surroundings of the mantle opening project slightly above the anterior region of the mantle. The two species have similar, though sufficiently distinct excrescences of the external cuticle. The species decidedly differ as far as concerns the male genital organs, which in *Loxothylacus amoenus* remain separated, and in *Loxothylacus sclerothrix* are largely united in their posterior extremities.

Summarizing it may be stated that the external shape of the parasites may furnish characters of sufficient specific value. Especially the presence or absence of a pronounced elevation for the mantle opening may form a specific character. Moreover a well developed system of grooves is peculiar to certain species only. If used with care some of the particulars dealt with above may serve as an aid for the distinction of the species.

Neurology. — *Positive and negative aspects of hypothalamic disorders.* By B. BROUWER. (From the neurological clinic and laboratory of the Wilhelmina Hospital and the Dutch Central Institute of Brain Research at Amsterdam.)

(Communicated at the meeting of October 25, 1947.)

In order to compare the experimental results with clinico-pathological findings I have studied with my co-workers nine cases of hypothalamic disorders after making serial sections through this region and the bordering parts of the brain. A tenth case, in which an adenoma of the hypophysis was grown into the hypothalamus causing total destruction of the tubero-infundibular region, is only partially studied microscopically. A full report of the relations in this case will be published later. The chief clinical symptoms and the pathological data are reproduced in the tables No. I and II. The extension of the lesions in the hypothalamus is reproduced in the figures No. I—V.

Some of these are already published more fully elsewhere, as indicated in our list of bibliography (No. 5, 6, 19). Of seven of these cases I have made glass-models, in which the extension of the lesions is visible in three demensions.

A striking point is, that extensive damages of the hypothalamus in man may be found, while in de clinical pictures many symptoms, usually seen in experimental investigations, were missing. Other authors have already stressed this point (ANDRÉ THOMAS c.s.[31]), BAILEY[2]), RIDDOCH[28]), ROUSSY and MOSINGER[29]) a.o.). Thus there are positive and negative aspects in hypothalamic lesions, a fact which has to be explained. In the following pages we try to check some of our clinico-anatomical findings with the experimental data.

a. Diabetes insipidus.

The experiments of BROERS[4]), FISHER, INGRAM and RANSON[8]) a.o. have shown that polyuria and polydipsia appear in dogs, cats and monkeys after bilateral destruction of the nuclei supra-optici. This syndrome was present in three of our cases (1, 2 and 3) and the nuclei supra-optici proved to be seriously damaged. In the last case there was also a tumour of the stalk and the neurohypophysis. There was no diabetes insipidus in our numbers 4, 5, 6, 7 and 8 in which these nuclei were intact. Problems arised however in the interpretation of the cases 5 and 9. In the last one the nuclei supra-optici were destroyed, while no diabetes insipidus had been present. The patient 5 showed cerebral disturbances of the bladder from the beginning, but no polyuria and polydipsia. There was a severe retrograde degeneration of the left nucleus supra-opticus. It has not been possible to make sure about the right side, owing to a technical failure in the slides of this region.

TABLE I.

1. G., man, 30 years	*a.* Dystrophia adiposo-genitalis. *b.* Diabetes insipidus. *c.* Psychosis of Korsakoff.	Chronic encephalitis of the hypothalamus.
2. D., man, 30 years	*a.* Diabetes insipidus. *b.* Mental disorders. *c.* Diminution of vision with bilateral syndrome of Argyll–Robertson. *d.* Paresis of the vertical movements of the eyeballs and vertical nystagmus. *e.* Sign of Babinski on the right side.	Carcinoma of the epiphysis, growing into the hypothalamus and the surroundings of the aqueductus Sylvii.
3. H., man, 20 years	*a.* Diabetes insipidus. *b.* Slight sexual and endocrinal symptoms. *c.* Dipoplia intermittens. *d.* Slight swelling of the optic disks.	Tumour of the neurohypophysis and the stalk, growing into the hypothalamus.
4. P., man, 34 years	*a.* Adiposity with sexual disorders without genital atrophy. *b.* Disturbance of the central regulation of body temperature. *c.* Attacks of hyperidrosis in the left side of the face, later spreading to the left side of the body. *d.* Facial tic on the left side. *e.* Cerebral disturbances of the bladder. *f.* Inversion of the sleep mechanism. *g.* Amblyopia on the left side with concentric depression of the visual field. *h.* Disorders of the movements of the eyeballs. *i.* Disorders of vestibular functions. *j.* Psychosis of Korsakoff.	Chronic encephalitis of the hypothalamus and bordering parts of the brain.

The whole tuber cinereum however was destroyed by the glioma. WITTERMANN [32]) also described cases, in which the nuclei supra-optici were practically absent while no diabetes insipidus had existed. On the other side ROUSSY, KOURILSKY and MOSINGER [30]) published two cases of polyuria and polydipsia in which no alterations in the hypophysis and the hypothalamus were found. One may conclude from these facts, that the relations regarding this syndrome in man are more complicated than could be inferred from the experimental data.

TABLE II.

5. N., girl, 18 years	*a.* Dystrophia adiposo-genitalis. *b.* Cerebral disorders of the bladder. *c.* Central vasomotor disorders. *d.* Swelling of the optic disks with circumscribed defect in the inferior basal quadrant of the visual field of the right eye.	Glioma of the chiasma and the hypothalamus.
6. B., man 20 years	*a.* Pubertas praecox. *b.* Adiposity. *c.* Psychopathia. *d.* Epilepsy. *e.* Polydactyly.	Circumscribed malformation of the hypothalamus.
7. L., boy, 3 years	*a.* Adiposity. *b.* Hypertension. *c.* Polyglobuly. *d.* Striae lividae. *e.* Glycosuria intermittens. *f.* Hypertrichosis. } Syndrome of Cushing without osteoporosis.	Circumscribed malformation in the tubero-infundibular region. No basophilic adenoma.
8. L., boy, 12 years	*a.* Pubertas praecox. *b.* Atrophy of the optic disks. *c.* Slight polyglobuly. *d.* Facial paresis on the left.	Astrocytoma of the chiasma and the hypothalamus, growing into the right neothalamus.
9. B., man, 55 years	*a.* Psychosis of Korsakoff. *b.* Blindness with atrophy of the optic disks. *c.* Absence of endocrinal disturbances.	Glioma of the chiasma and the hypothalamus.
10. L., man, 48 years	*a.* Blindness of the right eye in consequence of an old iridocyclitis and atrophy of the left optic disk. *b.* Psychosis of Korsakoff. *c.* Enlargement of the sella turcica. *d.* Absence of endocrinal disturbances.	Adenoma of the hypophysis growing into the hypothalamus with macroscopical total destruction of the tubero-infundibular region.

b. Glycosuria.

Hyperglycaemia and glycosuria are frequently seen after puncture in the tuberal region of animals and several authors assume that lesions of the paraventricular nuclei are responsible for this syndrome. The glycosuria however disappears after some days. LONG [22]), who recently analysed the various results in experimental physiology, concluded that up to the present (1940) there is little evidence to suggest that the control of the blood glucose level is *permanently* altered by hypothalamic lesions.

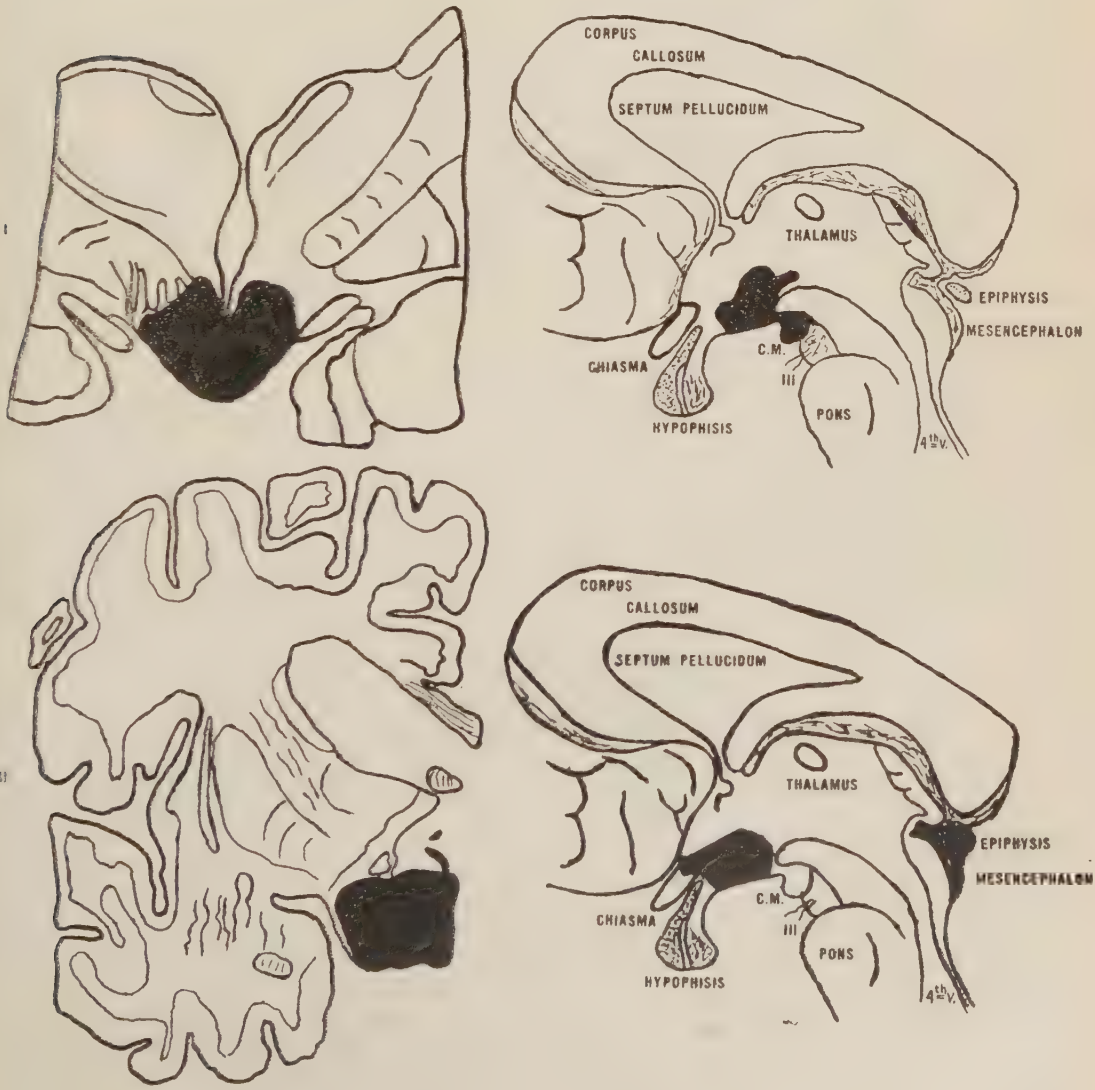
Only number 7 of our series showed intermittent glycosuria. There was

a congenital malformation in the infundibular region and the tuber cinereum was almost exclusively composed of neuroglia. The form of the nuclei paraventriculares showed alteration and the number of cells was diminished. In number 1 this nucleus was destroyed on the left side and the other was reduced to some cells. In the cases 2, 3, 4, 5 and 9 these nuclei were totally destroyed by the pathological process. In number 8 some cells were visible in this region, but they were pushed aside by the tumour mass. Hence it is clear, that a severe damage of the hypothalamus and particularly of the paraventricular nuclei does not necessarily cause disturbances of the carbohydrate metabolism in the chronic stage.

Fig. 1.

Maximal Extension of the lesion in vertico-frontal direction.

Extension of the lesion in sagittal direction



Disturbances of fat metabolism and of the genital organs.

Experimental and clinical experiences have shown, that adiposity, caused by a dysfunction of the hypophysis, is less intensive than in hypothalamic lesions. From which part of the tuber cinereum disturbances of fat metabolism originates, is not sufficiently known. BROBECK [3]) summarised recently our knowledge in this respect and referred-amongst others

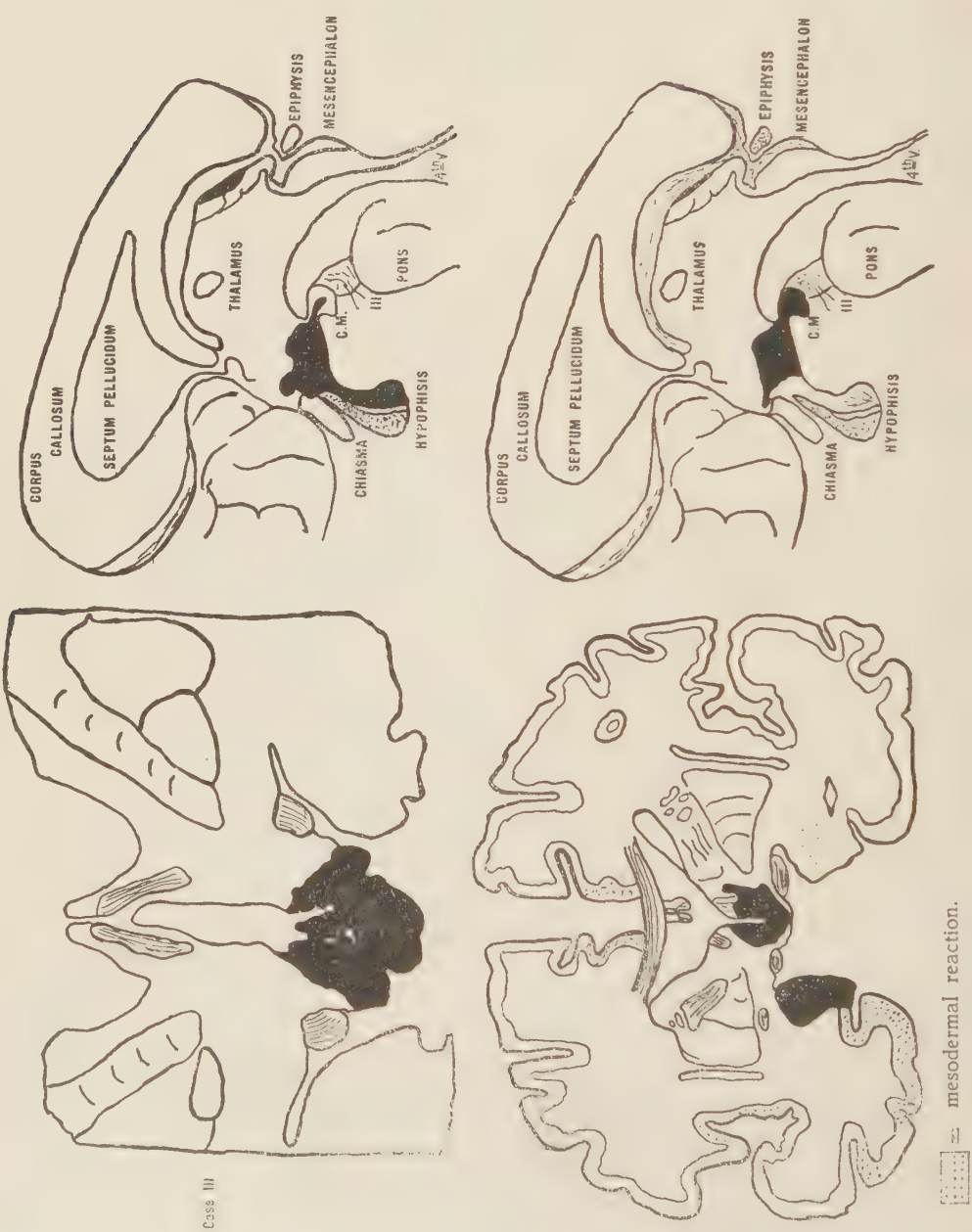


Fig. 2

— to the work of HETHERINGTON. He found that in the rat adiposity may be evoked more easily by making lesions in the ventro-medial and ventro-lateral hypothalamic nuclei, but also by destruction of the posterior part of the hypothalamus. Extensive lesions in the dorsal part and of the supra-chiasmatic and preoptic area do not cause such disturbances.

Adiposity in hypothalamic lesions in man is not always combined with genital atrophy, which is in agreement with experimental results. In our



Fig. 3.
Maximal Extension of the lesion in vertico-frontal direction.

series half of the cases showed obesity and in two the syndrome of dystrophia adiposo-genitalis was complete.

In two patients sexual disorders were present without adiposity or atrophy of the sexual organs. In two cases the hypothalamus was seriously damaged, while no adiposity or atrophy of the sexual organs had been observed.



Fig. 4.
Maximal Extension of the lesion in vertico-frontal direction.

d. Hypothalamic disturbances of vesical function.

The classical experiments of KARPLUS and KREIDL, showing that the hypothalamus controls the function of the bladder, have been confirmed in detail by many investigations. HESS and BRÜGGER [13]) evoked excretion of urine or defaecation or both together in cats by electrical stimulation, while the animals assured the posture of the body typical for the normal

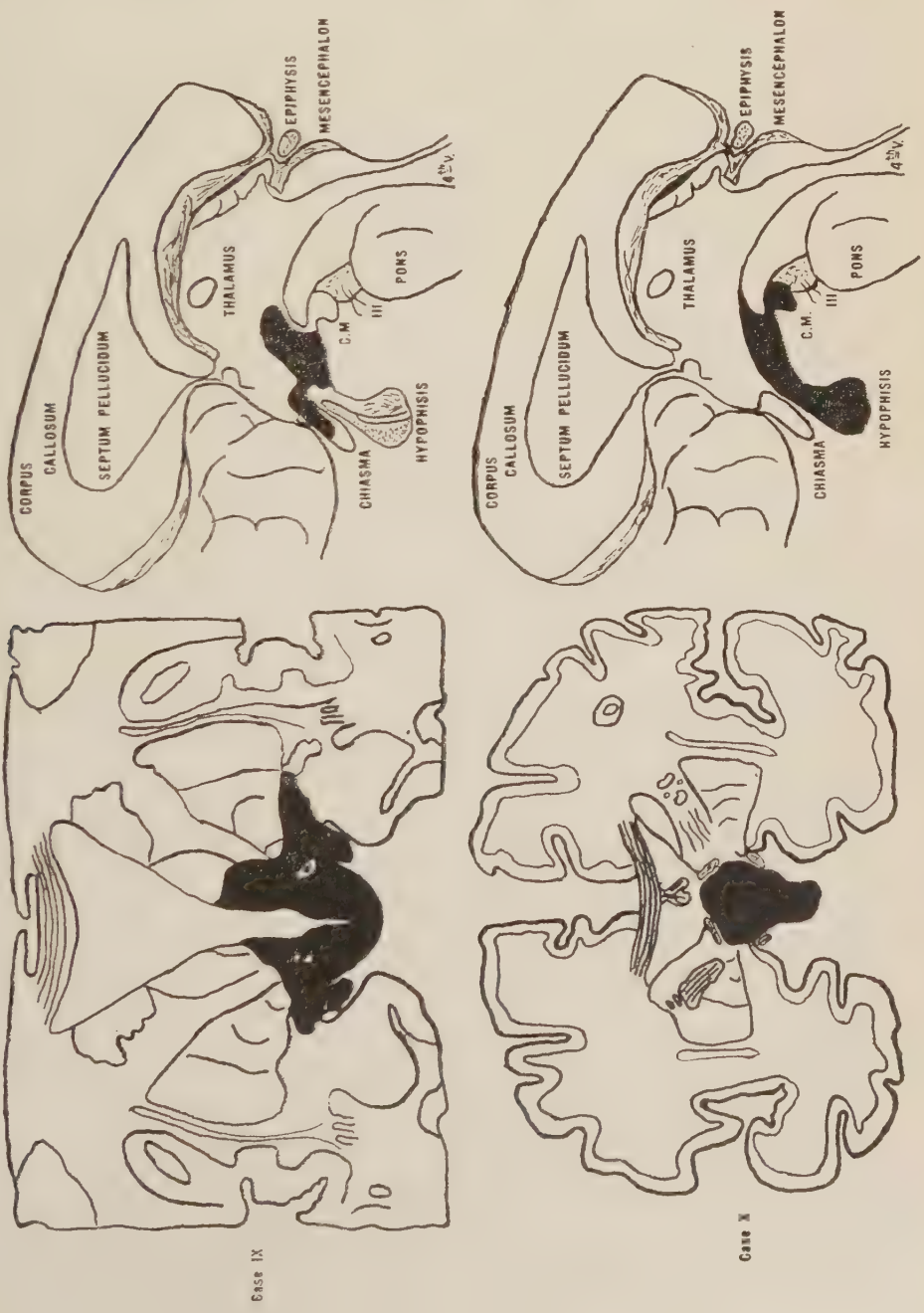


Fig. 5. Extension of the lesion in sagittal direction
Maximal Extension of the lesion in vertico-frontal direction.

evacuation of the bladder and the intestines. Cerebral disturbances of the bladder in man are usually caused by bilateral lesions of the hemispheres or by damage of both caudate nuclei with the bordering parts of the neothalamus. In our case 4 (chronic encephalitis) the involuntary evacuation of the bladder and the intestines was only found in the terminal stage of the disease. But in number 5 disturbed function of the bladder was one of the first symptoms and appeared with intervals regularly during many years. The figures of these two cases show that the pathological process extended in the sagittal direction through the whole hypothalamus and reached the preoptic area. Our investigations give no indication for a special place in the hypothalamus responsible for this symptom. The experimental results have shown that "centres" for the innervation of the bladder and the intestines are not only present in the oral part of the hypothalamus but must be widely spread through this part of the brain. The numerous negative cases in our series point to the same direction.

e. *Pubertas praecox.*

Many authors ascribe the cerebral form of macrogenitosomia praecox to a dysfunction of the epiphysis. In the conception of MARBURG this gland normally has an inhibitory influence on the development of the sexual organs. Experimental physiology however has not sufficiently supported this theory. KRABBE [18]) held an opposite view arguing that many epiphyseal tumours in young people did not show pubertas praecox, whereas an increasing number of hypothalamic lesions, having caused macrogenitosomia, is published. The most striking example is that of DRIGGS and SPATZ, in which a hyperplastic malformation in the tuber cinereum was found.

The syndrome of pubertas praecox has been present in the numbers 7 and 8 of our series. The first patient had already to be shaved at the age of 7 and showed hairs on the mons Veneris. A year later his penis and testes were much too large and adiposity developed. A circumscribed malformation of the tuber cinereum and hypoplastic mamillary bodies were found. In the second case, described by Prof. C. DE LANGE [19]), the typical picture of pubertas praecox developed at the age of ten. This boy showed also homo-sexual tendencies. At the post-mortem examination an astrocytoma in the hypothalamus was found penetrating the medial part of the right neothalamus. Many cells in the tuber cinereum had been destroyed but the mamillary bodies were intact. In both cases there was no tumour in the epiphysis, the adrenal glands or the testes, hence they are examples of the hypothalamic type of pubertas praecox. This suggests that it is the tuber cinereum, from where this syndrome originates.

f. *Disturbances of the central regulation of body temperature.*

Experimental physiology has shown that the disturbances of the central regulation of body temperature are more pronounced after large bilateral

destruction in the hypothalamus, by which the descending pathways to the tegmentum and the lower parts of the brain are destroyed (THAUER). RANSON [27] c.s. particularly evoked such disturbances by damage of the oral part of the hypothalamus, but found that such "centres" extend also more dorso-caudally.

FRAZIER, ALPERS and LEWY [11] concluded from their experiments in cats, that the grey substance in the region of the nucleus anterior hypothalami is indispensable for the normal regulation of body temperature.

Operations in the region of the hypothalamus in man always offer the danger of hyperthermia, caused by irritation of the walls of the third ventricle. Already in 1912 JACOBI and ROEMER [17] concluded from their experiments that excitation of the grey substance surrounding the third ventricle is followed by increase of temperature. Chronic cerebral disorders of thermo-regulation are rare. In our series 9 patients did not show these, although the tuber cinereum and its surroundings were seriously damaged. Only in nimber 4 this syndrome has been observed during one year and pathological changes in the internal organs, which could have explained the fever, were not found. The encephalitis had not only altered the cells in the tuber cinereum and the mamillary bodies, but extended especially in the grey substance of the walls of the third ventricle and reached the preoptic area. It is probable that the hyperthermia in this case is caused by irritation. In this respect it has to be noted that the patient suffered also from attacks of *hyperidrosis* on the left, while the encephalitis was more extensive on the right side. Furthermore he showed a typical leftsided *facial tic*, seen in psychasthenic individuals. In a later stage of his illness these involuntary quick movements spread also over neck and arms. The experimental work of HESS [16] has shown, that by electrical stimulation of the hypothalamus contractions could be evoked in the muscles of the head, the trunk and the extremities. HESS considers the hypothalamus as an important area of coordination between the somatic and vegetative central nervous system. At the present time we do not know very much about the tracts of the hypothalamus descending to the motor cells in the oblongata and the spinal cord, but one has to take into consideration that such reactions of the muscles may be the result of the influence which the hypothalamus exerts on the pallium.

g. Amblyopia and concentric depression of the field of vision.

These symptoms have been observed on the left side in our case 4. At the clinical examination myopic alterations in the retina were found, which must have existed for a long time. The patient however had no complaints about depressed vision until the last years before his entrance in the clinic. He often saw a cloud in front of his left eye. The vision on the right side was normal in the first month of the clinical observation, but gradually dropped to $\frac{1}{2}$ and the field of vision narrowed. The anatomical examination showed that the encephalitis had spared the optic nerves, the

chiasma and the optic tracts. This observation gains significance in connection with observations recently made in Switzerland. FRANCESCHETTI [10]) f.e. concluded from his observations that concentric depression of the field of vision may be the consequence of a lesion in the diencephalic-mesencephalic region of the brain outside the optic pathways. He founded this opinion on the experimental results of HESS [16]), who evoked diminution of vision by circumscribed lesions in the hypothalamus, in which the optic fibres were not affected. HESS believes that the hypothalamus has an indirect inductive influence on the power of the optic system by the hypothalamo-fugal fibres to the retina.

h. Syndrome of KORSAKOFF *and other mental disorders.*

The experiments of BARD, in which shamrage and other emotional disturbances were evoked in cats after exstirpation of the telencephalon, the striate bodies and the anterior part of the hypothalamus, have had a stimulating influence on clinical investigation. For a survey of the mental disorders, caused by affections of the hypothalamus we refer especially to the recent publications of ALPERS [1]) and of LHERMITTE [21]) c.s. In our series six cases showed severe mental disturbances, from which four belonged to the syndrome of KORSAKOFF. In the numbers 1 and 4 the mamillary bodies and their surroundings were seriously affected, in 9 and 10 the hypothalamic lesion was very extensive. Our material is not infavourable for the concept of GAMPER [12]) that the mamillary bodies and their surroundings have somethings to do with the syndrome of KORSAKOFF. However we cannot conclude much more, because this syndrome is also frequently seen in other localisations of pathological processes in the brain. From the physiologico-anatomical point of view the mamillary bodies have perhaps only significance for the elaboration of smell and of some qualities of sensation. But these ganglions are connected with many other parts of the brain and have to be regarded as association centres (ROUSSY and MOSINGER [29]), hence it is probable that they form an important point of junction in the cerebral chain reflexes, which make psychical functions possible.

Our material shows once more the great variety in the clinical pictures of hypothalamic lesions. This is caused by the variety in localisation, nature and tempo of development of the pathological process. Furthermore there are often accompanying disturbances in the brain outside the hypothalamus. The clinical anatomical observations seldom offer the opportunity to connect functional disturbances with the damage of special hypothalamic nuclei.

In experimental physiology the depression of function is caused by acute lesions in which the chain of neurons is suddenly interrupted. Furthermore most of the animals are studied only for a short time after the operation. The above described observations in man however were cases,

in which the pathological process developed slowly from the onset and compensation by other parts of the central nervous system could take place (RIDDOCH [28]) a.o.).

In order to explain the contradiction between the experimental and clinical facts one has to consider the experiences in other parts of the brain. An acute lesion in man may cause for severe disturbances in function, which gradually disappear for the greatest part after some time. The theory of the diaschisis of VON MONAKOW [23, 24]), which deepened our insight in the cerebral localisation of functions, has to be applied also to the hypothalamus. Many facts, shown by modern neuro-surgery, are in favour of this conception. FOERSTER [9]) for example showed that excision of circumscribed parts of the anterior central gyrus in man causes total paralysis of the corresponding extremity, but the main part of motility recovers very soon. Exstirpation of circumscribed parts of the posterior central gyrus evokes total loss of cutaneous and deep sensibility in the corresponding part of the body. Recovery however takes place and after some time the usual methods of clinical examination fail to show any diminution of sensitive function.

There is a luxury in the functional localisation in the brain, not the least in the vegetative central nervous system. Economic reasons incite nature to follow frequently used pathways. If one of the fuses ruptures, other redouble their activity. In the present days of scientific investigation attention is focussed on the hypothalamus, but vegetative functions are also represented in the cerebral cortex, although they are more widely spread. The different negative cases in hypothalamic affections point clearly in the direction of this functional luxury. Therefore positive and negative aspects have to be studied with equal care.

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Mathematics. — *A new generalization of* BORSUKs *theory of retracts.* By SZE-TSEN HU. (Communicated by Prof. L. E. J. BROUWER.)

(Communicated at the meeting of September 27, 1947.)

The important theory of retracts for compacta was created by BORSUK in the early thirties of this century. Later, by removing the condition of compactness, it was generalized to separable metric spaces by KURATOWSKI, LEFSCETZ, and others. An excellent account of this theory with a fairly comprehensive bibliography can be found in a monograph of LEFSCHETZ [5]. All works so far on retracts have been confined to separable metric spaces, as was explained by LEFSCHETZ [5, p. 58] that little is known on these questions for more general spaces.

The present paper is to show that an equally satisfactory and useful generalization of BORSUKs theory can be established for HAUSDORFF spaces if we retain the condition of compactness. Actually, we go one step further and define the new notions of absolute retracts and absolute neighbourhood retracts in terms of TYCHONOFF spaces (= completely regular spaces) [4, p. 28]; but the theory can be considered as satisfactory only in the case that the space is compact (= bicompact). In this paper, we study only the relations between the retracts and the extension properties of mappings and homotopies. Further developments will appear in later works.

We first recall that if Y is a topological space and $X \subset Y$, then a mapping (= continuous transformation) θ: $Y \to X$ such that $\theta \mid X = 1$ (i.e. the identity), is known as a *retraction* of Y onto X. If a retraction of Y onto X exists then X is known as a *retract* of Y; if X is a retract of some open set $U \supset X$ of Y then X is known as a *neighbourhood retract* of Y. Then we lay down the following definitions of absolute retracts and absolute neighbourhood retracts; (1.1) is the existing definition due to LEFSCHETZ.

(1.1) **Definition.** *A separable metric space X is said to be an AR [ANR] whenever a topological image of X as a closed subset X_1 of any other separable metric space Y is necessarily a retract [a neighbourhood retract] of Y.*

(1.2) **Definition.** *A* TYCHONOFF *space X is said to be an AR^* [ANR^*] whenever a topological image of X as a closed subset X_1 of any other* TYCHONOFF *space Y is necessarily a retract [a neighbourhood retract] of Y.*

It follows immediately from the definitions that every separable metric AR^* [ANR^*] is an AR [ANR]. Whether the converse holds in general is unknown; however, if X is compact, then the converse is true as shown by (4.2). Hence our new notion (1.2) is really a generalization of BORSUKs original definition.

(2.1) **Definition.** *A closed subset S_0 of a topological space S is said to have the extension property [the neighbourhood extension property] in S with respect to a topological space X, if every mapping $f: S_0 \to X$ can be extended over S [over some open set $U \supset S_0$ of S].*

Passing to BORSUK extension theorem, we first state the generalized form given by LEFSCHETZ [5, p. 60] and then prove an analogous theorem for our new theory. It should be noted that in the proof of (3.1) LEFSCHETZ did not use the separability of the space S.

(3.1) **Theorem.** *In order that a separable metric space X be an AR [ANR] it is necessary that every closed subset S_0 of an arbitrary metric space S has the extension property [the neighbourhood extension property] in S with respect to X.*

(3.2) **Theorem.** *In order that a* TYCHONOFF *space X be an AR^* [ANR^*] it is necessary that every closed subset S_0 of an arbitrary normal* HAUSDORFF *space S has the extension property [the neighbourhood extension property] in S with respect to X.*

Proof. According to TYCHONOFF imbedding theorem [4, p. 29], there is a homeomorphism $h: X \to X_1$ which maps X onto a subset X_1 of a compact parallelotope $Q = P_\alpha I_\alpha$, $\alpha \in A$, where I_α denotes the closed interval $(0, 1)$ of real numbers for each index $\alpha \in A$. Each point $q \in Q$ is represented by $q = \{q_\alpha \mid \alpha \in A\}$ where $0 \leq q_\alpha \leq 1$ are the coordinates of q. Denote by $\pi_\alpha: Q \to I_\alpha$ the projection of Q onto I_α.

Suppose $f: S_0 \to X$ be an arbitrary mapping. Define $\varphi: S_0 \to X_1$ by taking $\varphi = hf$ and let $\varphi_\alpha = \pi_\alpha \varphi$ for each $\alpha \in A$. Since S_0 is a closed subset of a normal space S, it follows from TIETZE extension theorem [4, p. 28] that φ_α has an extension $\varphi_\alpha^*: S \to I_\alpha$ for each $\alpha \in A$. Define a mapping $\varphi^*: S \to Q$ by taking $\varphi^*(s) = \{\varphi_\alpha^*(s) \mid \alpha \in A\}$ for each $s \in S$; then φ^* is an extension of φ, whose continuity follows from the topology of product spaces.

Since every normal HAUSDORFF space is also a TYCHONOFF space [4, p. 29], there is a homeomorphism $k: S \to S_1$ where S_1 is a subset of a compact parallelotope $Z = P_\mu I_\mu$, $\mu \in M$, whose points are $z = \{z_\mu \mid \mu \in M\}$ with coordinates $0 \leq z_\mu \leq 1$. Let us denote the point $k(s) \in Z$ by $\{s_\mu \mid \mu \in M\}$ for each $s \in S$. Let B denote the totality of finite subsets of M and let R denote the compact parallelotope $P_\beta I_\beta$, $\beta \in B$. For two arbitrary points $s, t \in S$, let us correspond a point $\varrho(s, t) = \{r_\beta \mid \beta \in B\}$ of R by taking

$$r_\beta = \max_{\mu \in \beta} |s_\mu - t_\mu|, \quad \beta \in B.$$

Let us denote by 0 the point of R whose coordinates $r_\beta = 0$ for each $\beta \in B$; then $\varrho(s, t) = 0$ if and only if $s = t$. Define a mapping $\psi: S \to R$ as follows, whose continuity can be easily seen. For each $s \in S$, we define $\psi(s) = \{r_\beta \mid \beta \in B\}$ by taking r_β to be the greatest lower bound of $\pi_\beta \varrho(s, t)$ for all $t \in S_0$, where $\pi_\beta: R \to I_\beta$ denotes the projection. It follows

easily from the topology of product spaces that $\psi(s) = 0$ if and only if $s \in S_0$.

Now let $T = Q \times R$ and define a mapping $g\colon S \to T$ by taking $g(s) = (\varphi^*(s), \psi(s))$ for each $s \in S$. Identifying Q with $Q \times 0 \subset T$, we observe that $g(S_0) \subset X_1$ and that $g(S - S_0) \subset T - Q$. Since Q is compact and $X_1 \subset Q$, necessarily $\overline{X_1} \subset Q$. Hence $g(S) \subset T - (\overline{X_1} - X_1)$ and so $\overline{X_1} \cap (X_1 \cup g(S)) = X_1$. Thus X_1 is a closed subset of $X_1 \cup g(S) = Y$, which is a TYCHONOFF space as every subset of a compact HAUSDORFF space should be.

Suppose X be an AR^*. Then by definition (1.2), there exists a retraction $\theta\colon Y \to X_1$. Define a mapping $f^*\colon S \to X$ by taking $f^*(s) = h^{-1}\theta g(s)$ for each $s \in S$. If $s \in S_0$, then we have

$$f^*(s) = h^{-1}\theta g(s) = h^{-1} g(s) = h^{-1}\varphi(s) = h^{-1} hf(s) = f(s);$$

hence f^* is an extension of f over S.

Next, suppose X to be an ANR^*. Then by (1.2), there exist an open set $U \supset X_1$ of Y and a retraction $\theta\colon U \to X_1$ of U onto X_1. Let $V = g^{-1}(U)$, then V is an open set of S containing S_0. Define a mapping $f^*\colon V \to X$ by taking $f^*(s) \to h^{-1}\theta g(s)$ for each $s \in V$; then it follows as above that f^* is an extension of f over V. This completes the proof.

Clearly the condition in (3.1) is also sufficient; but the condition in (3.2) is proved to be sufficient only for the case when X is compact, cf. our main theorem (5.1).

Now we are going to investigate the interesting particular case of a compact X. First, we note that the following strengthened form of (3.1) can be easily shown by the aid of TIETZE extension theorem.

(4.1) **Theorem.** *In order that a compactum X be an AR [ANR] it is necessary and sufficient that every closed subset S_0 of an arbitrary normal space S has the extension property [the neighbourhood extension property] in S with respect to X.*

(4.2) **Theorem.** *Every compact AR [ANR] is an AR^* [ANR^*].*

Proof. Let X be a compactum and $h\colon X \to X_1$ be an arbitrary homeomorphism of X onto a subset X_1 of a TYCHONOFF space Y. According to TYCHONOFF imbedding theorem, Y can be considered as a subset of some compact HAUSDORFF space Z. Since the image X_1 of a compact space X is compact, X_1 is closed both in Y and in Z. As a compact HAUSDORFF space, Z is normal [4, p. 26].

Suppose X be an AR; then it follows from (4.1) that the mapping $h^{-1}\colon X_1 \to X$ has an extension $k\colon Z \to X$. Define a mapping $\theta\colon Y \to X_1$ by taking $\theta(y) = hk(y)$ for each $y \in Y$. Clearly $\theta \mid X_1 = 1$, and so X is an AR^* by (1.2).

Next, suppose X be an ANR; then it follows from (4.1) that there exists an open set U of Z which contains X_1 such that the mapping

h^{-1}: $X_1 \to X$ has an extension k: $U \to X$. Let $V = U \cap Y$, then V is an open set of Y containing X_1. Define a mapping θ: $V \to X_1$ by taking $\theta(y) = hk(y)$ for each $y \in V$. Obviously θ is a retraction of V onto X_1; and hence X is an ANR^* by our definition (1.2). Thus our proof is completed.

We are in a position to establish our comprehensive main theorem of compact retracts stated as follows.

(5.1) **Theorem.** *For a compact* HAUSDORFF *space X, the following statements are equivalent:*

(i) *X is an AR^* [ANR^*].*

(ii) *X is homeomorphic with a retract [a neighbourhood retract] of some compact parallelotope Q.*

(iiii) *Every closed subset S_0 of an arbitrary normal space S has the extension property [the neighbourhood extension property] in S with respect to X.*

(iv) *Every closed subset S_0 of an arbitrary normal* HAUSDORFF *space S has the extension property [the neighbourhood extension property] in S with respect to X.*

(v) *Every compact subset S_0 of an arbitrary* TYCHONOFF *space S has the extension property [the neighbourhood extension property] in S with respect to X.*

Proof. Let us prove the theorem for ANR^*. The proof for AR^* is analogous and simpler.

(i) $\to$ (ii). Since a compact HAUSDORFF space is completely regular, there exists a homeomorphism h: $X \to X_1$ of X onto a subset X_1 of some compact parallelotope Q. Since X_1 is compact, it is closed in Q. Hence it follows from (1.2) that X_1 is a neighbourhood retract of the compact parallelotope Q.

(ii) $\to$ (iii). Suppose h: $X \to X_1$ be a homeomorphism of X onto a neighbourhood retract X_1 of a compact parallelotope Q. Let f: $S_0 \to X$ be an arbitrary mapping of a given closed subset S_0 of a normal space S into X. According to TIETZE extension theorem, the mapping $\varphi = hf$ has an extension φ^*: $S \to Q$. Let θ: $U \to X_1$ be a retraction of an open set $U \supset X_1$ of Q onto X_1 and let $V = \varphi^{*-1}(U)$. Then V is an open set of S containing S_0. Define a mapping f^*: $V \to X$ by taking $f^*(s) = h^{-1}\theta\varphi^*(s)$ for each $s \in V$; then f^* is an extension of f over V.

(iii) $\to$ (iv) is trivial.

(iv) $\to$ (v). Let f: $S_0 \to X$ be an arbitrary mapping of a compact subset S_0 of a TYCHONOFF space S into X. By TYCHONOFF imbedding theorem, S can be identified with a subset of a compact HAUSDORFF (and hence normal) space Z. Since S_0 is compact, it is a closed subset of Z. Therefore, it follows from (iv) that there exist an open set $U \supset S_0$ of Z and an extension $f^*: U \to X$ of f. Let $V = U \cap S$, then V is an open set of S containing S_0 and $f^* \mid V$ is an extension of f over V.

(v) → (i). Let h: $X \to X_1$ be a homeomorphism of X onto a subset X_1 of a TYCHONOFF space Y. By (v), the mapping h^{-1}: $X_1 \to X$ can be extended to a mapping k: $U \to X$, where U denotes some open set of Y containing X_1. Define a mapping θ: $U \to X_1$ by taking $\theta(y) = hk(y)$ for each $y \in U$; then θ is clearly a retraction of U onto X_1. Hence X is an ANR^* by our definition (1. 2). This completes our proof.

In the remainder of the present paper, we shall exhibit some theorems concerning the homotopy extension property with respect to an absolute neighbourhood retract.

(6. 1) **Definition.** *A closed subset S_0 of a topological space S is said to have the homotopy extension property in S with respect to a topological space X, if every partial homotopy f_t: $S_0 \to X$ $(0 \leqq t \leqq 1)$ of an arbitrary mapping f_0: $S \to X$ has an extension f_t^*: $S \to X$ $(0 \leqq t \leqq 1)$ such that $f_0^* = f_0$.*

(6. 2) **Definition.** *A normal space S is said to be binormal if $S \times I$ is normal, where I denotes the closed interval* $(0, 1)$.

In particular, every metric space is binormal. For another example, every paracompact [1, p. 66] HAUSDORFF space is binormal; for, according to a theorem of DIEUDONNÉ [1, Theorem 5] $S \times I$ is also a paracompact HAUSDORFF space and hence normal.

With some easy modifications of DOWKER's arguments [3, p. 86], the following BORSUK homotopy extension theorems can be proved.

(7. 1) **Theorem.** *If X is an ANR, then every closed subset S_0 of an arbitrary metric space S has the homotopy extension property in S with respect to X.*

(7. 2) **Theorem.** *If X is an ANR^*, then every closed subset S_0 of an arbitrary binormal HAUSDORFF space S has the homotopy extension property in S with respect to X.*

(7. 3) **Theorem.** *If X is a compact ANR^*, then every closed subset S_0 of an arbitrary binormal space S has the homotopy extension property in S with respect to X.*

For a compact ANR, (7. 3) overlaps with a recent theorem due to DOWKER [2, p. 232].

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Mathematics. — *Sur une formule d'inversion de* STIELTJES *et la théorie du potentiel.* By A. F. MONNA. (Communicated by Prof. W. VAN DER WOUDE.)

(Communicated at the meeting of October 25, 1947.)

§ 1. Soit donnée une répartition $m(e)$ de masses de signe variable sur la droite $y = 0$ du plan de la variable complexe $z = x + iy$. On entend par cela que $m(e)$ est une fonction additive au sens complet d'ensemble à variation bornée. Pour les ensembles d'un corps fermé la condition d'être à variation bornée est une conséquence de l'additivité [1]).

Considérons l'intégrale de STIELTJES

$$R(z) = \int_{y=0} \frac{d\,m\,(e_t)}{z-t} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

étendue sur la droite $y = 0$ (t parcourt cette droite). $R(z)$ est une fonction holomorphe de z pour les valeurs non-réelles de z. La formule d'inversion de STIELTJES, que nous avons en vue, fournit un moyen de déterminer les masses $m(e)$ par les valeurs de $R(z)$. D'ordinairement on écrit cette formule en partant d'une fonction de points à variation bornée au lieu d'une fonction d'ensembles. La relation entre ces deux points de vu est bien connue. Désignons par $I[R(z)]$ la partie imaginaire de $R(z)$. La formule d'inversion de STIELTJES prend alors la forme [2])

$$-\frac{1}{\pi} \lim_{y \to +0} \int_{x_1}^{x_2} I\,[R(z)]\,dx = \int_{x_1}^{x_2} d\,m\,(e_t) + \tfrac{1}{2}\,(m_1 + m_2), \quad . \quad . \quad (2)$$

où m_1 et m_2 désignent les sauts de $m(e)$ aux points x_1 et x_2, autrement dit les masses finis qui se trouvent en x_1 et x_2; si x_1 et x_2 sont des points de continuité on a $m_1 = m_2 = 0$ [3]). On a si $r_\varepsilon^2 = (x-t)^2 + (y-\varepsilon)^2$

$$I\,[R(z)] = -\,y \int_{y=0} \frac{d\,m\,(e_t)}{(x-t)^2 + y^2} = -\int \left(\frac{\partial \log \frac{1}{r_\varepsilon}}{\partial\,\varepsilon} \right)_{\varepsilon = 0} d\,m\,(e_t). \quad . \quad (3)$$

Il y a une relation entre la dernière intégrale et les potentiels des couches doubles si on généralise cette dernière notion. Dans la théorie du potentiel on définit une couche double sur une courbe Γ suffisamment régulière en

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[1]) S. SAKS. Theory of the integral (Warszawa 1937) p. 10.

[2]) Voir p. ex. A. WINTNER. Spektraltheorie der unendlichen Matrizen (Leipzig 1929) p. 96.

[3]) Les points de discontinuité constituent un ensemble dénombrable.

se donnant une fonction $\nu(t)$, définie sur Γ, appelée le moment de la couche. Le potentiel au point P vaut

$$\int_{\Gamma} \frac{\partial \log \frac{1}{r}}{\partial n} \nu(t)\, ds, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

où r désigne la distance de P aux points de Γ et $\frac{\partial}{\partial n}$ désigne la dérivé normale aux points de Γ suivant les normales de direction fixée en ces points. Cependant rien n'empêche de considérer des potentiels analogues à (4) qui ne puissent pas s'exprimer par un moment $\nu(t)$ et pour lesquels il faut prendre une intégrale de LEBESGUE–STIELTJES:

$$U(P) = \int_{\Gamma} \frac{\partial \log \frac{1}{r}}{\partial n}\, dm(e), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

$m(e)$ désignant une distribution sur Γ. On peut appeler U le potentiel de la couche double, déterminée par $m(e)$. Moyennant cette définition la dernière intégrale dans (3) vaut le potentiel au point $z = x + iy$ de la couche double sur $y = 0$, déterminée par $m(e)$. En le posant $U(z)$, la formule (2) devient

$$\lim_{y \to +0} \int_{x_1}^{x_2} U(z)\, dx = \pi \int_{x_1}^{x_2} dm(e_x) + \frac{\pi}{2}(m_1 + m_2). \quad . \quad . \quad . \quad (6)$$

L'intégrale du membre à gauche est étendue sur l'intervalle (x_1, y), (x_2, y) où $y > 0$. La formule exprime la masse en termes des valeurs du potentiel. La formule de STIELTJES ayant ainsi obtenu une forme où n'entrent que des notions, appartenant à la théorie du potentiel, il est naturel de chercheur une démonstration par des moyens adéquats. Or ceci est simple. On a

$$\int_{x_1}^{x_2} dx \int_{y=0} \frac{\partial \log \frac{1}{r}}{\partial n}\, dm(e_t) = \int_{y=0} dm(e_t) \int_{x_1}^{x_2} \frac{\partial \log \frac{1}{r}}{\partial n}\, dx. \quad . \quad . \quad (7)$$

Ici $y =$ constante > 0; $\frac{\partial}{\partial n} \log \frac{1}{r}$ est la dérivé normale de $\log \frac{1}{r}$ au point $(x, 0)$. Si r est la distance de $(x, 0)$ à (x', y), cette dérivé normale est égale à la dérivé normale de $\log \frac{1}{r}$ au point (x', y) par rapport à la normale de direction opposée en ce point. L'intégrale

$$\int_{x_1}^{x_2} \frac{\partial \log \frac{1}{r}}{\partial n}\, dx$$

a donc une signification simple: c'est l'angle entre la droite passant par les points $(t, 0)$ et (x_1, y) et celle passant par $(t, 0)$ et (x_2, y). Il s'ensuit

$$\lim_{y \to +0} \int_{x_1}^{x_2} \frac{\partial \log \frac{1}{r}}{\partial n} dx = \left\{ \begin{array}{lll} \pi & \text{si} & x_1 < t < x_2 \\ \frac{\pi}{2} & \text{si} & t = x_1 \text{ ou } t = x_2 \\ 0 & \text{si} & t < x_1 \text{ ou } t > x_2 \end{array} \right\} . \quad . \quad . \quad (8)$$

Puisque le passage à la limite sous le signe d'intégration est permis dans (7) — l'intégrand est uniformément borné par rapport à y — il en résulte (6).

§ 2. Examinons ce que devient de (6) en prenant pour la courbe Γ dans (5) un cercle de rayon R. Soit (α) un arc ouvert de Γ. Considérons les cercles concentriques Γ_i de rayon R_i intérieur à Γ et soient (α_i) les arcs découpés sur les Γ_i par les normales à Γ aux extrémités de (α); $(\alpha_i) \to (\alpha)$ si $\Gamma_i \to \Gamma$. Désignons par n la normale intérieure aux points de Γ, par r la distance de ces points aux points de (α_i). On a

$$\left. \begin{aligned} \int_{(\alpha_i)} U\, ds = \int_{(\alpha_i)} ds \int_{\Gamma} \frac{\partial \log \frac{1}{r}}{\partial n} dm(e) = \int_{\Gamma} dm(e) \int_{(\alpha_i)} \frac{\partial \log \frac{1}{r}}{\partial n} ds = \\ = \int_{(\alpha)} dm(e) \int_{(\alpha_i)} \frac{\partial \log \frac{1}{r}}{\partial n} ds + \int_{\Gamma - (\alpha)} dm(e) \int_{(\alpha_i)} \frac{\partial \log \frac{1}{r}}{\partial n} ds \end{aligned} \right\} . \quad (9)$$

Il s'agit de déterminer

$$\lim_{\Gamma_i \to \Gamma} \int_{(\alpha_i)} \frac{\partial \log \frac{1}{r}}{\partial n} ds \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

où r est la distance d'un point fixé P de Γ aux points de (α_i): $\frac{\partial}{\partial n}$ désigne la dérivé normale en P par rapport à la normale intérieure en P à Γ. Il faut donc déterminer

$$\lim_{\Gamma_i \to \Gamma} \int_{(\alpha_i)} \frac{\cos(n, r)}{r} ds . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

1. Soit P un point intérieur de (α). Soient C un cercle de centre P et de rayon ϱ suffisamment petit (destiné à tendre vers zéro). (α_i, ϱ) la partie de (α_i) intérieur à C (il faut prendre $R - R_i$ suffisamment petit) et (α_ϱ) la partie de (α) intérieur à C. On a

$$\int_{(\alpha_i)} \frac{\cos(n, r)}{r} ds = \int_{(\alpha_{i,\varrho})} \frac{\cos(n, r)}{r} ds + \int_{(\alpha_i) - (\alpha_{i,\varrho})} \frac{\cos(n, r)}{r} ds = I_1 + I_2 . \quad (12)$$

Considérons d'abord I_2. Alors $\frac{\cos(n, r)}{r}$ tend vers la dérivé normale de $\log \frac{1}{r}$ aux points de $(a) - (a_\varrho)$; r désigne la distance de P aux points de $(a) - (a_\varrho)$, $\frac{\partial}{\partial n}$ la dérivé normale suivant les normales à Γ en ces points. Le passage à la limite $\Gamma_i \to \Gamma$ peut, pour I_2, s'effectuer sous le signe d'intégration. Cette limite vaut donc la longueur de la projection de $(a) - (a_\varrho)$ sur le cercle-unité de centre P, donc, si aR est la longueur de (a), $\frac{1}{2}a - \delta$, où δ tend vers zéro si $\varrho \to 0$.

Afin d'effecteur $\lim I_1$, on considère d'abord l'intégrale

$$I_1' = \int \frac{\cos(n, r')}{r'} ds'$$

étendue sur la corde $K_{i,\varrho}$ de l'arc $a_{i,\varrho}$; r' désigne donc la distance de P aux points de $K_{i,\varrho}$. I'_1 vaut l'angle de sommet P et de base $K_{i,\varrho}$. Si $\Gamma_i \to \Gamma$ elle tend donc vers l'angle de sommet P et de base K_ϱ (la corde de (a_ϱ)). Si alors $\varrho \to 0$, cette angle tend vers π et on montre [4]) qu'on a pour des valeurs convenables de ϱ

$$\lim_{\substack{\Gamma_i \to \Gamma \\ \varrho \to 0}} I_1 = \lim_{\substack{\Gamma_i \to \Gamma \\ \varrho \to 0}} I_1' = \pi.$$

ϱ étant arbitraïre dans (12) on a donc

$$\lim_{\Gamma_i \to \Gamma} \int \frac{\cos(n, r)}{r} ds = \tfrac{1}{2} a + \pi,$$

si P est intérieur à (a).

2. Soit P un point intérieur de $\Gamma - (a)$. Comme pour I_2 ci-dessus on voit que dans ce cas (11) est égal à $\frac{1}{2}a$.

3. Si P est un point-frontière de (a) on trouve pour limite $\frac{1}{2}a + \frac{\pi}{2}$ (comparer les trois cas (8) dans § 1).

Il suit de (9)

$$\lim_{\Gamma_i \to \Gamma} \int_{(\alpha_i)} U ds = (\tfrac{1}{2}a + \pi) m((a)) + \tfrac{1}{2} a m(\Gamma - (a)) + \left(\frac{a}{2} + \frac{\pi}{2}\right)(m_1 + m_2) = \left.\begin{array}{l} \\ \\ = \pi m((a)) + \tfrac{1}{2} a m(\Gamma) + \tfrac{1}{2} \pi (m_1 + m_2) \end{array}\right\}, \quad (13)$$

Il est aisé de retourner de cette formule aux fonctions d'une variable com-

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[4]) On fait tendre ϱ et $R - R_i$ simultanément vers zéro tel que $\frac{R - R_i}{\varrho} \to 0$ et $\frac{\varrho^2}{R - R_i} \to 0$. On a

$$|I_1 - I_1'| \leqq C \frac{\varrho^2 - (R - R_i)^2}{R - R_i}.$$

C une constante convenable.

plexe. Supposons que Γ est le cercle-unité. On a alors, en désignant par $Re\ \Phi$ la partie réelle de Φ

$$U = Re\ \Phi(z),$$

$$\Phi(z) = \int_0^{2\pi} \frac{e^{i\varphi}}{e^{i\varphi} - z}\, dm(\varphi).$$

Puisque $R_i \to 1$, (13) se réduit à

$$\lim \int_{\varphi_1(\Gamma_i)}^{\varphi_2} Re\ \Phi(z)\, d\varphi = \pi\, m(a) + \frac{\varphi_2 - \varphi_1}{2}\, m(\Gamma) + \frac{\pi}{2}(m_1 + m_2),$$

ou encore si

$$\Psi(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{i\varphi} + z}{e^{i\varphi} - z}\, dm(\varphi),$$

$$\lim_{R_i \to 1} \int_{\varphi_1(\Gamma_i)}^{\varphi_2} Re\ \Psi(z)\, d\varphi = m(a) + \tfrac{1}{2}(m_1 + m_2).$$

On est ainsi conduit aux propriétés connues de l'intégrale de POISSON-STIELTJES [5]).

§ 3. On sait que le potentiel d'une couche double au sens classique subit un saut si l'on passe d'une côté de la couche à l'autre et que la valeur du saut dépend de la valeur du moment au point où la couche est traversée. Dans les cas considérés, où il n'existe pas un moment, cette propriété doit être remplacée par la suivante.

a. *Couche double sur la ligne droite.*

Si l'intégration au membre à gauche de (6) a lieu sur les droites $y < 0$ et si alors $y \to -0$, il faut remplacer cette formule par

$$\lim_{y \to -0} \int_{x_1}^{x_2} U(z)\, dx = -\pi \int_{x_1}^{x_2} dm(e_x) - \frac{\pi}{2}(m_1 + m_2) \quad . \quad . \quad . \quad (6a)$$

En désignant par e_{x_1, x_2} l'intervalle ouvert (x_1, x_2), il résulte de (6) et (6a)

$$\lim_{y \to +0} \int_{x_1}^{x_2} U(z)\, dx - \lim_{y \to -0} \int_{x_1}^{x_2} U(z)\, dx = 2\pi\, m(e_{x_1, x_2}) + \pi(m_1 + m_2). \quad (14)$$

[5]) R. NEVANLINNA. Eindeutige analytische Funktionen (Berlin 1936) p. 187 e.s.

b. *Couche double sur un cercle.*

Considérons les cercles concentriques Γ_e, extérieurs à Γ et les arcs (α_e), découpés sur Γ_e. Au lieu de (13) on trouve alors

$$\lim_{\Gamma_e \to \Gamma} \int_{(\alpha_e)} U\,ds = -\pi\, m(\alpha) + \tfrac{1}{2}\,\alpha\, m(\Gamma) - \frac{\pi}{2}\,(m_1 + m_2)\,, \quad . \quad (13a)$$

de sorte que

$$\lim_{\Gamma_i \to \Gamma} \int_{(\alpha_i)} U\,ds - \lim_{\Gamma_e \to \Gamma} \int_{(\alpha_e)} U\,ds = 2\pi\, m(\alpha) + \pi\,(m_1 + m_2)\,. \quad . \quad (15)$$

Les formules (14) et (15) expriment la même propriété:

En traversant la courbe de la couche double, l'intégrale du potentiel, étendue sur des arcs convenable situés à l'une côté et à l'autre de l'arc considéré de la couche, subit un saut dont la valeur est déterminée par la masse qui se trouve sur cet arc et aux extrémités.

Il serait intéressant de savoir si cette propriété subsiste pour des courbes plus générales et de la démontrer alors, comme ci-dessus, en n'utilisant que des moyens adéquats. On pourrait chercher s'il y a des extensions spatiales.

§ 4. Au moyen de (13) on peut résoudre le problème suivant.

Soit Ω l'intérieur d'un cercle Γ. On se donne une distribution de masse $\mu(e)$ sur Γ et on demande à chercher une fonction harmonique $U(P)$ dans Ω telle que pour chaque arc ouvert α de Γ et pour les arcs α_i des cercles Γ_i, tels qu'ils ont été utilisés ci-dessus, on a

$$\lim_{\alpha_i \to \alpha} \int_{\alpha_i} U\,ds = \mu(\alpha) + \frac{\mu_1 + \mu_2}{2}\,. \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

On obtient une solution en prenant pour U le potentiel d'une couche double sur Γ, dont la distribution $m(e)$ est déterminée ainsi:

$m(e)$ et $\mu(e)$ ont le même ensemble de points de discontinuité; si μ est la masse finie qui se trouve en Q, la distribution $m(e)$ y a la masse $\dfrac{1}{2\pi}\,\mu$. Si α est un arc ouvert, il faut que

$$\pi\, m(\alpha) + \tfrac{1}{2}\,\alpha\, m(\Gamma) = \mu(\alpha).$$

Pour $\alpha \equiv \Gamma$ donc

$$2\pi\, m(\Gamma) = \mu(\Gamma).$$

Il s'ensuit

$$m(\alpha) = \frac{1}{\pi}\,\mu(\alpha) - \frac{1}{4\pi^2}\,\mu(\Gamma).$$

On peut généraliser ce problème en prenant pour Ω un ensemble ouvert à frontière Σ. Il sera alors nécessaire de modifier d'une façon convenable la condition (16), imposée à la fonction harmonique U cherchée. En particulier il sera nécessaire de préciser le caractère des ensembles α_i, tendant vers l'ensemble donné α de Σ.

Un tel problème aura sans doute une solution si la distribution donnée $\mu(e)$ sur Σ admet une densité $f(Q)$, fonction continue sur Σ, et si Ω est tel que le problème de DIRICHLET, posé pour Ω et valeurs-frontière $f(Q)$, admet une solution: en effet cette solution fournira aussi une solution du nouveau problème. En ce sens le problème de DIRICHLET apparaît comme un cas particulier du nouveau problème.

La Haye, septembre 1947.

Mathematics. — *Extension of* PEARSON's *Probability Distributions to two Variables.* I. By M. J. VAN UVEN. (Communicated by Prof. W. VAN DER WOUDE.)

(Communicated at the meeting of September 27, 1947.)

§ 1. Notations. Partial means. Regression equations. Regression coefficients. Partial standard deviations. Relations between the moments.

The present paper deals with a family of binary (bivariate) probability distributions obtained by extending PEARSON's differential equation $\frac{d \log \varphi}{dx} = \frac{x+\delta}{ax^2+\beta x+\gamma}$ to two variables. Some of these distributions have already been studied formerly (by L. N. G. FILON, L. ISSERLIS and K. PEARSON).

It will appear that, for a large group of distributions derived in the following sections, the regression equations are l i n e a r, and that the formulae for the regression coefficients and the partial standard deviations are the same as those which occur in normal correlation.

A. To begin with we introduce the following notations:

Probability density function of the variables x and y: $\varphi(x, y)$.

Probability differential: $d\,W = \varphi\, dx\, dy$.

Contour of the probability domain: O.

Limits of x with fixed y: ξ_α and ξ_ω (generally functions of y).

,, ,, y ,, ,, x: η_α ,, η_ω (,, ,, ,, x).

Extreme limits of x: x_α and x_ω (constants).

,, ,, ,, y: y_α ,, y_ω (,,).

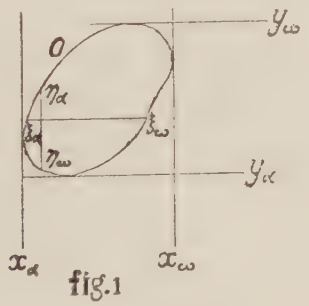


fig. 1

Hence: $$\int_{y_\alpha}^{y_\omega}\int_{\xi_\alpha}^{\xi_\omega} \varphi\, dx\, dy = \int_{x_\alpha}^{x_\omega}\int_{\eta_\alpha}^{\eta_\omega} \varphi\, dx\, dy = 1.$$

$$[\mathrm{F}(x)]_{\xi_\alpha}^{\xi_\omega} = \mathrm{F}(\xi_\omega) - \mathrm{F}(\xi_\alpha).$$

Probability density of x: $\mathbf{W}_x = \int_{\eta_\alpha}^{\eta_\omega} \varphi \, dy.$

,, ,, ,, y: $\mathbf{W}_y = \int_{\xi_\alpha}^{\xi_\omega} \varphi \, dx.$

Total (theoretical) mean of $\mathbf{f}(x, y)$ = mathematical expectation of $\mathbf{f}(x, y)$: $\langle \mathbf{f}(x, y) \rangle \equiv \widehat{\mathbf{f}} \equiv \int\int \mathbf{f}(x, y) \, \varphi \, dx \, dy$, e.g. $\widehat{x}, \widehat{y}$.

Partial mean of x with fixed y: $\widehat{x}^{(y)} = \dfrac{\int x \varphi \, dx}{\int \varphi \, dx}$.

,, ,, ,, y ,, ,, x: $\widehat{y}^{(x)} = \dfrac{\int y \varphi \, dy}{\int \varphi \, dy}$.

Regression equation of x with respect to y: $\widehat{x}^{(y)} = \chi_2(y)$.

,, ,, ,, y ,, ,, ,, x: $\widehat{y}^{(x)} = \chi_1(x)$.

Deviations from $\widehat{x}$ and $\widehat{y}$: $t_1 = x - \widehat{x}$, $t_2 = y - \widehat{y}$, whence $\widehat{t}_1 \equiv 0$, $\widehat{t}_2 \equiv 0$.

Moments with respect to $\widehat{x}$ and $\widehat{y}$:

$m^{i,j} \equiv \langle t_1^{\,i} \, t_2^{\,j} \rangle$, e.g. $m^{2,0} \equiv \langle t_1^{\,2} \rangle$, $m^{1,1} \equiv \langle t_1 \, t_2 \rangle$, $m^{0,2} \equiv \langle t_2^{\,2} \rangle$.

Total standard deviations: $\sigma_1 = \sqrt{m^{2,0}}$, $\sigma_2 = \sqrt{m^{0,2}}$.

Correlation coefficient: $\gamma = \dfrac{m^{1,1}}{\sigma_1 \, \sigma_2}$.

Partial standard deviations: $\sigma_{1;2} = \sqrt{\langle (x - \widehat{x}^{(y)})^2 \rangle}$, $\sigma_{2;1} = \sqrt{\langle (y - \widehat{y}^{(x)})^2 \rangle}$.

Functions will be denoted, besides by Greek letters, by Latin capitals. **A, B, C, D, P, Q, R, S** will represent linear functions of x and y: $\mathbf{A} \equiv a_0 + a_1 x + a_2 y, \ldots, \mathbf{S} \equiv s_0 + s_1 x + s_2 y$.

G and **H** will represent quadratic functions of x and y:

$$\mathbf{G} \equiv g_{00} + 2g_{01} x + 2g_{02} y + g_{11} x^2 + 2g_{12} xy + g_{22} y^2,$$

$$\mathbf{H} \equiv h_{00} + 2h_{01} x + 2h_{02} y + h_{11} x^2 + 2h_{12} xy + h_{22} y^2;$$

they become linear if $g_{11} = g_{12} = g_{22} = 0$, etc., constants, if besides $g_{01} = g_{02} = 0$, etc. $\mathbf{A}_1$, $\mathbf{B}_1$, ... will be functions only of x: $\mathbf{A}_1 \equiv a_0 + a_1 x, \ldots$; $\mathbf{A}_2, \mathbf{B}_2, \ldots$ will be functions only of y: $\mathbf{B}_2 \equiv b_0 + b_2 y, \ldots$.

Likewise $\mathbf{G}_1$, $\mathbf{H}_1$ will be functions only of x:

$$\mathbf{G}_1 \equiv g_{00} + 2g_{01} x + g_{11} x^2, \ldots,$$

$\mathbf{G}_2$, $\mathbf{H}_2$ only of y: $\mathbf{H}_2 \equiv h_{00} + 2h_{02} y + h_{22} y^2$.

$\mathbf{A}_0, \mathbf{B}_0, \ldots, \mathbf{K}_0, \ldots \mathbf{R}_0, \mathbf{S}_0$ will be constants.

B. Partial means.

As a starting point we consider PEARSON's differential equation for one variable

$$\frac{d \log \varphi}{dx} = \frac{x + \delta}{\alpha x^2 + \beta x + \gamma},$$

c.q. with $\alpha = 0$ (PEARSON's type III) or with $\alpha = 0$, $\beta = 0$ (PEARSON's type VII, the normal distribution). Extending this to two variables (x and y), we put

$$\frac{\partial \log \varphi}{\partial x} = \frac{\mathbf{P}}{\mathbf{G}}, \quad \frac{\partial \log \varphi}{\partial y} = \frac{\mathbf{Q}}{\mathbf{H}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

(where $\frac{\mathbf{P}}{\mathbf{G}}$ and $\frac{\mathbf{Q}}{\mathbf{H}}$ are assumed to be irreducible).

The functions $\mathbf{P}$, $\mathbf{G}$, $\mathbf{Q}$, $\mathbf{H}$ are linked by the relation

$$\frac{\partial^2 \log \varphi}{\partial x \, \partial y} = \frac{\partial}{\partial y}\left(\frac{\mathbf{P}}{\mathbf{G}}\right) = \frac{\partial}{\partial x}\left(\frac{\mathbf{Q}}{\mathbf{H}}\right),$$

or

$$\mathbf{H}\frac{\partial \mathbf{P}}{\partial y} - \mathbf{G}\frac{\partial \mathbf{Q}}{\partial x} = \frac{\mathbf{H}}{\mathbf{G}}\mathbf{P}\frac{\partial \mathbf{G}}{\partial y} - \frac{\mathbf{G}}{\mathbf{H}}\mathbf{Q}\frac{\partial \mathbf{H}}{\partial x} \quad . \quad . \quad . \quad . \quad . \quad (2)$$

From (1) follows

$$\int_{\xi_\alpha}^{\xi_\omega} \mathbf{G}\frac{\partial \varphi}{\partial x}\,dx = \int_{\xi_\alpha}^{\xi_\omega} \mathbf{P}\varphi\,dx, \text{ or } [\mathbf{G}\varphi]_{\xi_\alpha}^{\xi_\omega} - \int_{\xi_\alpha}^{\xi_\omega} \frac{\partial \mathbf{G}}{\partial x}\varphi\,dx = \int_{\xi_\alpha}^{\xi_\omega} \mathbf{P}\varphi\,dx, \text{ or}$$

$$\int_{\xi_\alpha}^{\xi_\omega} \left(\mathbf{P} + \frac{\partial \mathbf{G}}{\partial x}\right)\varphi\,dx = [\mathbf{G}\varphi]_{\xi_\alpha}^{\xi_\omega}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)_1$$

Likewise

$$\int_{\eta_\alpha}^{\eta_\omega} \left(\mathbf{Q} + \frac{\partial \mathbf{H}}{\partial y}\right)\varphi\,dy = [\mathbf{H}\varphi]_{\eta_\alpha}^{\eta_\omega}. \quad . \quad . \quad . \quad . \quad . \quad (3)_2$$

Putting

$$\mathbf{P} + \frac{\partial \mathbf{G}}{\partial x} \equiv \mathbf{R} \equiv r_0 + r_1 x + r_2 y, \quad \mathbf{Q} + \frac{\partial \mathbf{H}}{\partial y} \equiv \mathbf{S} \equiv s_0 + s_1 x + s_2 y, \quad . \quad (4)$$

we can write (3) as

$$\int_{\xi_\alpha}^{\xi_\omega} \mathbf{R}\varphi\,dx = [\mathbf{G}\varphi]_{\xi_\alpha}^{\xi_\omega}, \qquad \int_{\eta_\alpha}^{\eta_\omega} \mathbf{S}\varphi\,dy = [\mathbf{H}\varphi]_{\eta_\alpha}^{\eta_\omega}. \quad . \quad . \quad . \quad (3\text{bis})$$

or

$$r_0\int_{\xi_\alpha}^{\xi_\omega} \varphi\,dx + r_1\int_{\xi_\alpha}^{\xi_\omega} x\varphi\,dx + r_2 y\int_{\xi_\alpha}^{\xi_\omega} \varphi\,dx = [\mathbf{G}\varphi]_{\xi_\alpha}^{\xi_\omega},$$

$$s_0\int_{\eta_\alpha}^{\eta_\omega} \varphi\,dy + s_1 x\int_{\eta_\alpha}^{\eta_\omega} \varphi\,dy + s_2\int_{\eta_\alpha}^{\eta_\omega} y\varphi\,dy = [\mathbf{H}\varphi]_{\eta_\alpha}^{\eta_\omega},$$

or

$$r_0 + r_1\hat{x}^{(y)} + r_2 y = \frac{[\mathbf{G}\varphi]_{\xi_\alpha}^{\xi_\omega}}{\mathbf{W}_y} \equiv \Omega_2(y), \quad s_0 + s_1 x + s_2\hat{y}^{(x)} = \frac{[\mathbf{H}\varphi]_{\eta_\alpha}^{\eta_\omega}}{\mathbf{W}_x} \equiv \Omega_1(x). \quad (5)$$

C. Here we have obtained the regression equations:

$$\widehat{x}^{(y)} = \frac{1}{r_1}(\Omega_2 - r_0 - r_2 y), \qquad \widehat{y}^{(x)} = \frac{1}{s_2}(\Omega_1 - s_0 - s_1 x), \quad . \quad . \quad . \quad (5\,\text{bis})$$

which are valid, provided that $r_1 \neq 0$, $s_2 \neq 0$.

From (3 bis) we derive

$$\widehat{\mathbf{R}} = \int_{y_\alpha}^{y_\omega} [\mathbf{G}\varphi]_{\xi_\alpha}^{\xi_\omega}\, dy = \int_{y_\alpha}^{y_\omega} \Omega_2\, \mathbf{W}_y\, dy = \widehat{\Omega}_2, \qquad \widehat{\mathbf{S}} = \widehat{\Omega}_1 \quad . \quad . \quad . \quad (6)$$

Besides, from

$$\int_{\xi_\alpha}^{\xi_\omega} x\,\mathbf{G}\frac{\partial\varphi}{\partial x}\,dx = \int_{\xi_\alpha}^{\xi_\omega} x\,\mathbf{P}\varphi\,dx, \qquad \int_{\xi_\alpha}^{\xi_\omega} y\,\mathbf{G}\frac{\partial\varphi}{\partial x}\,dx = \int_{\xi_\alpha}^{\xi_\omega} y\,\mathbf{P}\varphi\,dx,$$

$$\int_{\eta_\alpha}^{\eta_\omega} x\,\mathbf{H}\frac{\partial\varphi}{\partial y}\,dy = \int_{\eta_\alpha}^{\eta_\omega} x\,\mathbf{Q}\varphi\,dy, \qquad \int_{\eta_\alpha}^{\eta_\omega} y\,\mathbf{H}\frac{\partial\varphi}{\partial y}\,dy = \int_{\eta_\alpha}^{\eta_\omega} y\,\mathbf{Q}\varphi\,dy$$

follows, by integrating the left members by parts, first

$$\int_{\xi_\alpha}^{\xi_\omega} (\mathbf{G} + x\mathbf{R})\,\varphi\,dx = [x\,\mathbf{G}\varphi]_{\xi_\alpha}^{\xi_\omega}, \qquad \int_{\xi_\alpha}^{\xi_\omega} y\,\mathbf{R}\varphi\,dx = [y\,\mathbf{G}\varphi]_{\xi_\alpha}^{\xi_\omega},$$

$$\int_{\eta_\alpha}^{\eta_\omega} x\,\mathbf{S}\varphi\,dy = [x\,\mathbf{H}\varphi]_{\eta_\alpha}^{\eta_\omega}, \qquad \int_{\eta_\alpha}^{\eta_\omega} (\mathbf{H} + y\,\mathbf{S})\,\varphi\,dy = [y\,\mathbf{H}\varphi]_{\eta_\alpha}^{\eta_\omega},$$

and next

$$\left.\begin{aligned} \widehat{\mathbf{G}} + \langle x\mathbf{R}\rangle &= \int_{y_\alpha}^{y_\omega} [x\mathbf{G}\varphi]_{\xi_\alpha}^{\xi_\omega}\, dy, \qquad \langle y\,\mathbf{R}\rangle = \int_{y_\alpha}^{y_\omega} [y\mathbf{G}\varphi]_{\xi_\alpha}^{\xi_\omega}\, dy, \\ \langle x\,\mathbf{S}\rangle &= \int_{x_\alpha}^{x_\omega} [x\mathbf{H}\varphi]_{\eta_\alpha}^{\eta_\omega}\, dx, \qquad \widehat{\mathbf{H}} + \langle y\,\mathbf{S}\rangle = \int_{x_\alpha}^{x_\omega} [y\mathbf{H}\varphi]_{\eta_\alpha}^{\eta_\omega}\, dx. \end{aligned}\right\} \quad . \quad (7)$$

We remark by the way that the equations

$$\langle(\mathbf{G}+x\mathbf{R})\rangle^{(y)} = \frac{[x\,\mathbf{G}\,\varphi]_{\xi_\alpha}^{\xi_\omega}}{\mathbf{W}_y} = \Omega_2^{1,0} \text{ and } \langle(\mathbf{H}+y\,\mathbf{S})\rangle^{(x)} = \frac{[y\,\mathbf{H}\,\varphi]}{\mathbf{W}_x} = \Omega_1^{0,1}$$

can serve as regression equations, provided that $g_{11} + r_1 = 0$, resp. $h_{22} + s_2 = 0$, so, in the case that $r_1 = 0$ and $s_2 = 0$, provided that $g_{11} = 0$, $h_{22} = 0$.

Introducing the deviations $t_1 = x - \widehat{x}$, $t_2 = y - \widehat{y}$ from the total means, we get

$$r_1 t_1 + r_2 t_2 = \mathbf{R} - \widehat{\mathbf{R}} \quad , \quad s_1 t_1 + s_2 t_2 = \mathbf{S} - \widehat{\mathbf{S}} \quad . \quad . \quad . \quad . \quad (8)$$

When diminishing the equations (7) resp. by $x\mathbf{R}$, $y\mathbf{R}$, $x\mathbf{S}$, $y\mathbf{S}$, we obtain

$$\left.\begin{aligned} \langle t_1\,\mathbf{R}\rangle &= \int_{y_\alpha}^{y_\omega} [t_1\,\mathbf{G}\varphi]_{\xi_\alpha}^{\xi_\omega}\, dy - \widehat{\mathbf{G}} \equiv \mathbf{R}_0^{1,0}, \qquad \langle t_2\,\mathbf{R}\rangle = \int_{y_\alpha}^{y_\omega} [t_2\,\mathbf{G}\varphi]_{\xi_\alpha}^{\xi_\omega}\, dy \equiv \mathbf{R}_0^{0,1}, \\ \langle t_1\,\mathbf{S}\rangle &= \int_{x_\alpha}^{x_\omega} [t_1\,\mathbf{H}\varphi]_{\eta_\alpha}^{\eta_\omega}\, dx \equiv \mathbf{S}_0^{1,0} \quad , \quad \langle t_2\,\mathbf{S}\rangle = \int_{x_\alpha}^{x_\omega} [t_2\,\mathbf{H}\varphi]_{\eta_\alpha}^{\eta_\omega}\, dx - \widehat{\mathbf{H}} \equiv \mathbf{S}_0^{0,1}, \end{aligned}\right\} \quad . \quad (9)$$

or

$$\left.\begin{aligned} r_1\langle t_1^2\rangle + r_2\langle t_1 t_2\rangle &= \mathbf{R}_0^{1,0}, \quad r_1\langle t_1 t_2\rangle + r_2\langle t_2^2\rangle = \mathbf{R}_0^{0,1}, \\ s_1\langle t_1^2\rangle + s_2\langle t_1 t_2\rangle &= \mathbf{S}_0^{1,0}, \quad s_1\langle t_1 t_2\rangle + s_2\langle t_2^2\rangle = \mathbf{S}_0^{0,1}, \end{aligned}\right\} \text{(9bis)}$$

whence

$$\left.\begin{aligned} m^{2,0} \equiv \langle t_1^2\rangle &= \frac{s_2\mathbf{R}_0^{1,0} - r_2\mathbf{S}_0^{1,0}}{r_1 s_2 - r_2 s_1}, \quad m^{1,1} \equiv \langle t_1 t_2\rangle = \\ = \frac{s_2\mathbf{R}_0^{0,1} - r_2\mathbf{S}_0^{0,1}}{r_1 s_2 - r_2 s_1} &= \frac{r_1\mathbf{S}_0^{1,0} - s_1\mathbf{R}_0^{1,0}}{r_1 s_2 - r_2 s_1}, \; m^{0,2} \equiv \langle t_2^2\rangle = \frac{r_1\mathbf{S}_0^{0,1} - s_1\mathbf{R}_0^{0,1}}{r_1 s_2 - r_2 s_1}. \end{aligned}\right\} \text{(10)}$$

We consider the case, that Ω_1 and Ω_2 are linear functions:

$$\Omega_2 \equiv \omega_0 + \omega_2 y, \quad \Omega_1 \equiv \omega_0' + \omega_1 x. \tag{11}$$

Eq. (6) now passes into

$$\widehat{\mathbf{R}} = \omega_0 + \omega_2 \widehat{y} \quad , \quad \widehat{\mathbf{S}} = \omega_0' + \omega_1 \widehat{x}. \quad . \quad . \quad . \quad . \tag{12}$$

On the other hand, the equations (5bis) become

$$\widehat{x}^{(y)} = \frac{1}{r_1}\{(\omega_0 - r_0) + (\omega_2 - r_2)\, y\} \quad , \quad \widehat{y}^{(x)} = \frac{1}{s_2}(\{\omega_0' - s_0) + (\omega_1 - s_1)\, x\}. \tag{13}$$

So in this case the regression equations are linear.

By $[\varphi]_{\xi_\alpha}^{\xi_\omega} = \Omega_2 \mathbf{W}_y$, $[\varphi]_{\eta_\alpha}^{\eta_\omega} = \Omega_1 \mathbf{W}_x$ (see (5)), the 2nd and 3rd equations (7) pass into

$$\left.\begin{aligned} \langle y\mathbf{R}\rangle &= \int_{y_\alpha}^{y_\omega} y\,[\mathbf{G}\varphi]_{\xi_\alpha}^{\xi_\omega}\, dy = \int_{y_\alpha}^{y_\omega} y\,(\Omega_2 \int_{\xi_\alpha}^{\xi_\omega} \varphi\, dx)\, dy = \\ &= \langle y\,\Omega_2\rangle = \omega_0 \widehat{y} + \omega_2\langle y^2\rangle, \quad \langle x\mathbf{S}\rangle = \omega_0' \widehat{x} + \omega_1\langle x^2\rangle. \end{aligned}\right\} \quad . \quad (7')$$

As (by 12)) $\widehat{y}\widehat{\mathbf{R}} = \omega_0\widehat{y} + \omega_2\widehat{y^2}$, $\widehat{x}\widehat{\mathbf{S}} = \omega_0'\widehat{x} + \omega_1\widehat{x^2}$, we find by subtraction from (7'):

$$\langle t_2\mathbf{R}\rangle = \omega_2(\langle y^2\rangle - \widehat{y^2}) = \omega_2\langle t_2^2\rangle = \omega_2 m^{0,2},$$

$$\langle t_1\mathbf{S}\rangle = \omega_1(\langle x^2\rangle - \widehat{x^2}) = \omega_1\langle t_1^2\rangle = \omega_1 m^{2,0},$$

or

$$r_0\widehat{t_2} + r_1\langle t_1 t_2\rangle + r_2\langle t_2^2\rangle = 0 + r_1 m^{1,1} + r_2 m^{0,2} = \omega_2 m^{0,2},$$

$$s_0\widehat{t_1} + s_1\langle t_1^2\rangle + s_2\langle t_1 t_2\rangle = 0 + s_1 m^{2,0} + s_2 m^{1,1} = \omega_1 m^{2,0},$$

whence

$$\frac{\omega_2 - r_2}{r_1} = \frac{m^{1,1}}{m^{0,2}} = \gamma\frac{\sigma_1}{\sigma_2} \quad , \quad \frac{\omega_1 - s_1}{s_2} = \frac{m^{1,1}}{m^{2,0}} = \gamma\frac{\sigma_2}{\sigma_1}. \quad . \quad . \quad . \tag{14}$$

D. Thus we obtain (see (13)) for the regression coefficients:

$$\varrho_{1|2} = \frac{\omega_2 - r_2}{r_1} = \gamma\frac{\sigma_1}{\sigma_2} \quad , \quad \varrho_{2|1} = \frac{\omega_1 - s_1}{s_2} = \gamma\frac{\sigma_2}{\sigma_1},$$

which are the same formulae as those with normal correlation.

For the deviations from the partial means we have in the present case (see (5 bis), (12) and (14))

$$x-\hat{x}^{(y)} = \frac{1}{r_1}(r_0 + r_1 x + r_2 y - \Omega_2) = \frac{1}{r_1}\{\mathrm{R}-(\omega_0 + \omega_2 y)\} =$$

$$= \frac{1}{r_1}\{\hat{\mathrm{R}} + r_1 t_1 + r_2 t_2 - (\omega_0 + \omega_2 \hat{y}) - \omega_2 t_2\} = t_1 - \frac{\omega_2 - r_2}{r_1} t_2,$$

or

$$x-\hat{x}^{(y)} = t_1 - \frac{m^{1,1}}{m^{0,2}} t_2, \qquad \text{likewise: } y-\hat{y}^{(x)} = t_2 - \frac{m^{1,1}}{m^{2,0}} t_1.$$

E. Hence the partial standard deviations $\sigma_{1;2}$ and $\sigma_{2;1}$ are determined by

$$\left.\begin{aligned} \sigma_{1;2}{}^2 = \langle (x-\hat{x}^{(y)})^2 \rangle = \langle t_1{}^2 \rangle - 2\frac{m^{1,1}}{m^{0,2}}\langle t_1 t_2 \rangle + \left(\frac{m^{1,1}}{m^{0,2}}\right)^2 \langle t_2{}^2 \rangle = \\ = m^{2,0} - \frac{(m^{1,1})^2}{m^{0,2}} = (1-\gamma^2)\,\sigma_1{}^2, \\ \sigma_{2;1}{}^2 = \langle (y-\hat{y}^{(x)})^2 \rangle = (1-\gamma^2)\,\sigma_2{}^2, \end{aligned}\right\} \quad . \; . \quad (15)$$

just as with normal correlation.

So it appears, that in the case that $\Omega_2 = \omega_0 + \omega_2 y$, $\Omega_1 = \omega_0' + \omega_1 x$, the formulae for the regression coefficients and for the partial standard deviations wholly agree with those found in normal correlation.

Clearly this concordance also exists, if $\Omega_2 = 0$ and $\Omega_1 = 0$, in particular if $[\mathrm{G}\varphi]_{\xi_\alpha}^{\xi_\omega} = 0$ and $[\mathrm{H}\varphi]_{\eta_\alpha}^{\eta_\omega} = 0$, i.e. if $\mathrm{G}\varphi$ and $\mathrm{H}\varphi$ are zero at the boundary of the probability domain.

Thus we met with a large group of non-normal binary probability distributions with linear regression equations, and even sharing the formulae for the regression coefficients and the partial standard deviations with the normal binary distribution. This implies a warning against using the term "linear correlation" as a synonym for normal correlation.

F. Relations between the moments.

We shift the zero-points of x and y to $\hat{x}$ and $\hat{y}$ and so operate with t_1 and t_2.

The functions φ, P, Q, G, H, R, S will then undergo some modifications. So we write eq. (1) as

$$\frac{\partial \log \varphi}{\partial t_1} = \frac{\mathrm{P}}{\mathrm{G}} \quad , \quad \frac{\partial \log \varphi}{\partial t_2} = \frac{\mathrm{Q}}{\mathrm{H}}. \quad . \; . \; . \; . \; . \quad (1')$$

In order to obtain relations between the moments, we start from

$$t_1{}^i t_2{}^j \mathrm{G}\frac{\partial \varphi}{\partial t_1} = t_1{}^i t_2{}^j \mathrm{P}\varphi \quad , \quad t_1{}^i t_2{}^j \mathrm{H}\frac{\partial \varphi}{\partial t_2} = t_1{}^i t_2{}^j \mathrm{Q}\varphi.$$

Hence

$$\int_{\tau_{1\alpha}}^{\tau_{1\omega}} t_1^{\,i}\, t_2^{\,j}\, \mathbf{G}\, \frac{\partial \varphi}{\partial t_1}\, dt_1 = [t_1^{\,i}\, t_2^{\,j}\, \mathbf{G}\, \varphi]_{\tau_{1\alpha}}^{\tau_{1\omega}} -$$

$$-\int_{\tau_{1\alpha}}^{\tau_{1\omega}} \left(i\, t_1^{\,i-1}\, t_2^{\,j}\, \mathbf{G} + t_1^{\,i}\, t_2^{\,j}\, \frac{\partial \mathbf{G}}{\partial t_1} \right) \varphi\, dt_1 = \int_{\tau_{1\alpha}}^{\tau_{1\omega}} t_1^{\,i}\, t_2^{\,j}\, \mathbf{P}\, \varphi\, dt_1,$$

and by subsequent integration over t_2:

$$i\, \langle t_1^{\,i-1}\, t_2^{\,j}\, \mathbf{G} \rangle + \langle t_1^{\,i}\, t_2^{\,j}\, \mathbf{R} \rangle = \int_{t_{2\alpha}}^{t_{2\omega}} [t_1^{\,i}\, t_2^{\,j}\, \mathbf{G}\, \varphi]_{\tau_{1\alpha}}^{\tau_{1\omega}}\, dt_2. \quad . \quad . \quad (16)_1$$

Similarly

$$j\, \langle t_1^{\,i}\, t_2^{\,j-1}\, \mathbf{H} \rangle + \langle t_1^{\,i}\, t_2^{\,j}\, \mathbf{S} \rangle = \int_{t_{1\alpha}}^{t_{1\omega}} [t_1^{\,i}\, t_2^{\,j}\, \mathbf{H}\, \varphi]_{\tau_{2\alpha}}^{\tau_{2\omega}}\, dt_1. \quad . \quad . \quad (16)_2$$

As a rule only those functions φ are acceptable as probability densities, for which $\mathbf{G}\varphi$ and $\mathbf{H}\varphi$ are zero at the boundary of the probability domain. Confining ourselves, accordingly, to these functions φ, we can simplify

(6) to $$\hat{\mathbf{R}} = 0 \quad , \quad \hat{\mathbf{S}} = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6')$$

(8) to $$\mathbf{R} = r_1\, t_1 + r_2\, t_2 \quad , \quad \mathbf{S} = s_1\, t_1 + s_2\, t_2, \quad . \quad . \quad . \quad . \quad . \quad (8')$$

(9) to $$\left.\begin{aligned} \mathbf{R}_0^{1,0} &= \langle t_1\, \mathbf{R} \rangle = -\hat{\mathbf{G}}, \quad \mathbf{R}_0^{0,1} = \langle t_2\, \mathbf{R} \rangle = 0, \\ \mathbf{S}_0^{1,0} &= \langle t_1\, \mathbf{S} \rangle = 0, \quad \mathbf{S}_0^{0,1} = \langle t_2\, \hat{\mathbf{S}} \rangle = -\hat{\mathbf{H}}, \end{aligned}\right\} \quad . \quad . \quad . \quad (9')$$

(10) to $$\left.\begin{aligned} m^{2,0} &= \frac{-s_2\, \hat{\mathbf{G}}}{\delta}, \quad m^{1,1} = \frac{+r_2\, \hat{\mathbf{H}}}{\delta} = \frac{+s_1\, \hat{\mathbf{G}}}{\delta}, \\ m^{0,2} &= \frac{-r_1\, \hat{\mathbf{H}}}{\delta}, \quad \text{with } \delta = r_1\, s_2 - r_2\, s_1, \end{aligned}\right\} \quad . \quad . \quad . \quad (10')$$

whence $$\gamma^2 = \frac{r_2\, s_1}{r_1\, s_2}, \quad 1 - \gamma^2 = \frac{\delta}{r_1\, s_2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (10'')$$

(16) to $$\left.\begin{aligned} \langle t_1^{\,i}\, t_2^{\,j}\, \mathbf{R} \rangle &= r_1\, m^{i+1,j} + r_2\, m^{i,j+1} = -i\, \langle t_1^{\,i-1}\, t_2^{\,j}\, \mathbf{G} \rangle, \\ \langle t_1^{\,i}\, t_2^{\,j}\, \mathbf{S} \rangle &= s_1\, m^{i+1,j} + s_2\, m^{i,j+1} = -j\, \langle t_1^{\,i}\, t_2^{\,j-1}\, \mathbf{H} \rangle. \end{aligned}\right\} \quad . \quad (16')$$

From these latter equations the higher moments can be expressed in the coefficients of the differential equations (1).

In particular, we obtain for $i = 0$, resp. $j = 0$:

$$r_1\, m^{1,j} + r_2\, m^{0,j+1} = 0 \quad , \quad s_1\, m^{i+1,0} + s_2\, m^{i,1} = 0, \quad . \quad . \quad (17)$$

whence

$$\frac{m^{1,1}}{m^{0,2}} = \frac{m^{1,2}}{m^{0,3}} = \frac{m^{1,3}}{m^{0,4}} \left(= -\frac{r_2}{r_1} \right) \quad , \quad \frac{m^{1,1}}{m^{2,0}} = \frac{m^{2,1}}{m^{3,0}} = \frac{m^{3,1}}{m^{4,0}} \left(= -\frac{s_1}{s_2} \right). \quad (17\text{bis})$$

So we find for the characteristics of skewness $S^{3,0}$, $S^{2,1}$, $S^{1,2}$, $S^{0,3}$ and for the characteristics of excess $E^{4,0}$, $E^{3,1}$, $E^{2,2}$, $E^{1,3}$, $E^{0,4}$:

$$S^{2,1} = \frac{m^{2,1}}{m^{2,0}\,(m^{0,2})^{1/2}} = \frac{\dfrac{m^{1,1}\,m^{3,0}}{m^{2,0}}}{m^{2,0}\,(m^{0,2})^{1/2}} = \frac{m^{1,1}}{\sqrt{m^{2,0}\,m^{0,2}}} \cdot \frac{m^{3,0}}{(m^{2,0})^{3/2}},$$

or

$$S^{2,1} = \gamma S^{3,0}; \quad \text{likewise} \quad S^{1,2} = \gamma S^{0,3}; \ldots\ldots \quad (18)$$

$$E^{3,1} = \frac{m^{3,1}}{(m^{2,0})^{3/2}\,(m^{0,2})^{1/2}} - 3\gamma = \frac{\dfrac{m^{1,1}\,m^{4,0}}{m^{2,0}}}{(m^{2,0})^{3/2}\,(m^{0,2})^{1/2}} - 3\gamma =$$

$$= \frac{m^{1,1}}{\sqrt{m^{2,0}\,m^{0,2}}} \cdot \frac{m^{4,0}}{(m^{2,0})^2} - 3\gamma = \gamma\left(\frac{m^{4,0}}{(m^{2,0})^2} - 3\right),$$

or

$$E^{3,1} = \gamma E^{4,0}; \quad \text{likewise} \quad E^{1,3} = \gamma E^{0,4}. \ldots\ldots \quad (19)$$

Mathematics. — *A study of Bessel functions in connection with the problem of two mutually attracting circular discs.* By C. J. BOUWKAMP. (Natuurkundig Laboratorium der N.V. Philips' Gloeilampenfabrieken, Eindhoven, Netherlands.) (Communicated by Prof. J. G. VAN DER CORPUT.)

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Summary.

In this note I discuss the application of Bessel functions to the physical problem of the mutual attraction of two homogeneous circular discs lying in the same plane. It is assumed that the law of force, which describes the interaction of two unit point masses, is derivable from the potential function $V(r)$ depending only on the distance r between the masses. So, the force problem is reducible to a scalar problem: the calculation of the mutual potential energy of the discs.

Special attention is paid to potential functions varying with the distance as r^{-n} where n is any positive number. This includes the gravitational force ($n = 1$) as well as the London–Van der Waals force ($n = 6$).

The paper is entirely mathematical.

1. *Formulation of the problem in terms of Bessel functions.*

The reader is, of course, familiar with the two-dimensional, logarithmic, potential: $V(r) = \log r$. In this case the mutual energy of two non-overlapping homogeneous discs in the same plane is equal to that obtained when the total masses of the discs are concentrated at the respective centres.

A somewhat more general question arises almost at once. Namely, whether there exists a particular law of interaction such that two non-overlapping discs (radii: a, b; distance between centres: $c > a + b$) in the same plane shall attract each other as if certain reduced masses were located at the centres. More precisely, whether it is possible to choose $V(r)$ such that the mutual energy of the two discs is given by $\varphi(a, b)\, V(c)$ where $\varphi(a, b)$ is a (symmetric) function of a and b, *not depending on* c.

The answer to the question above is affirmative, even if the trivial case of the logarithmic potential, for which $\varphi(a, b) = 1$ (both discs having unit mass), is excluded. As will be shown in due course, the modified Bessel function $K_0(rt)$ serves the purpose, for all values of the parameter t. Once we have succeeded to represent the potential function $V(r)$ as a sum or integral (the weight function depending on t) of the 'invariant' function $K_0(rt)$, it is easy to calculate the interaction energy in question.

We now proceed to the invariance — with respect to the transition from point mass to disc — of the function $K_0(rt)$.

Let R be the distance between a unit point mass and the centre of a disc of radius a. The point mass is assumed in the plane of the disc and lying outside the latter. Further, let polar coordinates (ϱ, θ) be introduced at the centre of the disc, the polar axis being directed to the unit point mass. Then the mutual energy of the point mass and the disc, which will be supposed homogeneous and of unit mass, is given by

$$u_t(a, 0; R) = \frac{1}{\pi a^2} \int_0^a \varrho\, d\varrho \int_0^{2\pi} K_0(t\sqrt{\varrho^2 - 2\varrho R\cos\theta + R^2})\, d\theta$$

$$= \frac{2}{a^2} K_0(Rt) \int_0^a I_0(\varrho t)\, \varrho\, d\varrho$$

$$= \frac{2}{at} I_1(at)\, K_0(Rt), \qquad (a < R)$$

whereby use is made of some well-known formulae of the theory of Bessel functions.

Therefore, the interaction between the point mass and the disc is as if the mass

$$\varphi(a, 0) = \frac{2}{at} I_1(at)$$

were located at the centre of the disc, for all values of the distance $R > a$. This is the invariance property of the Bessel function K_0 as referred to above.

It will further be obvious, by twice applying this process of reduction, that

$$\varphi(a, b) = \varphi(a, 0)\ \varphi(b, 0).$$

Hence, the energy of interaction of the two discs under consideration is given by

$$u_t(a, b; c) = \frac{4}{abt^2} I_1(at)\, I_1(bt)\, K_0(ct)$$

when $V(r) = K_0(rt)$.

Let us now assume that the given potential function $V(r)$ can be represented by an integral of the following type:

$$V(r) = \int_0^\infty f(t)\, K_0(rt)\, dt. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Then, since the energy is additive, the mutual potential energy of the discs becomes

$$U(a, b; c) = \frac{4}{ab} \int_0^\infty I_1(at)\, I_1(bt)\, K_0(ct)\, f(t)\, t^{-2}\, dt. \quad . \quad . \quad . \quad (2)$$

2. *Expression for the energy of two non-overlapping discs when* $V(r) = r^{-n}$ $(n > 0)$.

The 'generating' function $f(t)$ occurring in (1) and (2) is known for the particular potential $V(r) = r^{-n}$ $(n > 0)$, as follows from [1])

$$\frac{1}{r^n} = \frac{2^{2-n}}{\left\{\Gamma\left(\frac{n}{2}\right)\right\}^2} \int_0^\infty K_0(rt)\, t^{n-1}\, dt. \qquad (n > 0)$$

Consequently, the interaction energy of the two discs in response to the law $V(r) = r^{-n}$ is given by

$$U_n(a, b; c) = -\frac{2^{4-n}}{ab\left\{\Gamma\left(\frac{n}{2}\right)\right\}^2} \int_0^\infty I_1(at)\, I_1(bt)\, K_0(ct)\, t^{n-3}\, dt. \quad . \quad . \quad (3)$$

This function will be discussed in sections 4, 5, 6, 7 for $b = 0$, $a = b$, $a \neq b \neq 0$, $b \to \infty$, respectively.

3. *Differential relations.*

Let x, y denote rectangular cartesian coordinates in the plane of a disc of arbitrary shape and mass distribution. Then, the potential outside the disc under action of the law $V(r)$ is given by

$$U(x, y) = \iint D(\xi, \eta)\, V(r)\, d\xi\, d\eta \qquad (r^2 = (x-\xi)^2 + (y-\eta)^2)$$

where the integration has to be carried out over the surface of the disc, and where $D(\xi, \eta)$ stands for the local mass density.

Obviously,

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = \iint D(\xi, \eta) \left\{\frac{\partial^2 V}{\partial \xi^2} + \frac{\partial^2 V}{\partial \eta^2}\right\} d\xi\, d\eta.$$

Furthermore,

$$\frac{\partial^2 V}{\partial \xi^2} + \frac{\partial^2 V}{\partial \eta^2} = \left\{\frac{d^2 V}{dr^2} + \frac{1}{r}\frac{dV}{dr}\right\}_{r = \{(x-\xi)^2 + (y-\eta)^2\}^{1/2}}.$$

We have thus proved the following

Theorem I:

If U is the potential of an arbitrary disc in response to the potential law $V(r)$, then $\triangle U$ is the potential of the same disc in response to the law $V''(r) + V'(r)/r$.

Especially for centre-symmetric discs, $\triangle U = U''(c) + U'(c)/c$ where c denotes the distance from the centre. When in this case the theorem is applied twice in succession, we obtain

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[1]) Cf. G. N. WATSON, A treatise on the theory of Bessel functions, Cambridge 1922/1944, p. 388, formula (2).

Theorem II:

If $U(a, b; c)$ denotes the energy of interaction of two non-overlapping circular discs with centre-symmetric mass distributions under influence of the potential law $V(r)$, then

$$U^*(a, b; c) = \frac{\partial^2 U}{\partial c^2} + \frac{1}{c}\frac{\partial U}{\partial c} = 4\frac{\partial}{\partial (c^2)}\left\{c^2 \frac{\partial U}{\partial (c^2)}\right\} = \frac{1}{c}\frac{\partial}{\partial c}\left(c\frac{\partial U}{\partial c}\right)$$

is the energy of interaction of the same discs under influence of the potential law

$$V^*(r) = \frac{d^2 V}{dr^2} + \frac{1}{r}\frac{dV}{dr}.$$

This way of reasoning I owe to Prof. N. G. DE BRUIJN. An alternative proof of Theorem II is based on the integral representation (2). In view of the differential equation satisfied by K_0, the function U^* corresponds to the generating function $t^2 f(t)$ when U corresponds to $f(t)$, and, in its turn, $t^2 f(t)$ corresponds by (1) to the potential law $V^*(r)$. In the latter way I originally found Theorem II.

Theorem II is of particular interest with respect to the potential functions r^{-n} since it reduces the interval of n to be investigated to $0 < n \leq 2$. For greater values of n the function can be found by a process of differentiation, namely,

$$U_{n+2}(a, b; c) = \frac{1}{n^2}\left\{\frac{\partial^2}{\partial c^2} + \frac{1}{c}\frac{\partial}{\partial c}\right\} U_n(a, b; c). \qquad (n > 0) \quad . \quad . \quad (4)$$

Of course, equation (4) is also easily proved directly from (3) when use is made of Bessel's differential equation for $K_0(ct)$.

4. *The potential energy of a point mass outside a circular disc when* $V(r) = r^{-n}$.

The discussion of the function $U_n(a, b; c)$ defined by (3) is comparatively simple when one of the discs reduces to a point mass. In that case we have

$$U_n(a, 0; c) = \frac{2^{3-n}}{a\left\{\Gamma\left(\frac{n}{2}\right)\right\}^2}\int_0^\infty I_1(at)\, K_0(ct)\, t^{n-2}\, dt. \quad . \quad . \quad . \quad (5)$$

This integral is expressible in terms of hypergeometric functions. When use is made of the modified WEBER–SCHAFHEITLIN integral [2]) it is found that

$$U_n(a, 0; c) = \frac{1}{c^n} F\left(\frac{n}{2}, \frac{n}{2}; 2; \frac{a^2}{c^2}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

In passing, it may be noted that (6) holds for *all* values of n, not necessarily positive, as from physical considerations it is obvious that $U_n(a, b; c)$ is an analytic function of the variable n.

[2]) Cf. WATSON, *loc. cit.*, p. 410, formula (1).

For even values of n the function $U_n(a, 0; c)$ is elementary. For instance,

$$U_2(a, 0; c) = -\frac{1}{a^2}\log\left(1-\frac{a^2}{c^2}\right), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

$$U_4(a, 0; c) = (c^2-a^2)^{-2} \quad , \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

$$U_6(a, 0; c) = (c^2 + \tfrac{1}{2}a^2)(c^2-a^2)^{-4} \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

Further, the function is expressible in terms of complete elliptic integrals of the first and second kinds when n is an odd integer. For example, let us take $n = 1$; then

$$F(\tfrac{1}{2}, \tfrac{1}{2}; 2; k^2) = \frac{\Gamma(2)}{\Gamma(\frac{1}{2})\,\Gamma(\frac{3}{2})}\int_0^1 u^{-\frac{1}{2}}(1-u)^{\frac{1}{2}}\,(1-k^2u)^{-\frac{1}{2}}\,du$$

$$= \frac{4}{\pi}\int_0^{\pi/2}\frac{\cos^2\varphi\,d\varphi}{\sqrt{1-k^2\sin^2\varphi}}$$

$$= \frac{4}{\pi}\left[\left(1-\frac{1}{k^2}\right)\int_0^{\pi/2}\frac{d\varphi}{\sqrt{1-k^2\sin^2\varphi}} + \frac{1}{k^2}\int_0^{\pi/2}\sqrt{1-k^2\sin^2\varphi}\,d\varphi\right]$$

$$= \frac{4}{\pi}\left\{\left(1-\frac{1}{k^2}\right)K(k) + \frac{1}{k^2}E(k)\right\},$$

in the usual notation of elliptic integrals. Consequently,

$$U_1(a, 0; c) = \frac{4c}{\pi a^2}\left\{E\left(\frac{a}{c}\right) - \left(1-\frac{a^2}{c^2}\right)K\left(\frac{a}{c}\right)\right\}. \quad . \quad . \quad (10)$$

This result is not new [3].

In a similar way one can evaluate the function U_n when $n = 3$, *viz.*:

$$F(\tfrac{3}{2}, \tfrac{3}{2}; 2; k^2) = \frac{\Gamma(2)}{\Gamma(\frac{3}{2})\,\Gamma(\frac{1}{2})}\int_0^1 u^{\frac{1}{2}}(1-u)^{-\frac{1}{2}}(1-k^2u)^{-\frac{3}{2}}\,du$$

$$= \frac{4}{\pi}\int_0^{\pi/2}\frac{\sin^2\varphi\,d\varphi}{(1-k^2\sin^2\varphi)^{3/2}} = \frac{4}{\pi k}\frac{dK(k)}{dk}$$

$$= \frac{4}{\pi k^2}[(1-k^2)^{-1}E(k) - K(k)].$$

We therefore obtain that

$$U_3(a, 0; c) = \frac{4}{\pi a^2 c}\left[\left(1-\frac{a^2}{c^2}\right)^{-1}E\left(\frac{a}{c}\right) - K\left(\frac{a}{c}\right)\right]. \quad . \quad . \quad (11)$$

When comparing (10) with (11), we see that

$$U_1(a, 0; c) = (c^2-a^2)\,U_3(a, 0; c), \quad . \quad . \quad . \quad . \quad . \quad (12)$$

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[3] Cf. H. BATEMAN, Partial differential equations of mathematical physics, Cambridge 1932 — New York 1944, p. 417, example 2.

which might also have been obtained by EULER's transformation of hypergeometric functions:

$$F(a, b; c; x) = (1-x)^{c-a-b} F(c-a, c-b; c; x). \quad . \quad . \quad (13)$$

The result I found when taking $n = 5$ is

$$U_5(a, 0; c) = \frac{4}{9\pi a^2 c^3}\left[\frac{1+7\frac{a^2}{c^2}}{\left(1-\frac{a^2}{c^2}\right)^3} E\left(\frac{a}{c}\right) - \frac{1+3\frac{a^2}{c^2}}{\left(1-\frac{a^2}{c^2}\right)^2} K\left(\frac{a}{c}\right)\right]. \quad (14)$$

Generally, $U_{2m+1}(a, 0; c)$ (m = non-negative integer) is expressible as a linear combination of $E(a/c)$ and $K(a/c)$ with coefficients rational in $a\,c$.

It is very interesting that also for half-integral values of n the function $U_n(a, 0; c)$ is expressible in terms of elementary functions and complete elliptic integrals. This is easily proved with the aid of KUMMER's relation [4]):

$$F(\tfrac{3}{4}, \tfrac{3}{4}; 2; \sin^2\theta) = F(\tfrac{3}{2}, \tfrac{3}{2}; 2; \sin^2\tfrac{1}{2}\theta) \quad . \quad . \quad . \quad . \quad (15)$$

The hypergeometric function on the right has already been evaluated; therefore we have at once:

$$U_{\frac{1}{2}}(a, 0; c) = \frac{16 c^{\frac{1}{2}}}{\pi a^2}\left[E\left\{\sqrt{\frac{1}{2}\left(1-\sqrt{1-\frac{a^2}{c^2}}\right)}\right\} - \frac{1}{2}\left(1+\sqrt{1-\frac{a^2}{c^2}}\right) K\left\{\sqrt{\frac{1}{2}\left(1-\sqrt{1-\frac{a^2}{c^2}}\right)}\right\}\right]. \quad (16)$$

Moreover, the relation (15) immediately leads to the following interesting equation

$$U_3(a, 0; c) = c^{-\frac{1}{2}} U_{\frac{1}{2}}\left(2a\sqrt{1-\frac{a^2}{c^2}}, 0; c\right). \quad . \quad . \quad . \quad . \quad (17)$$

When $n = \frac{5}{2}$ the calculation runs as follows. First, by EULER's transformation (13),

$$F\left(\tfrac{5}{4}, \tfrac{5}{4}; 2; \frac{a^2}{c^2}\right) = \left(1-\frac{a^2}{c^2}\right)^{-\frac{1}{2}} F\left(\tfrac{3}{4}, \tfrac{3}{4}; 2; \frac{a^2}{c^2}\right),$$

and, consequently,

$$U_{\frac{5}{2}}(a, 0; c) = (c^2-a^2)^{-\frac{1}{2}} U_{\frac{3}{2}}(a, 0; c), \quad . \quad . \quad . \quad . \quad . \quad (18)$$

which is known by (16).

To determine $U_{\frac{1}{2}}(a, 0; c)$, I once more apply EULER's transformation (13):

$$F\left(\tfrac{1}{4}, \tfrac{1}{4}; 2; \frac{a^2}{c^2}\right) = \left(1-\frac{a^2}{c^2}\right)^{\frac{1}{2}} F\left(\tfrac{3}{4}, \tfrac{3}{4}; 2; \frac{a^2}{c^2}\right)$$

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[4]) Cf. E. T. WHITTAKER–G. N. WATSON, A course of modern analysis, Cambridge 1935, p. 298, example 12.

which leads to the relation

$$U_{\frac{1}{2}}(a, 0; c) = (c^2 - a^2)^{\frac{1}{2}} U_{\frac{1}{2}}(a, 0; c). \quad . \quad . \quad . \quad . \quad . \quad (19)$$

On the other hand, we have by (4)

$$U_{\frac{3}{2}}(a, 0; c) = \frac{4}{3}\left\{\frac{\partial^2}{\partial c^2} + \frac{1}{c}\frac{\partial}{\partial c}\right\} U_{\frac{1}{2}}(a, 0; c).$$

Consequently, both $U_{\frac{1}{2}}$ and $U_{\frac{3}{2}}$ are expressible in terms of elementary functions and complete elliptic integrals; and so is $U_{m-\frac{1}{2}}(a, 0; c)$ for any positive integer m in virtue of (4).

5. *The case of equal radii.*

In the second place I shall consider the function U_n for discs of equal radii: $a = b$. Thus

$$U_n(a, a; c) = \frac{2^{4-n}}{a^2\left\{\Gamma\left(\frac{n}{2}\right)\right\}^2}\int_0^\infty I_1^2(at)\, K_0(ct)\, t^{n-3}\, dt. \quad . \quad . \quad (20)$$

The function (20) can be evaluated in terms of the generalized hypergeometric function ${}_3F_2$. To prove this, let us expand [5]) the square of the Bessel function into ascending powers of t, *viz.*:

$$I_1^2(at) = \sum_{m=0}^{\infty}\left(\frac{at}{2}\right)^{2+2m}\frac{\Gamma(2m+3)}{m!\{\Gamma(m+2)\}^2\Gamma(m+3)},$$

and integrate term by term. One then finds that

$$U_n(a, a; c) = \frac{1}{c^n}\sum_{m=0}^{\infty}\frac{\Gamma(2m+3)}{\Gamma(m+1)\,\Gamma(m+3)}\left\{\frac{\Gamma\left(m+\frac{n}{2}\right)}{\Gamma\left(\frac{n}{2}\right)\Gamma(m+2)}\right\}^2\left(\frac{a^2}{c^2}\right)^m$$

$$= \frac{1}{c^n}\frac{\Gamma(1)\,\Gamma(2)\,\Gamma(3)}{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{n}{2}\right)}\sum_{m=0}^{\infty}\frac{\Gamma\left(m+\frac{3}{2}\right)\Gamma\left(m+\frac{n}{2}\right)\Gamma\left(m+\frac{n}{2}\right)}{\Gamma(m+1)\,\Gamma(m+2)\,\Gamma(m+3)}\left(\frac{4a^2}{c^2}\right)^m.$$

That is to say,

$$U_n(a, a; c) = \frac{1}{c^n}\,{}_3F_2\left(\frac{3}{2}, \frac{n}{2}, \frac{n}{2}; 2, 3; \frac{4a^2}{c^2}\right), \quad . \quad . \quad . \quad . \quad (21)$$

which, as (6), is true for all values of n not necessarily positive.

As in the preceding section, the function is elementary when n is an even integer. Particularly simple are the cases $n = 4$, $n = 6$, since then the

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[5]) Cf. WATSON, *loc. cit.*, p. 147, formula (6).

generalized hypergeometric function (21) reduces to an ordinary hypergeometric function, namely,

$$U_4(a, a; c) = \frac{1}{c^4} F\left(\tfrac{3}{2}, 2; 3; \frac{4a^2}{c^2}\right),$$

$$U_6(a, a; c) = \frac{1}{c^6} F\left(\tfrac{3}{2}, 3; 2; \frac{4a^2}{c^2}\right).$$

Now,

$$F(\tfrac{3}{2}, 2, 3; x) = 2\int_0^1 u(1-xu)^{-\frac{3}{2}}\, du,$$

which is easily integrated. The result is found to be

$$U_4(a, a; c) = \frac{1}{c^4} \frac{1}{\sqrt{1-\frac{4a^2}{c^2}}} \left\{ \frac{2}{1+\sqrt{1-\frac{4a^2}{c^2}}} \right\}^2 \quad . \quad . \quad . \quad (22)$$

Moreover, by EULER's formula (13),

$$F(\tfrac{3}{2}, 3; 2; x) = (1-x)^{-\frac{1}{2}} F(\tfrac{1}{2}, -1; 2; x).$$

The hypergeometric series on the right consists of two terms; accordingly,

$$U_6(a, a; c) = \frac{1}{c^6} \frac{1-\frac{a^2}{c^2}}{\left(1-\frac{4a^2}{c^2}\right)^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

The evaluation of $U_2(a, a; c)$ can be accomplished by an integration, either by means of (4) or more directly as follows. First, it is easily verified that

$$\frac{d}{dx}\{x\, {}_3F_2(\tfrac{3}{2}, 1, 1; 2, 3; x)\} = F(\tfrac{3}{2}, 1; 3; x)$$

$$= (1-x)^{\frac{1}{2}} F(\tfrac{3}{2}, 2; 3; x) = \left\{\frac{2}{1+\sqrt{1-x}}\right\}^2$$

$$= -8\frac{d}{dx}\left\{\frac{1}{1+\sqrt{1-x}} + \log(1+\sqrt{1-x})\right\}.$$

Therefore, upon integrating while suitably accounting for the constant of integration, we obtain

$${}_3F_2(\tfrac{3}{2}, 1, 1; 2, 3; x) = \frac{8}{x}\left\{\tfrac{1}{2} - \frac{1}{1+\sqrt{1-x}} - \log\left(\frac{1+\sqrt{1-x}}{2}\right)\right\}$$

from which it follows that

$$U_2(a, a; c) = \frac{1}{a^2}\left[1 - \frac{2}{1+\sqrt{1-\frac{4a^2}{c^2}}} - 2\log\left\{\frac{1+\sqrt{1-\frac{4a^2}{c^2}}}{2}\right\}\right] \quad . \quad . \quad (24)$$

The analysis is much more complicated when n is an odd integer. As a

matter of fact, only the case $n = 3$ seems tractable; that is to say, I have not been able to sum the series (21) when $n = 1$, this of course being possible for $n = 5, 7, \ldots$, in virtue of (4), once $U_3(a, a; c)$ is known.

To evaluate $U_3(a, a; c)$, I use a relation due to CLAUSEN [6]); it follows from this relation that

$$ {}_3F_2\left(\tfrac{3}{2}, \tfrac{3}{2}, \tfrac{3}{2}; 2, 3; x\right) = \left\{F\left(\tfrac{3}{4}, \tfrac{3}{4}; 2; x\right)\right\}^2. $$

Therefore,

$$ U_3(a, a; c) = \frac{1}{c^3}\left\{F\left(\tfrac{3}{4}, \tfrac{3}{4}; 2; \frac{4a^2}{c^2}\right)\right\}^2. $$

On comparing this with (6) we see that

$$ U_3(a, a; c) = \left\{U_{\frac{3}{2}}(2a, 0; c)\right\}^2 \quad \ldots\ldots\ldots \quad (25) $$

Consequently, from (16),

$$ U_3(a, a; c) = \frac{16c}{\pi^2 a^4}\left[E(k) - (1-k^2)K(k)\right]^2, \quad \ldots\ldots \quad (26) $$

where the modulus of the complete elliptic integrals is given by

$$ k = \frac{1}{\sqrt{2}}\left\{1 - \sqrt{1 - \frac{4a^2}{c^2}}\right\}^{\frac{1}{2}} \quad \ldots\ldots\ldots\ldots \quad (27) $$

It may be remarked that (26) is equivalent to

$$ \int_0^\infty I_1^2(xt)\, K_0(2t)\, dt = \frac{\pi}{16}\left\{P_{-\frac{1}{2}}^{-1}\left(\sqrt{1-x^2}\right)\right\}^2, \quad \ldots\ldots \quad (28) $$

$$ (-1 < x < 1) $$

which is closely related to MEIJER's integral representations of the product of two Legendre functions [7]).

6. *The case of unequal discs.*

In this section, the function U_n will be investigated for general values of a and b. However, n will again be restricted to integers since I have not succeeded to evaluate the integral (3) for other values of n, apart from the fact that (3) is expressible in terms of APPELL's hypergeometric function F_4, *viz.*:

$$ U_n(a, b; c) = \frac{1}{c^n} F_4\left(\frac{n}{2}, \frac{n}{2}; 2, 2; \frac{a^2}{c^2}, \frac{b^2}{c^2}\right), \quad \ldots\ldots \quad (29) $$

which is readily proved by expanding $I_1(at)\ I_1(bt)$ into ascending powers of t and integrating term by term [8]).

To begin with, let us take $n = 4$. Then from (3),

$$ U_4(a, b; c) = \frac{1}{ab}\int_0^\infty I_1(at)\, I_1(bt)\, K_0(ct)\, t\, dt. $$

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[6]) Cf. WHITTAKER–WATSON, *loc. cit.*, p. 298, example 11.

[7]) Cf. C. S. MEIJER, Nieuw Arch. Wiskunde **19**, 207—234, 1938.

[8]) For the definition of APPELL's function, cf. P. APPELL–J. KAMPÉ DE FÉRIET, Fonctions hypergéométriques et hypersphériques — Polynomes d'Hermite, Paris 1926.

This integral can be evaluated with the aid of a result due to MACDONALD [9]):

$$\int_0^\infty K_\mu(at)\,J_\nu(bt)\,J_\nu(ct)\,t^{\mu+1}\,dt = -\frac{a^\mu \cos\nu\pi\,(X^2-1)^{-\frac{1}{4}\mu-\frac{1}{8}}}{\sqrt{2\pi}\,(bc)^{\mu+1}\sin(\mu+\nu)\pi}\,Q_{\nu-\frac{1}{2}}^{\mu+\frac{1}{2}}(X), \quad (30)$$

in which $2bcX = a^2+b^2+c^2$ and Re $(a \pm ib \pm ic) > 0$; Q is the Legendre function of the second kind in the sense of BARNES.

As is well known, the right-hand side of (30) can be written in terms of hypergeometric functions. The result is found to be

$$\frac{a^\mu\,\Gamma(\mu+\nu+1)}{2^{\nu+1}(bc)^{\mu+1}\,\Gamma(\nu)\,X^{\mu+\nu+1}}\,F\left(\frac{\mu+\nu+2}{2},\,\frac{\mu+\nu+1}{2};\,\nu+1;\,X^{-2}\right).$$

Upon taking $\mu = 0$, $\nu = 1$ we obtain

$$\int_0^\infty K_0(at)\,J_1(bt)\,J_1(ct)\,tdt = \frac{1}{4bcX^2}\,F(\tfrac{3}{2},1;2;X^{-2})$$

$$= \frac{1}{4bcX^2}\int_0^1\left(1-\frac{u}{X^2}\right)^{-\frac{3}{2}}du = \frac{1}{2bc}\left\{\left(1-\frac{1}{X^2}\right)^{-\frac{1}{2}}-1\right\}.$$

Now replace a, b, c by c, ia, ib respectively. It has thus been proved that

$$\int_0^\infty I_1(at)\,I_1(bt)\,K_0(ct)\,tdt = \frac{1}{2ab}\left[\left\{1-\frac{4a^2b^2}{(c^2-a^2-b^2)^2}\right\}^{-\frac{1}{2}}-1\right],$$

and, consequently,

$$U_4(a,b;c) = \frac{1}{2a^2b^2}\left[\frac{c^2-a^2-b^2}{\{c^2-(a-b)^2\}^{\frac{1}{2}}\{c^2-(a+b)^2\}^{\frac{1}{2}}}-1\right]. \quad (31)$$

As is easily verified, formula (31) is in accordance with the special case $a = b$ given before at (22).

For greater, even, values of n the function $U_n(a,b;c)$ can be calculated by differentiation of (31) in virtue of (4). Omitting the rather tedious, though elementary, intermediate computations, I merely state the final result when $n = 6$:

$$U_6(a,b;c) = \frac{c^2(2c^2-a^2-b^2)-(a^2-b^2)^2}{2\{c^2-(a-b)^2\}^{\frac{11}{2}}\{c^2-(a+b)^2\}^{\frac{11}{2}}}, \quad . \quad . \quad . \quad (32)$$

which, by the way, is in complete agreement with the result obtained by DUBE and DASGUPTA [10]).

Whereas U_6 has been obtained by differentiation of U_4, the function U_2 can be calculated by the inverse process, that is, by integration of

$$U_4(a,b;c) = \frac{\partial}{\partial(c^2)}\left\{c^2\,\frac{\partial U_2(a,b;c)}{\partial(c^2)}\right\}.$$

---

[9]) Cf. WATSON, *loc. cit.*, p. 412, formula (6).

[10]) G. P. DUBE and H. K. DASGUPTA, On the London–Van der Waals forces between two disc-like particles, Indian J. Phys. **13**, 411—416, 1939.

For this purpose, let

$$u=c^2; \quad W(u)=\sqrt{u^2-2(a^2+b^2)u+(a^2-b^2)^2}.$$

Then, by integrating once,

$$u\frac{\partial U_2}{\partial u}=\frac{1}{2a^2b^2}\int_0^u\left\{\frac{t-a^2-b^2}{W(t)}-1\right\}dt$$
$$=\frac{1}{2a^2b^2}[a^2+b^2-u+W(u)],$$

where the constant of integration has been chosen so as to make $u\frac{\partial U_2}{\partial u}$ vanish at infinity (as a matter of fact, this condition is sufficient; the more stringent, and physically necessary, condition $u^2\frac{\partial U_2}{\partial u}\to -1$ $(u\to\infty)$ is then automatically fulfilled).

Further, a second integration yields:

$$2a^2b^2U_2=W(u)-u+a^2+b^2-(a^2+b^2)\log\left\{\frac{W(u)+u-a^2-b^2}{2u}\right\}$$
$$+(a^2-b^2)^2\int_\infty^u\frac{dt}{tW(t)},$$

where the constant of integration is taken such that $U_2(\infty)=0$.

Moreover we have

$$(a^2-b^2)^2\int_\infty^u\frac{dt}{tW(t)}=-|a^2-b^2|\log\left\{\frac{u(a^2+b^2)-(a^2-b^2)^2-|a^2-b^2|\,W(u)}{u(a^2+b^2-|a^2-b^2|)}\right\}.$$

Accordingly, the final result is found to be:

$$U_2(a,b;c)=\frac{1}{2a^2b^2}\left[W-c^2+a^2+b^2-(a^2+b^2)\log\left\{\frac{W+c^2-a^2-b^2}{2c^2}\right\}-\right.$$
$$\left.-|a^2-b^2|\log\left\{\frac{c^2(a^2+b^2)-(a^2-b^2)^2-|a^2-b^2|\,W}{c^2(a^2+b^2-|a^2-b^2|)}\right\}\right], \quad (33)$$

in which W is an abbreviation for

$$W=W(c^2)=\sqrt{\{c^2-(a-b)^2\}\{c^2-(a+b)^2\}} \quad . \quad . \quad . \quad . \quad (34)$$

As may have been anticipated from section 5, the function $U_n(a,b;c)$ is expressible in terms of complete elliptic integrals when n is an odd integer greater than 1.

To begin with, let us take $n=3$. Then, the fourth type of APPELL's hypergeometric function of two variables reduces to a product of two ordinary hypergeometric functions, as follows from BAILEY's formula [11])

$$F_4(a,b;c,a+b-c+1;\ x(1-y),\ y(1-x))=$$
$$=F(a,b;c;x)\ F(a,b;a+b-c+1;y);$$

---

[11]) W. N. BAILEY, Generalized hypergeometric series, Cambridge 1935, p. 81.

namely,

$$F_4\left(\tfrac{3}{2}, \tfrac{3}{2}; 2, 2; x(1-y), y(1-x)\right) = F\left(\tfrac{3}{2}, \tfrac{3}{2}; 2; x\right) F\left(\tfrac{3}{2}, \tfrac{3}{2}; 2; y\right).$$

Since the remaining ordinary hypergeometric function is known from the preceding section, we at once have

$$U_3(a, b; c) = \frac{16c}{\pi^2 a^2 b^2} \{E(k_1) - (1-k_1^2) K(k_1)\} \{E(k_2) - (1-k_2^2) K(k_2)\}, \quad (35)$$

in which the moduli of the complete elliptic integrals are defined by

$$k_1^2 (1-k_2^2) = a^2/c^2,$$
$$k_2^2 (1-k_1^2) = b^2/c^2;$$

thus

$$\left.\begin{aligned} k_1^2 &= \frac{1}{2c^2} \{c^2 + a^2 - b^2 - W\} \\ k_2^2 &= \frac{1}{2c^2} \{c^2 - a^2 + b^2 - W\} \end{aligned}\right\} \quad \ldots\ldots \quad (36)$$

where W is as defined in (34).

It is to be remarked that (35) is completely symmetric in a and b; further, it reduces to (26) when $a = b$.

By means of (4), the functions $U_5, U_7, \ldots$, can be expressed in terms of elliptic integrals also. Unfortunately, I have not succeeded to sum the series (29) in the newtonian case $n = 1$.

7. *Energy of interaction between a circular disc and a half-plane.*

In conclusion, it may be worth while to consider briefly the degenerated problem of one of the discs becoming infinitely large. Thus, let us calculate the mutual energy of a disc of radius a, of homogeneous mass distribution and *total mass* equal to 1, and a homogeneous half-plane of *mass density* equal to 1.

Let $\triangle$ denote the distance between the centre of the disc and the boundary of the half-plane. Confining ourselves to the work function $V(r) = r^{-n}$ $(n > 2)$, we have for the energy in question

$$u_n(a; \triangle) = \lim_{b \to \infty} \{\pi b^2 U_n(a, b; \triangle + b)\} \quad \ldots\ldots \quad (37)$$

This limit is most easily evaluated by replacing the functions $I_1(bt)$ and $K_0(ct)$ in (3) by their asymptotic expressions for $bt \to \infty$. The result is found to be

$$u_n(a; \triangle) = \frac{2^{3-n} \pi}{a \left\{\Gamma\left(\frac{n}{2}\right)\right\}^2} \int_0^\infty e^{-\triangle t} I_1(at) t^{n-4} dt \quad \ldots \quad (38)$$

The analogue of the differential relation (4) is very simple; it now connects the functions of orders n and $n+1$, namely,

$$u_{n+1}(a;\triangle) = -\tfrac{1}{2}\left\{\frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right)}\right\}^2 \frac{\partial u_n(a;\triangle)}{\partial \triangle} \quad . \quad . \quad . \quad (39)$$

as is readily verified by differentiation of (38) with respect to $\triangle$.

The integral (38) is generally expressible in terms of Legendre functions:

$$u_n(a;\triangle) = -\frac{\sqrt{2\pi a}}{2^{n-3}a^2}\,\frac{(\triangle^2-a^2)^{\frac{7}{4}-\frac{n}{2}}}{\sin n\pi\left\{\Gamma\left(\frac{n}{2}\right)\right\}^2}\,Q_{\frac{1}{2}}^{n-\frac{7}{2}}(\triangle/a) \quad . \quad . \quad (40)$$

In terms of hypergeometric functions, one has

$$u_n(a;\triangle) = \frac{\pi}{(2\triangle)^{n-2}}\,\frac{\Gamma(n-2)}{\left\{\Gamma\left(\frac{n}{2}\right)\right\}^2}\,F\left(\frac{n-2}{2},\frac{n-1}{2};2;\frac{a^2}{\triangle^2}\right), \quad . \quad (41)$$

or, alternatively,

$$u_n(a;\triangle) = \frac{\pi}{(2\sqrt{\triangle^2-a^2})^{n-2}}\,\frac{\Gamma(n-2)}{\left\{\Gamma\left(\frac{n}{2}\right)\right\}^2}\,F\left(\frac{n-2}{2},\frac{5-n}{2};2;-\frac{a^2}{\triangle^2-a^2}\right), \quad (42)$$

by means of which the function $u_n(a;\triangle)$ is easily evaluated for integral values of n. The latter functions are elementary; for instance,

$$u_3(a;\triangle) = \frac{4}{\triangle+\sqrt{\triangle^2-a^2}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (43)$$

$$u_4(a;\triangle) = \frac{\pi}{2a^2}\left\{\frac{\triangle}{\sqrt{\triangle^2-a^2}}-1\right\}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (44)$$

$$u_5(a;\triangle) = \frac{4}{9}\frac{1}{(\triangle^2-a^2)^{3/2}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (45)$$

$$u_6(a;\triangle) = \frac{3\pi}{32}\frac{\triangle}{(\triangle^2-a^2)^{5/2}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (46)$$

$$u_7(a;\triangle) = \frac{4}{75}\frac{4\triangle^2+a^2}{(\triangle^2-a^2)^{7/2}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (47)$$

Acknowledgement. I am indebted to Dr D. POLDER for suggesting the problem.

Eindhoven, July 1947.

Mathematics. — *A matric representation of binary modular congruence groups of degree m.* (Second communication.) By F. VAN DER BLIJ. (Communicated by Prof. J. G. VAN DER CORPUT.)

(Communicated at the meeting of October 25, 1947.)

3.1. *The matric representation.*

In our first communication bearing the same title we determined the behaviour of the functions $X(Z \mid T; P)$ under the group $G(\varepsilon_n)$ of modular operators $\mathbf{U} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ with a matrix C of degree m. If P runs through a complete system of non-congruent special matrices $P^{(n,m)}$ to modulus ν_0, a set of $\nu^{nm} \triangle^m$ linearly independent functions $X(Z \mid T; P)$ is obtained. So the correspondence

$$\mathbf{U} \rightarrow (A(P, R)) \tag{3.01}$$

where

$$\begin{cases} A(P,R) = \Lambda_0 \, e\left\{\dfrac{PAB'P' + 2PBC'X' + XDC'X'}{\nu_0}\right\} \\ \text{if there exists a special matrix } X \text{ satisfying } R \equiv PA + XC \pmod{\nu_0}, \\ A(P,R) = 0 \text{ otherwise;} \end{cases} \tag{3.02}$$

and where we denote

$$\Lambda_0 = \nu^{-\frac{1}{4}nr} \, \delta^{\frac{1}{2}n} \left(\frac{\triangle}{\nu_+}\right) e\left[\tfrac{1}{4} n \sum_{j=1}^{r} (\nu_j - 1)\right], \tag{3.03}$$

is a matric representation of the group $G(\varepsilon_n)$.

As a special case let $\mathbf{U}$ be an element of the subgroup $G(\nu\varepsilon_n)$ of $G(\varepsilon_n)$, that is to say let $\mathbf{U} \equiv \mathbf{E} \pmod{\nu\varepsilon_n}$. If $P \equiv R \pmod{\nu_0}$ we have $A(P,R) = A(P,P) = 1$ and if $P \not\equiv R \pmod{\nu_0}$ we have $A(P,R) = 0$ in this special case.

We thus proved that in the homomorphic correspondence (3.01) the identy matrix corresponds to all elements $\mathbf{U}$ of $G(\nu\varepsilon_n)$. The functions $X(Z \mid T; P)$ are invariant under $G(\nu\varepsilon_n)$. Therefore the representation (3.01) can also be considered as a matric representation of the quotient group $G(\varepsilon_n)/G(\nu\varepsilon_n)$. This quotient group is simply isomorphic to the binary modular congruence group G_m of degree m modulo ν. The last group consists of matrices $\mathbf{U}^* = \begin{pmatrix} A^* & B^* \\ C^* & D^* \end{pmatrix}$ where the elements of A^*, B^*, C^* and D^* are classes of residues to modulus ν, such that, if E^* and N^*

are the unit matrix and the zero matrix in the classes of residues, the following equation between the elements is satisfied

$$\mathbf{U}^{*\prime}\,\mathbf{I}^{*}\,\mathbf{U}^{*}=\mathbf{I}^{*} \text{ where } \mathbf{I}^{*}=\begin{bmatrix} N^{*} & E^{*} \\ -E^{*} & N^{*} \end{bmatrix}. \tag{3.04}$$

Thus we have proved:

Theorem 1.

The correspondence (3.01) *where* $A(P,R)$ *is given by* (3.02) *and* (3.03) *is a matric representation of the quotient group* $G(\varepsilon_n)/G(\nu\,\varepsilon_n)$, *which is simply isomorphic with the binary modular congruence group of degree m modulo* ν.

The study of this representation R_m and especially its decomposition into a sum of irreducible components would be interesting, but there are still some difficulties. The representation space of R_m is a $\nu^{nm}\,\Delta^{m}$ dimensional linear manifold (vector-space) over the field of complex numbers. It will sometimes also be denoted by R_m. Apart of its meaning as a function of Z and T we interprete the symbol $X(P)$ sometimes as a system of basic vectors for R_m.

A first reduction of R_m is rather trivial. It may be seen readily that if $P \not\equiv R \pmod{\Delta}$ we have $A(P,R)=0$. Let P_0 be a special matrix and let $R_m(P_0)$ be the subspace of R_m, which is spanned by all those vectors $X(P)$ for which $P \equiv P_0\ (\Delta)$. Then $R_m(P_0)$ is an invariant subspace of R_m under the operators $\mathbf{U}$. The representation space R_m is the direct sum of the representation spaces $R_m(P_0)$, if P_0 runs through a complete system of non-congruent special matrices to modulus Δ. The number of these spaces is Δ^{m}. It is now easily proved that the trace of $R_m(P_0)$, being defined by

$$T(\mathbf{U};P_0)=\sum_{\substack{P \bmod \nu_0 \\ P\equiv P_0(\Delta)}} A(P,P) \tag{3.05}$$

is independent of P_0. For let L be a special matrix, we have

$$A(P+\nu L,\ R+\nu L)=A(P,R).$$

Now we chose L such that $P_0+\nu L \equiv N(\Delta)$. Then

$$T(\mathbf{U};P_0)= \\ =\sum_{\substack{P \bmod \nu_0 \\ P\equiv P_0(\Delta)}} A(P,P)=\sum_{\substack{P \bmod \nu_0 \\ P+\nu L\equiv N(\Delta)}} A(P+\nu L,P+\nu L)=\sum_{\substack{P \bmod \nu_0 \\ P\equiv N(\Delta)}} A(P,P)=T(\mathbf{U};N).$$

Therefore the representations $R_m(P_0)$ are all aequivalent to one another and we consider only the representation $R_m(N)$.

Then $A(P,R)$ does not vanish if and only if a special matrix X exists such that $R \equiv PA-XC \pmod{\nu_0}$; $XC \equiv N(\Delta)$. If such a matrix does exist it can always be chosen $\equiv N \pmod{\Delta}$; we denote

$$A(\Delta P,\Delta R)=C(P,R). \tag{3.06}$$

The representation $R_m(N)$, which will henceforth simply be denoted by R_m is defined by the formulas

$$R_m\colon\ X(P)\cdot \mathbf{U} = \sum_{R \bmod \nu} C(P,R)\,X(R). \tag{3.07}$$

We introduce now the symbol $e(X)$ by

$$e(X) = e\left\{\frac{\triangle X}{\nu}\right\} = e\left[\frac{\sigma(QX)}{\nu}\right] = e^{\frac{\pi i}{\nu}\sigma(QX)} \tag{3.08}$$

Then we denote

$$\begin{cases} C(P,R) = \Lambda_0\, e(P\,A\,B'\,P' + 2\,P\,B\,C'\,X' + X\,D\,C'\,X'), \\ \text{if there exists an integral matrix } X \text{ with } R \equiv PA + XC \pmod{\nu}. \\ C(P,R) = 0 \text{ otherwise.} \end{cases} \tag{3.09}$$

These formulas can still be simplified. After replacing, if necessary the modular matrix $\mathbf{U}$ by a congruent modular matrix modulo ν, we may suppose that there exists a matrix $\overline{C} = \overline{C}^{(m)}$ such that $C\overline{C} \equiv E \pmod{\nu}$ and $\nu\overline{C}$ is an integral matrix.

If we replace X in (3.09) by $(R - PA)\overline{C}$, we get after some calculations

$$\begin{cases} C(P,R) = \Lambda\, e(P\,A\,\overline{C}\,P' - 2\,R\,\overline{C}\,P' + R\,\overline{C}\,D\,R'), \\ \text{if there exists an integral matrix } X \text{ with } R \equiv PA + XC \pmod{\nu}. \\ C(P,R) = 0 \text{ otherwise.} \end{cases} \tag{3.10}$$

And since now $r = m$ we have

$$\Lambda = \nu^{-\frac{1}{2}nm}\,\delta^{\frac{1}{2}n}\left(\frac{\triangle}{\nu^m\,\delta}\right) e\left[\tfrac{1}{4}\,n\sum_{j=1}^{m}(\nu_j - 1)\right]. \tag{3.11}$$

Here γ_j is the j-th elementary divisor of C; $(\gamma_j, \nu) = \delta_j$; $\prod_j \delta_j = \delta$; $\nu = \delta_j\,\nu_j$.

Thus we have proved

Theorem 2.

The correspondence $\mathbf{U} \longleftrightarrow C(P,R)$ where $C(P,R)$ is given by (3.10) *and* (3.11) *is a matric representation of the quotient group $G(\varepsilon_n)\,|\,G(\nu\,\varepsilon_n)$ which is simply isomorphic with the binary modular congruence group of degree m modulus ν.*

4.1. *A new proof of theorem* 2.

In this part the characters P and R denote integral, not necessarily special matrices of n rows and m columns.

Let $\mathbf{U}$ be a modular matrix, $\mathbf{U} \in G(\varepsilon_n)$. We may suppose, after multiplying if necessary with a modular matrix $\mathbf{U}_0 \in G(\nu\,\varepsilon_n)$ that the element C of $\mathbf{U}$ satisfies the condition that a matrix C with $C\,C \equiv E\,(\nu^3)$ and integral $\nu\overline{C}$ can be found.

We consider modular matrices $\mathbf{U}$, $\mathbf{U}_1$ and $\mathbf{U}_2$ which satisfy these conditions and the equation $\mathbf{U} = \mathbf{U}_1 \cdot \mathbf{U}_2$.

The formulas of theorem 2 define a matric representation of the quotient group $G(\varepsilon_n)/G(\nu\varepsilon_n)$ if the equation

$$\sum_{R \bmod \nu} C(P, R; \mathbf{U}_1)\, C(R, S; \mathbf{U}_2) = C(P, S; \mathbf{U}) \tag{4.01}$$

is satisfied and if we have for every $\mathbf{U}_0 \in G(\nu\varepsilon_n)$

$$C(P, R; \mathbf{U}_0) = \begin{matrix} 1 \text{ if } P \equiv R \pmod{\nu}, \\ 0 \text{ if } P \not\equiv R \pmod{\nu}. \end{matrix} \tag{4.02}$$

It may be seen readily that this last condition is satisfied.

We define $\overline{C}_1$ and $\overline{C}_2$ to C_1 and C_2 as the matrix $\overline{C}$ was defined to C. The numbers Λ_1 and Λ_2 depend on $\mathbf{U}_1$ and $\mathbf{U}_2$ in the same way as the number Λ depends on $\mathbf{U}$, see (3.11).

Hereafter we shall use consistently the identities: (4.03)

$$\mathbf{U} = \mathbf{U}_1 \mathbf{U}_2; \quad \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A_1 A_2 + B_1 C_2 & A_1 B_2 + B_1 D_2 \\ C_1 A_2 + D_1 C_2 & C_1 B_2 + D_1 D_2 \end{pmatrix}.$$

$$\mathbf{U}_1 = \mathbf{U}\, \mathbf{U}_2^{-1}; \quad \begin{pmatrix} A_1 B_1 \\ C_1 D_1 \end{pmatrix} = \begin{pmatrix} A\, D_2' - B\, C_2' & -A\, B_2' + B\, A_2' \\ C\, D_2' - D\, C_2' & -C\, B_2' + D\, A_2' \end{pmatrix}.$$

$$\mathbf{U}_2 = \mathbf{U}_1^{-1}\, \mathbf{U}; \quad \begin{pmatrix} A_2 B_2 \\ C_2 D_2 \end{pmatrix} = \begin{pmatrix} D_1' A - B_1' C & D_1' B - B_1' D \\ -C_1' A + A_1' C & -C_1' B + A_1' D \end{pmatrix}.$$

We abbreviate the lefthand side of (4.01) to Σ. Thus we have

$$\begin{aligned} \Sigma = \Lambda_1 \Lambda_2 \Sigma e(&P A_1 \overline{C}_1 P' - 2 R \overline{C}_1 P' + R \overline{C}_1 D_1 R' + \\ &+ R A_2 \overline{C}_2 R' - 2 S \overline{C}_2 R' + S \overline{C}_2 D_2 S'), \end{aligned} \tag{4.04}$$

where the summation must be extended over those matrices R, for which there exist two integral matrices X and Y satisfying

$$R \equiv P A_1 + X C_1 \pmod{\nu} \quad \text{and} \quad R \equiv S D_2' + Y C_2' \pmod{\nu}.$$

(It may be seen readily that the condition for R to be satisfied in order to obtain an integral matrix Y with $R \equiv S D_2' + Y C_2'$ (ν) and that to be satisfied in order to obtain an integral matrix X_1 with $S \equiv R A_2 + X_1 C_2 (\nu)$ are the same condition.)

Now we introduce

$$C^*(P, S; \mathbf{U}) = \Lambda e(P A \overline{C} P' - S \overline{C} P' + S \overline{C} D S'). \tag{4.05}$$

Thus

$$\left\{ \begin{aligned} \Sigma = \frac{\Lambda_1 \Lambda_2}{\Lambda} C^*(P, S; \mathbf{U}) \Sigma e(&P(A_1 \overline{C}_1 - A \overline{C}) P' + \\ &+ R(\overline{C}_1 D_1 + A_2 \overline{C}_2) R' + S(\overline{C}_2 D_2 - \overline{C} D) S' - \\ &- 2 R \overline{C}_1 P' - 2 S \overline{C}_2 R' + 2 S \overline{C} P'), \end{aligned} \right. \tag{4.06}$$

$$(4.07)\quad \begin{cases} \Sigma = \dfrac{\Lambda_1 \Lambda_2}{\Lambda} C^*(P,S;\mathbf{U})\, \Sigma e\,([R-P\overline{C}'C_2'-S\overline{C}C_1] \\ \qquad\qquad \overline{C}_1 C \overline{C}_2 [R-P\overline{C}'C_2'-S\overline{C}C_1]'). \end{cases}$$

(Here the summation must be extended over those matrices R, which satisfy the conditions mentioned above.)

The matrix $C_1 C C_2'$ is symmetric. Let $\varkappa$ the least positive integer such that the matrix $\varkappa\overline{C}$ is integral. Hence $\varkappa$ is a divisor of ν. We introduce a new matrix of summation M by

$$(4.08)\qquad M = \varkappa R - \varkappa P\overline{C}'C_2' - \varkappa S\overline{C}C_1.$$

We have

$$(4.09)\qquad \Sigma = \frac{\Lambda_1\Lambda_2}{\Lambda} C^*(P,S;U)\,\Sigma e\left(\frac{M\overline{C}_1 C \overline{C}_2 M'}{\varkappa^2}\right).$$

where the sum must be extended over those matrices M mod $\varkappa\nu$ for which there exist integral matrices X and Y such that

$$(4.10)\quad \begin{cases} M \equiv \varkappa([PA-S]\overline{C}\ C_1 + XC_1) & (\mathrm{mod}\ \varkappa\nu), \\ M \equiv \varkappa([SD'-P]\overline{C}'C_2' + YC_2') & (\mathrm{mod}\ \varkappa\nu), \end{cases}$$

$$(4.11)\qquad M \equiv -\varkappa(P\overline{C}'C_2' + S\overline{C}C_1) \qquad (\mathrm{mod}\ \varkappa).$$

We deduced these congruences from

$$M \equiv \varkappa(PA_1 + XC_1 - P\overline{C}'C_2' - S\overline{C}C_1) \qquad (\mathrm{mod}\ \varkappa\nu),$$
$$M \equiv \varkappa(SD_2' + YC_2' - P\overline{C}'C_2' - S\overline{C}C_1) \qquad (\mathrm{mod}\ \varkappa\nu).$$

The congruence (4.11), being a consequence of the congruences (4.10) can be omitted. We replace the congruences (4.10) by

$$\varkappa\nu K + M = \varkappa((PA-S)\overline{C}C_1 + XC_1),$$
$$\varkappa\nu L + M = \varkappa((SD'-P)\overline{C}'C_2' + YC_2'),$$

with arbitrary integral matrices K and L.

Multiplying these congruences with $D_1' C' C_2'$ and $A_2\overline{C}C_1$ resp. and adding, we find with an integral matrix W

$$\begin{aligned} \nu W + M(D_1'C'C_2' + A_2\overline{C}C_1) = {} & \varkappa P(A\overline{C}C_1 D_1' C' C_2' - \overline{C}'C_2' A_2 \overline{C}C_1) + \\ & + \varkappa S(D'\overline{C}'C_2' A_2\overline{C}C_1 - \overline{C}C_1 D_1'\overline{C}'C_2') + \\ & + \varkappa X C_1 D_1' C' C_2' + \varkappa Y C_2' A_2 C C_1. \end{aligned}$$

Now we use the congruence

$$D_1'\overline{C}'C_2' + A_2\overline{C}C_1 \equiv \overline{C}_2'(C_2'D_1' + A_2'C_1')\overline{C}'C_2' \equiv E \pmod{\nu}.$$

and write as an abbreviation $G = C_2\overline{C}C_1$. Then we have

$$\nu W_1 + M = \varkappa P(A\overline{C}D_1 - C'A_2')G + \varkappa S(D'C'A_2' - CD_1)G + \varkappa(XD_1 + YA_2')G,$$
$$M \equiv \varkappa(PB_1 + SB_2 + XD_1 + YA_2')G.$$

We introduce a new matrix of summation Z by $M = \varkappa Z G$

$$\Sigma = \frac{A_1 A_2}{A} C^* (P, S; U) \Sigma e (Z G Z'). \tag{4.12}$$

Here the sum must be extended over a set of integral matrices Z, such that the matrices $\varkappa Z G$ are non-congruent to the modulus $\varkappa \nu$ and that it is possible to determine integral matrices X and Y satisfying

$$Z C_2 \equiv (P A - S) + X C \pmod{\nu},$$
$$Z C_1' \equiv (S D' - P) + Y C' \pmod{\nu}.$$

These congruences are not linearly independent. This can be seen readily after multiplying the second congruence with the matrix A and adding it to the first congruence.

Now we consider

$$\Sigma e (Z G Z'), \tag{4.13}$$

where the sum must be extended over a set of integral matrices Z such that the matrices $\varkappa Z G$ are non-congruent modulus $\varkappa \nu$ and that we can determine an integral matrix X satisfying $Z C_2 \equiv (P A - S) + X C \pmod{\nu}$.

We first prove the

Lemma.

Let Z_0 be an integral matrix and G_0 and F are integral matrices, such that there exist integral matrices μG_0 and $\mu \overline{F}$ with $G_0 \overline{G}_0 \equiv F \overline{F} \equiv E(\mu)$. Further let the matrix $\mu \overline{F}' \overline{G}_0 \overline{F}$ be integral and

$$e_\mu (X) = e^{\frac{\pi i}{\mu} \sigma (Q X)}$$

$$\sigma = \sum_{Z G_0 \bmod \mu} e_\mu (Z G_0 Z'),$$

where the sum must be extended over a set of integral matrices Z, such that the matrices $Z G_0$ are non-congruent modulus μ and that we are able to determine an integral matrix Y satisfying $Z \equiv Z_0 + Y F \pmod{\mu}$. Then we have either $\sigma = 0$ or $Z_0 \equiv W_0 F \pmod{\mu}$ with a suitably chosen integral matrix W_0.

In order to prove this lemma we write

$$\sigma = e_\mu (Z_0 G_0 Z_0') \sum_{U F G_0 \bmod \mu} e_\mu (U F G_0 F' U') e_\mu (2 U F G_0 Z_0').$$

Now we replace U by $U + \mu K F' G_0 F$ with an arbitrary integral matrix K:

$$\sigma = e_\mu (Z_0 G_0 Z_0') e_\mu (2 \mu K F' Z_0') \sum_{U F G_0 \bmod \mu} e_\mu (U F G_0 F' U') e_\mu (2 U F G_0 Z_0').$$

Thus we have either $\sigma = 0$ or $e_\mu (2 \mu K \overline{F}' Z_0') = 1$ for every integral matrix K, thus there exists an integral matrix W_0 with $Z_0 \equiv W_0 F \pmod{\mu}$.

In order to calculate the sum over Z in (4.12) we use this lemma. We

choose $G_0 = \varkappa G = \varkappa C_2 \overline{C} C_1$; $\mu = \varkappa\nu$; $F = C\overline{C}_2$; $\varkappa \overline{G}_0 = \overline{C}_1 C \overline{C}_2$; $\overline{F} = C_2 \overline{C}$ and thus the matrices $\mu \overline{G}_0 = \nu \overline{C}_1 (C_1 A_2 + D_1 C_2)\overline{C}_2$ and $\mu \overline{F} = \nu\varkappa C_2 \overline{C}$ are integral. The matrix $\mu \overline{F}' \overline{G}_0 \overline{F} = \nu \overline{C}' C_2' \overline{C}_1 = = \nu(A_1 \overline{C}_1 - \overline{C}' A')$ is an integral matrix.

The sum over Z vanishes unless there exists an integral matrix X satisfying $S \equiv P A + X C \pmod{\nu}$.
If this condition is satisfied we must calculate

$$\Sigma e(Z G Z'), \tag{4.14}$$

where the sum must be extended over as set of integral matrices Z such that the matrices $\varkappa Z G$ are non-congruent to modulus $\varkappa\nu$.

We replace this sum by

$$\Sigma e\left(\frac{X \overline{G} X'}{\varkappa^2}\right), \tag{4.15}$$

where the sum must be extended over integral matrices $X \bmod \varkappa\nu$ which satisfy $\nu X \overline{G} \equiv 0 \pmod{\varkappa\nu}$.

Since ν is an odd integer we may suppose the matrix $\overline{C}_1 C \overline{C}_2$ to be a diagonal matrix. We denote the elements of this diagonal matrix by $\lambda_j \delta_j \delta_{1j}^{-1} \delta_{2j}^{-1}$ $(j = 1, \dots m)$ where $(\lambda_j, \nu) = 1$ and δ_j, δ_{1j}, δ_{2j} are the greatest common divisors of ν and the j-th elementary divisor of the matrices C, C_1 and C_2 resp.

Since $(\nu, \triangle) = 1$ we use the following formula for the Gaussian sums:

$$\sum_{Y \bmod \mu} e\left[\frac{h y' Q y}{\mu}\right] = \mu^{\frac{1}{2}n} \left(\frac{h^n \triangle}{\mu}\right) e\left[\tfrac{1}{4} n (1 - \mu)\right]. \tag{4.16}$$

Thus we have, since $\nu \delta_j \equiv 0$ $(\delta_{1j} \delta_{2j})$ and n is even

$$\sum_{\substack{X \bmod \varkappa\nu \\ \nu X G \equiv 0 (\varkappa\nu)}} e\left(\frac{X \overline{C}_1 C C_2 X'}{\varkappa^2}\right) =$$

$$= \prod_{j=1}^{m} \left[\frac{\nu \delta_j}{\delta_{1j} \delta_{2j}}\right]^{\frac{1}{2}n} \left(\frac{\triangle}{\nu \delta_j \delta_{1j} \delta_{2j}}\right) e\left[\tfrac{1}{4} n (1 - \nu \delta_j \delta_{1j} \delta_{2j})\right].$$

If we denote

$$\delta = \prod_j \delta_j, \quad \delta_1 = \prod_j \delta_{1j}, \quad \delta_2 = \prod_j \delta_{2j}$$

we write

$$\sum_{\substack{X \bmod \varkappa\nu \\ \nu X G \equiv 0 (\varkappa\nu)}} e\left(\frac{X \overline{C}_1 C C_2 X'}{\varkappa^2}\right) =$$

$$= \frac{\nu^{\frac{1}{2}nm} \delta^{\frac{1}{2}n}}{\delta_1^{\frac{1}{2}n} \delta_2^{\frac{1}{2}n}} \left(\frac{\triangle}{\nu^m \delta \delta_1 \delta_2}\right) e\left[\tfrac{1}{4} n \sum_j (1 - \nu \delta_j \delta_{1j} \delta_{2j})\right].$$

It follows at once from the definition of Λ (3.11) that

$$\frac{\Lambda}{\Lambda_1 \Lambda_2} = \frac{\nu^{-\frac{1}{2}nm}\,\delta^{\frac{1}{2}n}\left(\frac{\Delta}{\nu^m\,\delta}\right) e\left[-\frac{1}{4}n\,\Sigma(1-\nu\,\delta_j)\right]}{\nu^{-nm}\,\delta_1^{\frac{1}{2}n}\,\delta_2^{\frac{1}{2}n}\left(\frac{\Delta}{\delta_1\,\delta_2}\right) e\left[-\frac{1}{4}n\,\Sigma(2-\nu\,\delta_{1j}-\nu\,\delta_{2j})\right]}.$$

We use the congruence $1-\nu\,\delta_{1j}-\nu\,\delta_{2j}+\nu\,\delta_j \equiv 1-\nu\,\delta_j\,\delta_{1j}\,\delta_{2j} \pmod 4$;

$$\frac{\Lambda}{\Lambda_1 \Lambda_2} = \frac{\nu^{\frac{1}{2}nm}\,\delta^{\frac{1}{2}n}}{\delta_1^{\frac{1}{2}n}\,\delta_2^{\frac{1}{2}n}}\left(\frac{\Delta}{\nu^m\,\delta\,\delta_1\,\delta_2}\right) e\left[\tfrac{1}{4}n\sum_j(1-\nu\,\delta_j\,\delta_{1j}\,\delta_{2j})\right].$$

If we substitute these results in formula (4.12), we have proved, since $C(P,S;\mathbf{U}) = C^*(P,S;\mathbf{U})$ if there exists an integral matrix X with $S \equiv PA + XC\ (\nu)$, and $C(P,S;\mathbf{U}) = 0$ otherwise,

$$\Sigma = \frac{\Lambda_1 \Lambda_2}{\Lambda} C(P,S;\mathbf{U}) \cdot \frac{\Lambda}{\Lambda_1 \Lambda_2} = C(P,S;\mathbf{U}).$$

And we have proved the formula (4.01).

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Mathematics. — *On the principles of intuitionistic and affirmative mathematics.* II. By D. VAN DANTZIG. (Communicated by Prof. J. G. VAN DER CORPUT.)

(Communicated at the meeting of June 28, 1947.)

Ch. 3. *The strong interpretation. Affirmative mathematics. (Intuitionistic mathematics from a formal standpoint.)*

Some classical mathematicians might perhaps say that this weak interpretation, scetched roughly in Ch. 2, were exactly what they needed. and that they therefore had no need of BROUWER's strong interpretation. Against this view however several remarks can be made.

First: scientific interest, and even mere curiosity makes us desire to obtain other than only negative results. If it is true that "totally normal" numbers "exist", we want to know such a number, and if every real number except those of a set of measure 0 possesses this property. we might know whether particular numbers like $\sqrt{2}$, or e possess it. Then: for the *applications* of mathematics these negative statements are entirely worthless. There we have no use for "existence" of a real number in the weak sense; we need effective approximations, the possible errors of which we can estimate explicitly. Third: the use of the weak interpretation may lead to difficulties with regard to the *foundations* of mathematics and physics. E.g. this is the case with the theory of probability, as I have tried to show in a recent conference [20]). Finally it is a good and current habit of mathematicians, always to prove their theorems in as strong a form and with as few undefined notions as can be done without unduly augmenting the difficulties. There is no reason, not to follow this habit with respect to the *logical* part also.

In this chapter we shall follow the inverse way as in Ch. 2, and take the standpoint of a "classical" mathematician who wants to deduce the fundamental theorems of arithmetics and analysis from as few hypotheses and undefined notions and in as strong a form and as rigorously as he can. We shall not assume here anything about intuitionism but proceed in a purely formal way, though here also only a rough scetch can be given.

Of course we begin our analysis with a theory of natural numbers as a base for a theory of real numbers.

Before doing so, we have to say a few words about the logical notations used here. All constant symbols are printed in vertical type; all variables are denoted by letters in italics. The implication, conjunction and

[20]) D. VAN DANTZIG [2].

disjunction of two statements, denoted by A and B are denoted by

$$|A \supset B|$$
$$|A \wedge B|$$
$$|A \vee B|$$

respectively *inclusive* the vertical strokes, which replace brackets or dots. This notation has the advantage, that complex formulae are obtained by direct *substitution* of these formulae for the cursive letters, so that no *counting* of dots or strokes is necessary. e.g.

$$|||A \supset C| \wedge |B \supset C|| \supset ||A \vee B| \supset C||$$

and that dots and brackets may retain their original algebraical meaning. In the same way the sum and product of two numbers x and y are denoted by $(x + y)$ and $(x \cdot y)$ respectively, *inclusive* the brackets. This makes every further convention with respect to "strength" of brackets superfluous [21]). As, however, we don't intend to give here a complete formalisation, in particular of the logical relations, we sometimes omit brackets or strokes which are not necessary for reading the formulae.

In order to avoid a calculus of classes, we write $\mathrm{N}[x]$ and $\mathrm{R}[x]$ instead of $x \in \mathrm{N}$ and $x \in \mathrm{R}$ respectively. We read these symbols as "*x denotes* (not: "is") a natural (real) number" so that we may avoid defining what a natural or real number "really is". Constant symbols (e.g. N, C, T, R, d) are printed in vertical type.

The axioms which we introduce are: the axioms *A* 1, *A* 2, *A* 3 the latter of which is introduced below, and the axioms $E_1, \ldots, E_4$ of equality. They are:

A 1. $\quad \vdash \mathrm{N}[0]$

A 2. $\quad \vdash| \mathrm{N}[x] \supset \mathrm{N}[x']|$

E 1. $\quad \vdash| \mathrm{N}[x] \supset| x = x||$

E 2. $\quad \vdash| \mathrm{N}[x, y] \supset|| x = y| \supset| x' = y'||$

E 3. $\quad \vdash| \mathrm{N}[x, y, z] \supset||| x = y| \wedge| y' = z'|| \supset| z = x|||$

E 4. $\quad \vdash| \mathrm{N}[x] \supset|| x = y| \supset \mathrm{N}[y]||$

Here we wrote for abbrevation $\mathrm{N}[x, y]$ and $\mathrm{N}[x, y, z]$ in stead of $|\mathrm{N}[x] \wedge \mathrm{N}[y]|$ and $||\mathrm{N}[x] \wedge \mathrm{N}[y]| \wedge \mathrm{N}[z]||$ respectively.

We did not — as it is sometimes done — *define* equality of x and y by requiring $A[x]$ and $A[y]$ to be equivalent for any statement A, depending on one variable. In fact, this condition is too strong, as we may consider statements, concerning the logical calculus itself. If e.g. $A[x]$ is replaced by the statement that the statement $x = 4$ may be

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[21]) Strictly spoken the use of two kinds of brackets (opening and closing ones) is superfluous, though of course it makes reading easier. For the same reason we might replace the strokes by inverted or ordinary comma's, writing e.g. $,A \supset B,$ in stead of $|A \supset B|$. The advantage however is not important.

obtained from the axioms and definitions by at most a given number of applications of the deduction schemes, we may very well have $A\,[4]$ without having $A\,[(2+2)]$. It would therefore be necessary to limit in advance the set of statements for which the equivalence of $A\,[x]$ and $A\,[y]$ must hold. We therefore preferred to define equality implicitly by four axioms, from which the other properties are easily deduced.

Complete induction is introduced as a deduction-scheme (not as an axiom), viz

$$\frac{\vdash A\,[0] \qquad \vdash N\,[x] \supset A\,[x] \supset A\,[x']}{\vdash N\,[x] \supset A\,[x]}$$

for any statement $A\,[x]$ such that $A\,[y]$ is independent of x. We can then define in the ordinary way sums and products of natural numbers and deduce their formal algebraic properties.

We remark that we did not introduce PEANO's fourth axiom:

$$\vdash N\,[x] \supset \neg\, x' = 0,$$

nor did we until now use any negation or disjunction. The question then naturally arises whether we can do entirely without them.

From the formal point of view mathematics is a system of formulae, the mathematician is willing to "accept" or "admit", whatever these words may mean. The symbol $A \supset B$ means that he is prepared to take B in his list of accepted formulae as soon as A is admitted. The symbol $A \wedge B$ means that A as well as B is admitted.

The symbol $\neg\, A$, however, has quite another nature. It does not describe the admittance of any formula, but the rejection of A, i.e. the mathematician's refusal to accept A. Of course he *may* refuse A, but why should he mention the fact at all? We may make our list without telling anything about formulae we reject, or we eventually or conditionnally would refuse to admit [22]).

And if we *can* do so — and we shall see we can — the mathematician's principle not to introduce any superfluous primitive notion urges us to avoid the negation at all. Such a mathematical (or logical) system in which negation occurs *not* — of course it may occur in discussions *about* (the "metamathematics" of) the system — will be called *affirmative* [23]).

With respect to the disjunction the situation is still somewhat different.

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[22]) From the intuitionist's standpoint the situation is not very different. For him $\neg\, A$ denotes the impossibility ("absurdity") of a construction as stated by A. This is a warning, useful for future investigators, that attempts to construct A necessarily must fail, but strictly indispensible it is not.

[23]) The principles of affirmative mathematics were first exposed in my lectures at the university of Amsterdam and in a subsequent conference before the Amsterdam mathematical society in the spring of 1939. I found since that I. JOHANSSEN [1] had put forward more or less analogous ideas before. Recently the idea of eliminating negations occurred independently to Dr. G. F. C. GRISS also, who, however, does not avoid unrestricted existence statements.

The disjunction $|A \vee B|$ describes our will to admit either A or B (or both). But if we are prepared to do so, we still have not admitted anything at all, nor have we — if we avoid negations — engaged ourselves to any definite acceptation. It expresses therefore a certain hesitation, so that one would be inclined to avoid it also. On the other hand we may remark that, if we do so, we necessarily loose a large number of statements of ordinary mathematics. The simplest example is the theorem

$$||(a \cdot b) = 0| \supset || a = 0| \vee | b = 0|||$$

for natural numbers [24]), and all its consequences.

On another occasion I hope to show that avoidance of the disjunction also is possible without losing anything essential of mathematics. But as the procedure becomes somewhat inelegant, we shall *here* retain the disjunction. For the same reason we introduce here the *restricted* existential symbol, though SKOLEM [1] showed that it can be avoided. It is defined as follows

$$|\exists_\xi^0 A[\xi]| \overset{\text{df}}{\equiv} A[0]$$

$$|\exists_\xi^{n'} A[\xi]| \overset{\text{df}}{\equiv} ||\exists_\xi^n A[\xi]| \vee A[n']|$$

The unrestricted existence-symbol can not be defined by complete induction and will not be introduced at all.

Instead of PEANO's fourth axion we introduce the axiom

A 3. $$\vdash |N[x, y] \supset || (x + y) = 0| \supset | y = 0|||$$

It evidently is independent of the preceding axioms, which are satisfied e.g. in a prime field of characteristic $p > 1$.

The system of axioms introduced here is certainly exempt of formal contradictions, as it would become trivial if the statement

T $$|N[x] \supset | x = 0||$$

were introduced as another axiom. Evidently T would follow already from

$$|0' = 0|$$

That this statement is not a consequence of the other axioms follows from the fact that this would imply a contradiction in intuitive mathematics, a possibility, the admittance of which does not seem to have any meaning. A formal proof of non-triviality of the set of axioms still stands out. Perhaps it might be less hopeless than the proof of non-contradictority in the case of HILBERT's axioms.

One might perhaps think of *defining* the negation by

$$\neg A \overset{\text{df}}{\equiv} |A \supset T|$$

analogous to the proceduce of GENTZEN [1] and I. JOHANSSON [1]. But

[24]) For real numbers it is not valid under the strong interpretation.

apart from the fact that the negation then were not a primitive but a defined notion, one should remind that T has *not* the nature of a contradiction, whereas GENTZEN's λ *does* denote a contradiction. For one may build several arithmetic systems, e.g. by replacing the first axiom N [0] by N [$\square$] and varying the other axioms in some way or another. The relations $0' = 0$ and $\square' = \square$ may then be entirely independent. The fact that some formal arithmetical system reduces to trivialty is of no more importance than the fact that some group consists of its unit-element only. If we would insist upon calling $|A \supset T|$ the "negation" of A, this negation would loose the "universal" character usually ascribed to it and become a relative notion [25]). Of course this does not prevent that in a *single* formal arithmetic system $|A \supset T|$ is *practically* equivalent with $\neg A$ [26]).

The further development of the theory of natural numbers, as well as the introduction of integers and rational numbers does not lead to any essential difficulties. We omit it here, and mention only the fact that the relations $\leqq$, $<$ and $\neq$ are defined by

$$|a \leqq b| \overset{\text{df}}{\equiv} |\exists_\xi^b |a = \xi||$$

$$|a < b| \overset{\text{df}}{\equiv} |a' \leqq b|$$

$$|a \neq b| \overset{\text{df}}{\equiv} ||a < b| \vee |b < a||,$$

the first of which is equivalent with

$$|a \leqq 0| \overset{\text{df}}{\equiv} |a = 0|$$

$$|a \leqq b'| \overset{\text{df}}{\equiv} ||a \leqq b| \vee |a = b'||$$

Real numbers may be introduced by means of a constant function symbol d, such that $r_n(x) = 2^{-n} \, d_n(x)$ is the n^{th} approximating rational dual number of x. If we denote the statements "x represents a real number" and "x represents an integer" by R [x] and I [x] respectively, the definition becomes:

$$R[x] \overset{\text{df}}{\equiv} \forall_n |N[n] \supset |I[d_n(x)] \wedge |d_n'(x) - 2\, d_n(x)| \leqq 1|||$$

Expressed by the rational numbers $r_n(x)$ the inequality means

$$|r_{n+1}(x) - r_n(x)| \leqq 2^{-n-1}.$$

Though less general in form, the definition is substantially equivalent

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[25]) This does *not* imply that the "exclusion-negation" (according to MANNOURY's terminology [5], p. 333, [10]) would become a "choice-negation".

[26]) Anyhow we can not define $|A \vee B|$ here by $\neg |\neg A \wedge \neg B|$ i.e. by $||A \supset T| \wedge |B \supset T|| \supset T|$, as this would lead us back to the weak interpretation.

with BROUWER's definition, the main difference being that BROUWER's λ-intervals [27]) have been replaced by their centers, and that the set of sequences, defining equal real numbers ("coinciding points") is somewhat narrower than with BROUWER. The relevant point is that the real numbers are obtained by a limit-procedure with a prescribed "velocity of convergence [28]).

Equality of real numbers is defined by

$$\|R[x] \wedge R[y]| \supset \| x = y| \overset{\text{df}}{\equiv} \bigwedge_n |N[n] \supset | |d_n(x) - d_n(y)| \leqq 1 |||$$

The reflexivity and symmetry of the relation are trivial, the transitivity is easily proved by substituting n' for n and considering that $d_n(x)$ etc. are integers. Also the definitions and the demonstration of the algebraic properties of sums and products of real numbers do not lead to any essential difficulties. Nomore is this the case with the identification of particular real numbers with integers p or dual rationals $p \cdot 2^{-k}$ by means of the definitions

$$|d_n(p) \overset{\text{df}}{=} p \cdot 2^n|$$

and

$$|d_n(p \cdot 2^{-k}) \overset{\text{df}}{=} \mathfrak{E}(p \cdot 2^{n-k})|$$

respectively, where $\mathfrak{E}(r)$ denotes the entier of the *rational* number r.

The relation $\leqq$ for real numbers is defined as follows:

$$\|R[x] \wedge R[y]| \supset \| x \leqq y| \overset{\text{df}}{\equiv} \bigwedge_n |N[n] \supset | d_n(x) \leqq d_n(y) + 1 |||$$

Here also the properties $\|x = y| \supset |x \leqq y\|$,

$$\text{hence } |x \leqq x| \text{ and } \|x \leqq y| \wedge |y \leqq x\| \supset |x = y\|$$

are trivial, whereas $\|\|x \leqq y| \wedge |y \leqq z\| \supset |x \leqq z\|$ is easy to prove. The maximum $\text{Max}(x, y)$ of two real numbers x and y is defined by

$$|d_n(\text{Max}(x, y)) \overset{\text{df}}{=} \text{Max}(d_n(x), d_n(y))|$$

and has the properties

$$\text{Max}(x, y) = \text{Max}(y, x), \qquad x \leqq \text{Max}(x, y)$$

and

$$\|\|x \leqq z| \wedge | y \leqq z\| \supset | \text{Max}(x, y) \leqq z\|$$

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[27]) BROUWER [5] I p. 253. A λ-interval is an interval of length 2^{-n} for some natural n, the endpoints of which are integral multiples of 2^{-n-1}. A real number ("point of the linear continuum")is a sequence of λ-intervals, each of which is contained with its endpoints in the interior of the preceding one.

[28]) According to BROUWER's definition this "velocity" may be augmented, but not diminished arbitrarily. Therefore his definition contains an unrestricted existence statement which we have avoided.

Also the absolute value $|x|$ of x can now be defined as Max $(x, -x)$, and has the ordinary properties. We can, however, *not* prove affirmatively that

$$\|x| = x| \vee |x| = -x\|$$

Moreover we can *not* define affirmatively the relations $<$ and $\neq$. This is to be expected, as e.g. the relation $x \neq y$ means that a natural n *exists*, such that $|x - y| \geqq 2^{-n}$. This however implies an unrestricted existence-statement; this is connected with the fact that the relation $x \neq y$ can not lead to any affirmative result. For instance the statement $|y \neq 0|$ gives us no means to determine even the first approximation of y^{-1}, as long as no rational lower boundary for y is known. For the same reason the statement $y \neq 0$ for an *empirical* quantity y (measured, not counted!) for which no lower boundary is known, can not be used for any empirical conclusion concerning y^{-1}.

We have therefore to "disperse" relations like $<$ and $\neq$ into a sequence of relations, say $\overset{n}{<}$ and $\overset{n}{\neq}$ [29]), defined by

$$|x \overset{n}{<} y| \overset{\text{df}}{\equiv} |2^{-n} \leqq y - x|$$
$$|x \overset{n}{\neq} y| \overset{\text{df}}{\equiv} |2^{-n} \leqq |y - x|$$

We can then define xy^{-1} if $y \overset{n}{\neq} 0$ and prove the usual properties, in particular its independence of the value of n.

This "dispersion" of a statement into a "fine-structure" of other ones is rather characteristic for affirmative mathematics. As a further example we mention the limes-relation, e.g. $\lim_{n \to \infty} a_n = a$. It states e.g. that for every natural e a natural $n(e)$ exists, such that

$$|N[n] \supset |a - a_{n(e)+n}| \leqq 2^{-e}\|$$

Evidently this definition contains an unrestricted existential statement. In affirmative mathematics we can therefore only define limes-relations with a *given* [30]) function $n(e)$ where $|N[e] \supset N[n(e)]|$ i.e. with a given "velocity of convergence". Hence we can only define

$$|\lim_{n(e)} a_n = a| \overset{\text{df}}{\equiv} \mathsf{V}_e |N[e] \supset \mathsf{V}_{n|} N[n] \supset |a - a_{n(e)+n}| \leqq 2^{-e}\|\|$$

In an analogous way definitions like that of continuity, etc. have to be altered, as well, of course, as the theorems in which they are used. Generally speaking the "dispersion" consists only in a more *explicit*

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[29]) From which they can only be reobtained by means of an unrestricted existence-statement, e.g.

$$|x < y| \overset{\text{df}}{\equiv} |\mathsf{E}_\xi |N[\xi] \wedge |x \overset{\xi}{<} y\|\|$$

[30]) Or bounded, which comes to the same.

statement of theorems: in ordinary mathematics, if we have to *prove* a certain relation $\lim a_n = a$, we prove the existence of a certain corresponding $n(\varepsilon)$, and if we *use* such a relation, we use again the existence of such an $n(\varepsilon)$, so that the "fine structure" is only a less elliptic form of the theorem. But of course there are many theorems which don't admit of an affirmative form at all [31].

It need hardly be said that the results of affirmative mathematics express theorems which are correct according to the intuitionist's standard. It also is quite obvious that the strong interpretation, like the weak one, covers a part only of intuitionistic mathematics. I shall not go into further details now, as I hope to do so on another occasion.

Ch. 4. *Some remarks on formalism.*

In every human action — in particular in "acts of intercourse" by which human beings influence each other — we can recognize and distinguish: an *emotional* element, associated with the affections of joy and grief, and an *indicative* element, associated with the sensations of recognition and distinction, though in certain actions one or the other of these elements may heavily prevail [32].

In science — as opposed to poetry, mysticism, etc. — and in particular in mathematics, we strive to eliminate the emotional elements as far as possible. This is often done by associating with the objects under investigation certain signs or marks, i.e. other objects which can more easily be recognized individually and discerned from each other. This process is called *formalization* of science, and often is very useful, as recognition and distinction of the objects may be possible or easy *mediately* by means of the signs, if it were difficult or even impossible without them.

This stripping off of emotional elements, however, never can be done consequently, as long as we have to do with human beings: a man cannot be separated from his emotions, and every little child knows to "read between the lines" before it can read at all. A consequent formalization can only be obtained by replacing the human beings by machines, carrying out the formal parts of their actions for them. This analogy between a formalism and a mechanism which I scetched in 1932 [33] — later it was independently investigated in detail by A. M. TURING [1] — rests upon the fact that a machine has properties comparable with the possibility of recognition and distinction, but not or hardly such as are comparable with joy, grief, love, hate, rage, etc.

The principal difference between the Dutch "group of Significists" [34],

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[31]) Cf. e.g. D. VAN DANTZIG [3], [4].

[32]) G. MANNOURY, [2], [5] p. 292.

[33]) D. VAN DANTZIG [1].

[34]) The leading members of the group were L. E. J. BROUWER, G. MANNOURY and the late F. VAN EEDEN, J. VAN GINNEKEN, and J. I. DE HAAN. The group worked mainly

and some other groups of scientists (e.g. the "logical empirists") lies in the fact that the latter consider science etc. as a system of words or symbols, and the former as a kind of human activity.

The main disadvantage of the standpoint of these latter groups lies in their inconsequence. Their attitude with regard to words and symbols sometimes implies a kind of "existential absolutism" which they otherwise always try to avoid. If science is not interested in stars and animals, but only in their observations, or rather in *descriptions* of these observations, no reason can be seen, why it should be interested in words or symbols, instead of in their observations, or rather in descriptions of these observations, or rather in descriptions of these descriptions, etc. The significists on the contrary center their attention around scientific (and other) human activities, among wihch formalization may or may not occur.

Moreover such a formalization or mechanization never is *complete*: we may always discover new regularities in the *produce* of a machine, which can be formalized with the aid of a new mechanism or formalism, but not with the old one. Therefore also TURING's "universal machine" (as he himself shows) is not universal. In 1932 I illustrated this by the example of a linotype, by which every ordinary letter or combination of letters may be printed. It can, however, not ascertain whether the rhyme-scheme of a poem is *A B B A*, though one could very well imagine a "poets-controll-machine" verifying this property.

Some modern logicians sometimes forget this very restricted range of any formal system — restricted in as far as it is formal! — Some of them go as far as defining "knowledge" as "an interpreted system", "a calculus supplemented by an interpretation". This is an almost grotesque over-straining of the (in several cases and for restricted purposes undeniable) usefulness of formalization. It is rather analogous with defining "art" as "a catalogue, supplemented by a museum", and entirely opposed to our view [35]). We do *not* consider as an ideal mathematician the man who knows by heart PEANO's "Formulaire" or RUSSELL and WHITEHEAD's "Principia", but the man who discovers *new* properties, with or without a calculus, in or outside a formal system describing the old ones.

We also do not see mathematics as a "tautologie immense" [36]). On the contrary, we don't know of the existence of any tautology at all: saying twice the same thing is not saying twice the same thing. More precisely: if two things (or actions, statements, symbols, etc.) are recognized as being the "same" in certain respects, nevertheless as being "two", i.e. discerned as being different in other respects (and be it only in space or time), it may always become desirable to fix one's attention on the difference, even if they were treated (and formalized!) as "identical"

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in the jears 1917—1922. Its most active member since is G. MANNOURY, who edited the "significal dialogues" and also gave short descriptions of its history and its work [9].

[35]) D. VAN DANTZIG [5].

[36]) H. POINCARÉ [1] p. 10.

before. Therefore the paradox we wrote down above is not a contradiction: the second "same" is not the *same* "same" as the first one, at least for one who knows to "read between the lines".

Inasfar as progress of science consists of the discovery of new regularities of the *formal system*, the preceding formalization will be very useful, but it may be (even if one is willing to replace the old formalism by a new one) an impediment to the discovery of such new properties of the objects under investigation, which require finer distinctions ("fine structure") of relations hitherto regarded (and formalized!) as "identical" [37]).

It is to a large extent by such "finer distictions" and broader generalizations that progress of science proceeds, as numerous examples show. *After* they have been made, formalization may become useful again. Formalization therefore covers a small part of science only, in particular a part which to a certain extent is "ready" or "closed" at the moment, and therefore formalism is running *behind* actual science [38]).

If mathematics is not regarded as a formal tautology, no reason remains at all to claim any kind of "absoluteness" for it, either with regard to "certainty" or to "exactness". For, "recognizing A as B" may be called by other investigators or in a later stage of development: "not seeing the important difference between A and B", and "clearly distinguishing between A and B" is the same as "not yet having discovered the hidden resemblance between A and B". And, whether the processes involved are called "mental" ("splitting up a moment of life ...") or "physical" ("writing down a dash"), or both, and whether the mental processes are said to "accompany" the physical ones or vice-versa, is entirely irrelevant — if the distinction has any meaning at all! —. In any case, both formalists and intuitionists try to reduce mathematics to a system of actions which can be split up into a finite number of elements ("elementary steps"), for which the only relevant condition is that they can, practically almost without ambiguity, be recognized individually and distinguished from each other.

I have dwelled somewhat longer upon these generalities because I thereby hope to bridge the gulf between classical mathematics and logical empirism at one side and the apparently so distant shore of intuitionism and significs.

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[37]) The first case is implied in the second one, viz insofar as the observed objects are the signs of a calculus.

[38]) This, of course, does not deny the fact that, on the other hand, by the greater surveyability and the closer scrutinizing it allows, formalization often works as a usefull heuristic method for finding finer distinctions and new analogies.

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Mathematics. — *Bericht über die verschiedenen Methoden zur Lösung eines Systems linearer Gleichungen mit reellen Koeffizienten.* II. By E. BODEWIG. (Communicated by Prof. J. G. VAN DER CORPUT.)

(Communicated at the meeting of September 27, 1947.)

Die Methode von BOLTZ und BANACHIEWICZ.

BOLTZ (Lit. 2) hat seine Methode aus den Bedürfnissen der *Geodäten* heraus entwickelt, deren Gleichungen so beschaffen sind, dass nur die rechten Seiten eine Messung erfordern, während die Gleichungsmatrix ohne Messungen aufstellbar ist. Man kann daher, wenn man die Reziproke von $\mathfrak{A}$ kennt, den grössten Teil der Rechnung *vor* den Messungen erledigen. Ausserdem konnte er für die Reziproke von $\mathfrak{P}$ in diesem Falle von vorne herein Tabellen berechnen und damit die Arbeit bedeutend vereinfachen.

BOLTZ berechnet nämlich zunächst einen Teil der Unbekannten, etwa die m ersten und fügt dann sukzessive neue Unbekannte hinzu. Die Koeffizienten der rechten Seiten bleiben dabei unberücksichtigt. Schreiben wir daher das System in der Form:

$$\begin{aligned}
&p_{11}x_1 + \ldots + p_{1p}x_p + q_{11}y_1 + \ldots + q_{1,n-p}y_{n-p} = v_1\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&p_{p1}x_1 + \ldots + p_{pp}x_p + q_{p1}y_1 + \ldots + q_{p,n-p}y_{n-p} = v_p\\
&r_{11}x_1 + \ldots + r_{1p}x_p + s_{11}y_1 + \ldots + s_{1,n-p}y_{n-p} = w_1\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&r_{n-p,1}x_1 + \ldots + r_{n-p,p}x_p + s_{n-p,1}y_1 + \ldots + s_{n-p,n-p}y_{n-p} = w_{n-p}
\end{aligned}$$

oder in Matrizen:

$$\begin{pmatrix}\mathfrak{P} & \mathfrak{Q}\\ \mathfrak{R} & \mathfrak{S}\end{pmatrix}\begin{pmatrix}\mathfrak{x}\\ \mathfrak{y}\end{pmatrix} = \begin{pmatrix}\mathfrak{v}\\ \mathfrak{w}\end{pmatrix},\ \text{wo}\ \mathfrak{x} = \begin{pmatrix}x_1\\ \vdots\\ x_p\end{pmatrix},\ \mathfrak{y} = \begin{pmatrix}y_1\\ \vdots\\ y_{n-p}\end{pmatrix}$$

d.h.

$$\mathfrak{P}\mathfrak{x} + \mathfrak{Q}\mathfrak{y} = \mathfrak{v},\quad \mathfrak{R}\mathfrak{x} + \mathfrak{S}\mathfrak{y} = \mathfrak{w},\quad \text{also}$$
$$\mathfrak{x} = \mathfrak{P}^{-1}\mathfrak{v} - \mathfrak{P}^{-1}\mathfrak{Q}\mathfrak{y}.$$

Dies in die zweite Gleichung eingesetzt gibt

$$\mathfrak{R}\mathfrak{P}^{-1}\mathfrak{v} - \mathfrak{R}\mathfrak{P}^{-1}\mathfrak{Q}\mathfrak{y} + \mathfrak{S}\mathfrak{y} = \mathfrak{w} = \mathfrak{R}\mathfrak{P}^{-1}\mathfrak{v} + (\mathfrak{S} - \mathfrak{R}\mathfrak{P}^{-1}\mathfrak{Q})\mathfrak{y},\ \text{d.h.}$$
$$\mathfrak{y} = -\mathfrak{S}_1^{-1}\mathfrak{R}\mathfrak{P}^{-1}\mathfrak{v} + \mathfrak{S}_1^{-1}\mathfrak{w},\ \text{wo}\ \mathfrak{S}_1^{-1} = \mathfrak{S} - \mathfrak{R}\mathfrak{P}^{-1}\mathfrak{Q}.$$

Dies wiederum liefert

$$\mathfrak{x} = (\mathfrak{P}^{-1} + \mathfrak{P}^{-1}\mathfrak{Q}\mathfrak{S}_1^{-1}\mathfrak{R}\mathfrak{P}^{-1})\mathfrak{v} - \mathfrak{P}^{-1}\mathfrak{Q}\mathfrak{S}_1^{-1}\mathfrak{w}.$$

Da aber andrerseits

$$\begin{pmatrix}\mathfrak{x}\\ \mathfrak{y}\end{pmatrix} = \begin{pmatrix}\mathfrak{P} & \mathfrak{Q}\\ \mathfrak{R} & \mathfrak{S}\end{pmatrix}^{-1}\begin{pmatrix}\mathfrak{v}\\ \mathfrak{w}\end{pmatrix},$$

so folgt durch Vergleich mit dem vorhergehenden Wert von $\mathfrak{x}$ und $\mathfrak{y}$ die Relation (1).

BOLTZ hat also denselben Zusammenhang entdeckt wie SCHUR. Wenn ich also in einer früheren Arbeit (Lit. 1, p. 51) schrieb, dass erst BANACHIEWICZ 20 Jahre nach SCHUR dessen Relation neu entdeckt habe, so ist dies nicht ganz richtig, denn BOLTZ hat schon 1923, also 6 Jahre nach SCHUR, denselben Zusammenhang gefunden, ohne ihm freilich eine übersichtliche Matrizenform zu geben und ohne überhaupt zu wissen, worum es sich eigentlich handelt.

Ergebnis. Das BOLTZsche Verfahren ist sowohl im Prinzip wie hinsichtlich der auszuführenden Operationen identisch mit dem Verfahren, das die Relation von FROBENIUS–SCHUR benutzt.

Die Methode von JOSSA.

FRANCO JOSSA (Lit. 13) veröffentlichte ein Verfahren, welches die reziproke Matrix von $\mathfrak{A}$ sukzessive aus den Reziproken der ersten Hauptunterdeterminanten $\mathfrak{A}_1 = a_{11}, \mathfrak{A}_2, \mathfrak{A}_3, \ldots, \mathfrak{A}_{n-1}, \mathfrak{A}_n = \mathfrak{A}$ ersten, zweiten, ..., n-ten Grades berechnet. Dies ist genau der Gedankengang von BOLTZ mit der Beschränkung (die für die Rechnung unwesentlich ist), dass $\mathfrak{S}$ ein einzelnes Element ist. Die sich ergebenden Formeln lassen sich übersichtlicher somit in der Relation von SCHUR schreiben, ohne dass man dabei Summenzeichen nötig hat.

Die SCHMIDTsche Methode der Orthogonalisierung.

Ähnlich wie die GAUSSsche Methode darauf hinauskam, die Gleichungsmatrix in eine einfacher zu handhabende zu transformieren, so kann man auch bei den unbestimmten Methoden nach Transformationen suchen, welche die Matrix so verändern, dass die Reziproke der Transformierten sich leicht bilden lässt.

Die nächstliegende Methode wäre die Transformation von $\mathfrak{A}$ auf eine Diagonalmatrix. Darüber werden wir unten sprechen.

Eine zweite Matrizenart, deren Reziproke sich leicht bilden lässt, ist die *orthogonale Matrix* $\mathfrak{O}$, d.h. eine Matrix, für die $\mathfrak{O}'\mathfrak{O} = E$. Denn in diesem Falle ist die Reziproke nichts anderes als die Transponierte und lässt sich daher das System $\mathfrak{O}\mathfrak{x} = \mathfrak{w}$ sofort auflösen:

$$\mathfrak{x} = \mathfrak{O}^{-1}\mathfrak{w} = \mathfrak{O}'\mathfrak{w}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(II, 3)}$$

Bezeichnet daher $\mathfrak{s}_i$ die i-te Spalte von $\mathfrak{O}$, als Spalten-Vektor, d.h. als $(n, 1)$-Matrix geschrieben, so ist $x_i = \mathfrak{s}_i'\mathfrak{w}$.

Ähnlich nennen wir übrigens eine $(1, n)$-Matrix einen *Zeilenvektor*.

Die Methode von SCHMIDT besteht nun darin: die Matrix $\mathfrak{A}$ durch Multiplikation mit einer noch zu bestimmenden Matrix orthogonal zu machen. *Als Multiplikator genügt eine Dreiecksmatrix* $\mathfrak{D}_r$ *mit den Diagonalelementen* 1. $\mathfrak{D}_r$ ist also so zu wählen, dass

$$\mathfrak{D}_r\mathfrak{A} = \mathfrak{O} = \text{orthogonal}$$

wird. Wir lösen $\mathfrak{A}$ in seine n Zeilenvektoren $\mathfrak{z}_i$ auf. Die Elemente von $\mathfrak{T}_r$ seien r_{ik}. Dann sind die r_{ik} so zu wählen, dass die Zeilen der Produktmatrix

$$\begin{pmatrix} 1 & 0 & 0 & \cdot & \cdot & \cdot & 0 \\ r_{21} & 1 & 0 & \cdot & \cdot & \cdot & 0 \\ r_{31} & r_{32} & 1 & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ r_{n1} & r_{n2} & r_{n3} & \cdot & \cdot & \cdot & 1 \end{pmatrix} \begin{pmatrix} \mathfrak{z}_1 \\ \mathfrak{z}_2 \\ \mathfrak{z}_3 \\ \cdot \\ \mathfrak{z}_n \end{pmatrix} = \begin{pmatrix} \mathfrak{Z}_1 \\ \mathfrak{Z}_2 \\ \mathfrak{Z}_3 \\ \cdot \\ \mathfrak{Z}_n \end{pmatrix} \qquad \text{(II, 4)}$$

paarweise orthogonal sind. Dass eine solche Lösung möglich ist, d.h. dass man die r_{ik} so wählen kann, dass $\mathfrak{Z}_i \mathfrak{Z}'_k = 0$ für $i \neq k$, übersieht man sofort, denn für die Elemente $r_{i1}, r_{i2}, \ldots, r_{i,i-1}$ der i-ten Zeile hat man $i-1$ lineare Bedingungsgleichungen, wenn die vorhergehenden Zeilen alle bekannt sind. Es ist aber nicht zweckmässig, die r_{ik} auf diese Weise zu bestimmen. Vielmehr geht man folgendermassen vor.

Der Orthogonalisierungsprozess. Zunächst ist natürlich $\mathfrak{Z}_1 = \mathfrak{z}_1$. Dann soll $\mathfrak{Z}_2 = r_{21}\mathfrak{Z}_1 + \mathfrak{z}_2$ zu $\mathfrak{Z}_1$ orthogonal sein, d.h.

$$0 = \mathfrak{Z}_1 \mathfrak{Z}'_2 = r_{21}\mathfrak{Z}_1 \mathfrak{Z}'_1 + \mathfrak{Z}_1 \mathfrak{z}'_2.$$

Hieraus ergibt sich r_{21} durch Division: $r_{21} = -\mathfrak{Z}_1 \mathfrak{z}'_2 / \mathfrak{Z}_1 \mathfrak{Z}'_1$, denn Zähler und Nenner sind als Produkte einer $(1, n)$- mit einer $(n, 1)$-Matrix reine Zahlen, nämlich $(1, 1)$-Matrizen.

Nunmehr bestimmt man $\mathfrak{Z}_3$. Jedoch nicht nach der Definitionsgleichung als Produkt der dritten Zeile von $\mathfrak{T}_r$ mit der Matrix der $\mathfrak{z}_i$, also als lineare Kombination der Zeilen $\mathfrak{z}_i$, sondern vielmehr zunächst als lineare Kombination der beiden früheren Zeilen $\mathfrak{Z}_1$, $\mathfrak{Z}_2$ und $\mathfrak{z}_3$:

$$\mathfrak{Z}_3 = R_{31}\mathfrak{Z}_1 + R_{32}\mathfrak{Z}_2 + \mathfrak{z}_3,$$

woraus wir dann später die r_{31}, r_{32} berechnen. Die R_{3i} sind also so zu wählen, dass $\mathfrak{Z}_3$ sowohl zu $\mathfrak{Z}_1$ als $\mathfrak{Z}_2$ orthogonal ist, d.h. dass $\mathfrak{Z}_3 \mathfrak{Z}_1' = 0$ und $\mathfrak{Z}_3 \mathfrak{Z}_2' = 0$. Nun ist aber wegen $\mathfrak{Z}_1 \mathfrak{Z}_2' = \mathfrak{Z}_2 \mathfrak{Z}_1' = 0$:

$$0 = \mathfrak{Z}_3 \mathfrak{Z}'_1 = R_{31}\mathfrak{Z}_1 \mathfrak{Z}'_1 + \mathfrak{z}_3 \mathfrak{Z}'_1$$
$$0 = \mathfrak{Z}_3 \mathfrak{Z}'_2 = R_{32}\mathfrak{Z}_2 \mathfrak{Z}'_2 + \mathfrak{z}_3 \mathfrak{Z}'_2$$

woraus R_{31}, R_{32} folgen. Die r_{3i} bekommt man jetzt, indem man $\mathfrak{Z}_3$ als Matrizenprodukt und die Z als Kombinationen der z schreibt:

$$\mathfrak{Z}_3 = (R_{31}\ R_{32}\ 1)\begin{pmatrix} \mathfrak{Z}_1 \\ \mathfrak{Z}_2 \\ \mathfrak{z}_3 \end{pmatrix} = (R_{31}\ R_{32}\ 1)\begin{pmatrix} \mathfrak{z}_1 \\ R_{21}\mathfrak{z}_1 + \mathfrak{z}_2 \\ \mathfrak{z}_3 \end{pmatrix} =$$

$$= (R_{31}\ R_{32}\ 1)\left[\begin{pmatrix} 1 \\ R_{21} \\ 0 \end{pmatrix}\mathfrak{z}_1 + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\mathfrak{z}_2 + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\mathfrak{z}_3\right].$$

Die dritte Zeile der Matrix $\mathfrak{D}_r$ ist somit:

$$(r_{31}\ r_{32}) = (R_{31}\ R_{32})\begin{pmatrix} 1 & 0 \\ R_{21} & 1 \end{pmatrix}.$$

Darauf bestimmt man $\mathfrak{Z}_4$, und zwar zunächst als lineare Kombination von $\mathfrak{Z}_1$, $\mathfrak{Z}_2$, $\mathfrak{Z}_3$ und $\mathfrak{z}_4$:

$$\mathfrak{Z}_4 = R_{41}\,\mathfrak{Z}_1 + R_{42}\,\mathfrak{Z}_2 + R_{43}\,\mathfrak{Z}_3 + \mathfrak{z}_4,$$

derart dass dieser Zeilenvektor auf den vorhergehenden Zeilenvektoren $\mathfrak{Z}_1$, $\mathfrak{Z}_2$, $\mathfrak{Z}_3$ senkrecht steht, d.h. dass wegen $\mathfrak{Z}_h\,\mathfrak{Z}_i' = 0$ für $h < i < 4$:

$$\begin{aligned}
0 &= \mathfrak{Z}_4\,\mathfrak{Z}_1' = R_{41}\,\mathfrak{Z}_1\,\mathfrak{Z}_1' + \mathfrak{z}_4\,\mathfrak{Z}_1' \\
0 &= \mathfrak{Z}_4\,\mathfrak{Z}_2' = R_{42}\,\mathfrak{Z}_2\,\mathfrak{Z}_2' + \mathfrak{z}_4\,\mathfrak{Z}_2' \\
0 &= \mathfrak{Z}_4\,\mathfrak{Z}_3' = R_{43}\,\mathfrak{Z}_3\,\mathfrak{Z}_3' + \mathfrak{z}_4\,\mathfrak{Z}_3', \text{ kurz:} \\
0 &= \mathfrak{Z}_4\,\mathfrak{Z}_i' = R_{4i}\,\mathfrak{Z}_i\,\mathfrak{Z}_i' + \mathfrak{z}_4\,\mathfrak{Z}_i', \quad i = 1,\ 2,\ 3.
\end{aligned}$$

Aus jeder Gleichung ergibt sich dann ein Koeffizient R. Wiederum ist

$$\begin{aligned}
\mathfrak{Z}_4 = (R_{41}\,R_{42}\,R_{43}1)(\mathfrak{Z}_1\,\mathfrak{Z}_2\,\mathfrak{Z}_3\,\mathfrak{z}_4)' = (R_{41}\,R_{42}\,R_{43}1)(\mathfrak{z}_1,\ R_{21}\,\mathfrak{z}_1 + \mathfrak{z}_2, \\
R_{31}\,\mathfrak{z}_1 + R_{32}\,\mathfrak{z}_2 + \mathfrak{z}_3,\ \mathfrak{z}_4)',
\end{aligned}$$

also

$$(r_{41}\,r_{42}\,r_{43}) = (R_{41}\,R_{42}\,R_{43})\begin{pmatrix} 1 & 0 & 0 \\ R_{21} & 1 & 0 \\ R_{31} & R_{32} & 1 \end{pmatrix}.$$

So geht man weiter. Allgemein wählt man $\mathfrak{Z}_i$ so, dass es auf allen früheren $\mathfrak{Z}$ orthogonal ist. Die Koeffizienten R_{ih} von

$$\mathfrak{Z}_i = R_{i1}\,\mathfrak{Z}_1 + \ldots + R_{i,i-1}\,\mathfrak{Z}_{i-1} + \mathfrak{z}_i$$

folgen dann aus den Gleichungen

$$0 = \mathfrak{Z}_i\,\mathfrak{Z}_h' = R_{ih}\,\mathfrak{Z}_h\,\mathfrak{Z}_h' + \mathfrak{z}_i\,\mathfrak{Z}_h' \quad h = 1, 2, \ldots, i-1. \quad . \quad \text{(II, 5}$$

Daraus wieder ergibt sich die i-te Zeile von $\mathfrak{D}_r$ als

$$(r_{i1}\,r_{i2}\ldots r_{i,i-1}) = (R_{i1}\,R_{i2}\ldots R_{i,i-1})\begin{pmatrix} 1 & 0 & 0 & \ldots 0 \\ R_{21} & 1 & 0 & \ldots 0 \\ R_{31} & R_{32} & 1 & \ldots 0 \\ \ldots & \ldots & \ldots & \ldots \\ R_{i-1,1} & R_{i-1,2} & R_{i-1,3} & \ldots 1 \end{pmatrix} \quad \text{(II, 6)}$$

d.h. durch Multiplikation der i-ten Zeile der R mit der rechten Dreiecksmatrix der bisherigen R_{jk}.

Es ist somit stets

$$r_{i,i-1} = R_{i,i-1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(II, 7)}$$

Das Multiplikationsgesetz wird übersichtlicher, wenn wir nicht die ursprüngliche Dreiecksmatrix $\mathfrak{D}_r$ benutzen, sondern diejenige, die aus $\mathfrak{D}_r$ entsteht, wenn die Diagonalelemente alle gleich Null gesetzt werden und in der neuen Matrix $\mathfrak{D}_r - E$ die erste Zeile und die letzte Spalte gestrichen wird. Dasselbe tun wir mit der Matrix der R_{ik}. Die beiden neuen

Matrizen sind wiederum Dreiecksmatrizen mit den Diagonalelementen $r_{i,i-1}$ bzw. $R_{i,i-1}$, und es gilt

$$\left.\begin{array}{l}\begin{pmatrix} r_{21} & 0 & 0 & 0 & & 0 \\ r_{31} & r_{32} & 0 & 0 & & 0 \\ r_{41} & r_{42} & r_{43} & 0 & & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ r_{n1} & r_{n2} & r_{n3} & r_{n4} & \cdots & r_{n,n-1} \end{pmatrix} = \\ = \begin{pmatrix} R_{21} & 0 & 0 & \cdots & 0 \\ R_{31} & R_{32} & 0 & \cdots & 0 \\ R_{41} & R_{42} & R_{43} & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ R_{n1} & R_{n2} & R_{n3} & \cdots & R_{n,n-1} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & \cdots 0 \\ R_{21} & 1 & 0 & \cdots 0 \\ R_{31} & R_{32} & 1 & \cdots 0 \\ \cdot & \cdot & \cdot & \cdot \\ R_{n-1,1} & R_{n-1,2} & R_{n-1,3} & \cdots 1 \end{pmatrix}\end{array}\right\} \quad \text{(II, 8)}$$

Ausführung des Prozesses und der Lösung. Da die Zeilen $\mathfrak{Z}_i$ sukzessive gebildet werden, braucht man nur die Matrix der R_{ik} zu kennen und ist die Matrix der r_{ik} für unsern Zweck überflüssig. Natürlich muss der Kombinationsprozess, mittels dessen die $\mathfrak{Z}$ gebildet werden, auch auf die rechten Seiten v_i der Gleichungen ausgedehnt werden.

Hat man so ein Gleichungssystem $\mathfrak{M}\,\mathfrak{x} = \mathfrak{s}$ mit paarweise orthogonalen Zeilen bekommen, so multipliziere man beide Seiten mit $\mathfrak{M}'$: $\mathfrak{M}'\,\mathfrak{M}\,\mathfrak{x} = \mathfrak{M}'\,\mathfrak{s} = \mathfrak{t}$. Hier ist, zwar $\mathfrak{M}'\,\mathfrak{M}$ keine Einheitsmatrix, weil nämlich $\mathfrak{M}$ nur „semiorthogonal" ist, aber doch wenigstens eine Diagonalmatrix $\mathfrak{D}$. Wir haben somit

$$\mathfrak{D}\,\mathfrak{x} = \mathfrak{t}, \quad \text{(II, 9)}$$

wo die Bestimmung der x_i nur noch n Divisionen erfordert.

Anzahl der Operationen. Zur Berechnung von $\mathfrak{Z}_i$ hat man gemäss (5) $\mathfrak{Z}_i$ mit allen früheren $\mathfrak{Z}$ zu multiplizieren. Dadurch treten die Quadrate $\mathfrak{Z}_h\,\mathfrak{Z}_h'$ und die Produkte $\mathfrak{z}_i\,\mathfrak{Z}_h'$, wo $h < i$, auf. Die Quadrate $\mathfrak{Z}_h\,\mathfrak{Z}_h'$ mit $h < i-1$ kommen aber schon bei der Berechnung von $\mathfrak{Z}_{i-1}$ vor, so dass von den Quadraten nur $\mathfrak{Z}_{i-1}\,\mathfrak{Z}_{i-1}'$ neu ist. Zur Berechnung der $R_{i1}, R_{i2}, \ldots, R_{i,i-1}$ haben wir somit an neuen, bisher noch nicht ausgeführten Operationen i Produkte zweier Vektoren zu bilden und gemäss (5) $i-1$ Divisionen auszuführen. Das gibt

$$in + i - 1 \text{ neue Multiplikationen und } i(n-1) \text{ neue Additionen.}$$

Die Gesamtzahl der Operationen zur *Bildung der* R_{ik} beträgt demnach

$$\left.\begin{array}{l}(n+1)\sum_2^n i - \sum_2^n 1 = \tfrac{1}{2}\,n\,(n-1)\,(n+3) \text{ Multiplikationen und} \\ (n-1)\sum_2^n i = \tfrac{1}{2}\,(n-1)^2\,(n+2) = \tfrac{1}{2}\,(n^3 - 3n + 2) \text{ Additionen.}\end{array}\right\} \quad \text{(II, 10)}$$

Nunmehr müssen die $\mathfrak{Z}_i$ selber berechnet werden. Dazu haben wir so viele Multiplikationen eines Vektors vorzunehmen, wie es Koeffizienten $R_{ik} \neq 1$ gibt, nämlich $\sum_1^{n-1} i = \tfrac{1}{2}\,n(n-\,)$, also, da jeder Vektor n

Komponenten besitzt; innerhalb der Gleichungsmatrix:

$$\tfrac{1}{2}\, n^2 (n-1) \text{ Multiplikationen}$$

und auf den rechten Seiten $n(n-1)$ 2 Multiplikationen, im ganzen demnach

$$\tfrac{1}{2}\, n\,(n^2-1) \text{ Multiplikationen.} \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(II, 11)}$$

Was die Additionen angeht, so besteht $\mathfrak{Z}_i$ aus einer Summe von i Vektoren, erfordert daher, da jede Komponente $i-1$ Additionen kostet: $n(i-1)$ Additionen. Die Gesamtheit aller $\mathfrak{Z}_i$ erfordert also

$$n \sum_{1}^{n} (i-1) = \tfrac{1}{2}\, n^2 (n-1) \text{ Additionen.} \quad . \quad . \quad . \quad . \quad \text{(II, 12)}$$

Hinzu kommen die Additionen auf den rechten Seiten: $\sum_{1}^{n} (i-1)$, im ganzen somit

$$\tfrac{1}{2}\, n\,(n^2-1) \text{ Additionen.} \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(II, 13)}$$

Damit liegt das neue Gleichungssystem $\mathfrak{M}\,\mathfrak{x} = \mathfrak{s}$ fest. Es ist semi-orthogonal. Gemäss Obigem haben wir $\mathfrak{M}'\,\mathfrak{M}\,\mathfrak{x} = \mathfrak{M}'\,\mathfrak{s}$ zu bilden. Die Diagonalmatrix $\mathfrak{M}'\,\mathfrak{M} = \mathfrak{T}$ ist aber schon bestimmt, denn ihre Elemente sind gleich den Quadraten der $\mathfrak{Z}_i$. Hingegen erfordert $\mathfrak{M}'\,\mathfrak{s}$ noch n^2 Multiplikationen und $n^2 - n$ Additionen. Schliesslich kommen noch die n Divisionen durch die Diagonalelemente von $\mathfrak{T}$ hinzu. Damit ist *das System aufgelöst.*

Wollte man noch die Methode benutzen, um die *Reziproke von* $\mathfrak{A}$ zu bestimmen, so müsste man zunächst die Matrix der r_{ik} bestimmen aus der Matrix der R_{ik}. Dazu hat man in (5) die beiden rechtsstehenden Matrizen miteinander zu multiplizieren. In der i-ten Zeile erfordert

$$r_{i1} = R_{i1} + R_{i2} R_{21} + \ldots + R_{i,i-1} R_{i-1,1}$$

nur $i-2$ Multiplikationen und ebenso viele Additionen. r_{i2} kostet je eine Operation weniger. Usw. Insgesamt erfordert die i-te Zeile der Matrix $\mathfrak{Z}_r$ der r_{ik}:

$$\sum_{1}^{i-2} k = \tfrac{1}{2}\,(i-1)\,(i-2) \text{ Multiplikationen und Additionen,}$$

daher die Bestimmung aller r_{ik} aus den R_{ik}:

$$\tfrac{1}{6}\, n\,(n-1)(n-2) \text{ Multiplikationen und Additionen.} \quad . \quad \text{(II, 14)}$$

Ferner war $\mathfrak{M} = \mathfrak{Z}_r\,\mathfrak{A}$, was $\tfrac{1}{2}n^2(n-1)$ Multiplikationen und ebenso viele Additionen kostet. Sodann ist $\mathfrak{M}'\,\mathfrak{M} = \mathfrak{T}$, also $\mathfrak{M}^{-1} = \mathfrak{T}^{-1}\,\mathfrak{M}'$, was nochmals n^2 Multiplikationen erfordert. Schliesslich kostet die Berechnung von $\mathfrak{A}^{-1} = \mathfrak{M}^{-1}\,\mathfrak{Z}_r$ nochmals $\tfrac{1}{2}n^2(n-1)$ Multiplikationen und Additionen. Addieren wir die Anzahlen in (13, 14) usw., so bekommen wir unten stehende Zahlen.

Somit haben wir das

Ergebnis. 1. Die Orthogonalisierung einer Matrix nach SCHMIDT erfordert

$\frac{1}{2}n(n-1)(2n+3)$ Multipl. und $\frac{1}{2}(n-1)(2n^2+n-2)$ Additionen

2. Die Berechnung der Inversen einer Matrix mittels Orthogonalisierung ist ein Umweg. Denn sie kostet

$\frac{1}{6}n(n+1)(10n-7)$ Multipl. und $\frac{1}{6}(10n^3-9n^2-7n+6)$ Additionen,

also noch weit mehr als die Relation von SCHUR.

3. Die Lösung eines Gleichungssystems mittels Orthogonalisierung erfordert

n^3+2n^2-2n Multipl. und n^3+n^2-3n+1 Additionen,

also im allgemeinen Falle dreimal und bei symmetrischem Gleichungssystem 6-mal so viel wie die Methode von GAUSS.

4. Die weitgehend verbreitete Meinung, die SCHMIDTsche Orthogonalisierung sei die vorteilhafteste Methode zur Lösung eines Systems von linearen Gleichungen, ist somit falsch.

5. Die Methode hat allerdings den Vorteil der leichten Kontrolle. Dieser wiegt aber die grossen Nachteile der Methode bei weitem nicht auf.

Methode der Transformation auf Diagonalmatrix.

Ein anderer Weg zur Berechnung von $\mathfrak{A}^{-1}$, der hier zum ersten Male beschritten wird, ergibt sich aus der GAUSSschen Methode. Ihr Kern bestand, wie wir sahen, darin, eine rechte Dreiecksmatrix $\mathfrak{D}_r$ mit den Diagonalelementen 1 zu finden, so dass das Produkt mit $\mathfrak{A}$ eine linke Dreiecksmatrix wird:

$$\overline{\mathfrak{D}}_r \mathfrak{A} = \mathfrak{D}_l.$$

Die (linke) Multiplikation mit einer gewissen, eindeutig bestimmten rechten Dreiecksmatrix hat also die Eigenschaft, dass die Elemente links von der Diagonale verschwinden.

Ebenso kann man aber auch eindeutig eine linke Dreiecksmatrix finden, so dass ihre (linke) Multiplikation die Elemente rechts von der Diagonalen zum Verschwinden bringt. Auf $\mathfrak{D}_l$ angewandt heisst dies: Es lässt sich eine Matrix $\overline{\mathfrak{D}}_l$ mit den Diagonalelementen 1 finden, so dass

$$\overline{\mathfrak{D}}_l \mathfrak{D}_l = \mathfrak{D} = \text{Diagonalmatrix}$$

wird. Dann wird also $\overline{\mathfrak{D}}_l \mathfrak{D}_l \mathfrak{A} = \mathfrak{D}$, und daraus

$$\mathfrak{A}^{-1} = \mathfrak{D}^{-1} \overline{\mathfrak{D}}_l \overline{\mathfrak{D}}_r \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{II}, 15)$$

$\mathfrak{D}^{-1}$ lässt sich aber ohne weiteres angeben, denn seine Elemente sind die

Reziproken deren von $\mathfrak{D}$. Anderseits erhält man $\mathfrak{D}_l$ automatisch durch den GAUSSschen Prozess und $\mathfrak{D}_r$ gemäss (I, 9a) als Produkt gewisser Matrizen. Es handelt sich demnach noch um die Bestimmung von $\overline{\mathfrak{D}}_l$ mit den Elementen $\bar{l}_{ik}$, derart dass

$$\begin{pmatrix} 1 & \bar{l}_{12} & \bar{l}_{13} & \dots & \bar{l}_{1n} \\ 0 & 1 & \bar{l}_{23} & \dots & \bar{l}_{2n} \\ 0 & 0 & 1 & \dots & \bar{l}_{3n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix} \begin{pmatrix} l_{11} & l_{12} & l_{13} & \dots & l_{1n} \\ 0 & l_{22} & l_{23} & \dots & l_{2n} \\ 0 & 0 & l_{33} & \dots & l_{3n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \dots & l_{nn} \end{pmatrix} = \mathfrak{D} = \text{diagonal.}$$

Die Diagonalelemente von $\mathfrak{D}$ sind, wie man sofort sieht: $d_{ii} = l_{ii}$.

Die Berechnung von $\mathfrak{D}_l$. Die Matrix $\overline{\mathfrak{D}}_l$ kann man nun offenbar ebenso als Produkt darstellen wie oben die Matrix $\overline{\mathfrak{D}}_r$, welche $\mathfrak{A}$ zu einer linken Dreiecksmatrix machte. Um nämlich die letzte Spalte der Matrix $\mathfrak{D}_l$ der l_{ik} ausserhalb der Hauptdiagonalen zum Verschwinden zu bringen, hat man von der i-ten Zeile die mit l_{in}/l_{nn} multiplizierte letzte Zeile zu subtrahieren, wodurch alle übrigen Elemente jeder Zeile unverändert bleiben. Dieser Prozess ist aber nichts anderes als die linke Multiplikation von $\mathfrak{D}_l$ mit der Matrix $\mathfrak{D}_l^{(n)}$: $\mathfrak{D}_l^{(n)} \mathfrak{D}_l$, wo

$$\mathfrak{D}_l^{(n)} = \begin{pmatrix} 1 & 0 & 0 & \dots & -l_{1n}/l_{nn} \\ 0 & 1 & 0 & \dots & -l_{2n}/l_{nn} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \dots & -l_{n-1,n}/l_{nn} \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}$$

und es ist

$$\mathfrak{D}_l^{(n)} \mathfrak{D}_l = \begin{pmatrix} l_{11} & l_{12} & l_{13} & \dots & l_{1,n-1} & 0 \\ 0 & l_{22} & l_{23} & \dots & l_{2,n-1} & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \dots & l_{n-1,n-1} & 0 \\ 0 & 0 & 0 & \dots & 0 & l_{nn} \end{pmatrix}.$$

Diese Matrix behandelt man ganz ähnlich, d.h. man hat von der i-ten Zeile die mit $l_{i,n-1}/l_{n-1,n-1}$ multiplizierte vorletzte Zeile zu subtrahieren, was darauf hinauskommt, dass man sie links multipliziert mit einer Matrix $\mathfrak{D}_l^{(n-1)}$, welche aus der Einheitsmatrix entsteht, wenn man die vorletzte Spalte oberhalb der Diagonalen ersetzt durch

$$-l_{1,n-1}/l_{n-1,n-1}, \quad -l_{2,n-1}/l_{n-1,n-1}, \dots, \quad -l_{n-2,n-1}/l_{n-1,n-1}.$$

So geht man weiter und bekommt:

$$\overline{\mathfrak{D}}_l = \mathfrak{D}_l^{(2)} \dots \mathfrak{D}_l^{(n-1)} \mathfrak{D}_l^{(n)},$$

wo $\mathfrak{D}_l^{(i)}$ aus der Einheitsmatrix entsteht, wenn man die i-te Spalte e_i oberhalb der Diagonalen ersetzt durch

$$-l_{1i}/l_{ii}, -l_{2i}/l_{ii}, \dots, -l_{i-1,i}/l_{ii}.$$

Insgesamt wird nach Ausführung aller Multiplikationen

$$\overline{\mathfrak{D}}_l \mathfrak{D}_l = \begin{pmatrix} l_{11} & 0 & 0 & \dots & 0 \\ 0 & l_{22} & 0 & \dots & 0 \\ 0 & 0 & l_{33} & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \dots & l_{nn} \end{pmatrix}.$$

Anzahl der Operationen. Wollen wir jetzt nach (II, 15) $\mathfrak{A}^{-1}$ bestimmen, so haben wir zunächst $\overline{\mathfrak{D}}_r$ und $\overline{\mathfrak{D}}_l$ zu berechnen. Nun wurden durch den GAUSSschen Prozess die Elemente von $\mathfrak{D}_l$ automatisch geliefert und ebenso die Elemente der Matrizen $\overline{\mathfrak{D}}_r^{(i)}$. Man muss aber noch das Produkt $\overline{\mathfrak{D}}_r$ aller $\overline{\mathfrak{D}}_r^{(i)}$ bestimmen und die Zahl der dafür nötigen Operationen angeben. Bei $\overline{\mathfrak{D}}_l^{(i)}$ hingegen waren anders als bei den $\overline{\mathfrak{D}}_r^{(i)}$ die Elemente von vorne herein bekannt, und wir brauchen daher nur ihr Produkt $\overline{\mathfrak{D}}_l$ und die Anzahl der erforderlichen Operationen zu berechnen.

Zusammengefasst haben wir demnach noch zu bestimmen: *a*) das Produkt der $\overline{\mathfrak{D}}_r^{(i)}$; *b*) das Produkt der $\overline{\mathfrak{D}}_l^{(i)}$; *c*) das Produkt $\overline{\mathfrak{D}}_l \overline{\mathfrak{D}}_r$; *d*) das Produkt von $\mathfrak{D}^{-1}$ mit letzterem Produkt. Die hierzu nötigen Operationen werden nun abgezählt.

a) Zum Produkt aller $\overline{\mathfrak{D}}_r^{(i)}$ haben wir zunächst $\overline{\mathfrak{D}}_r^{(2)} \overline{\mathfrak{D}}_r^{(1)}$ zu bestimmen. Nun lässt $\overline{\mathfrak{D}}_r^{(2)}$ die zwei ersten Zeilen von $\overline{\mathfrak{D}}_r^{(1)}$ unverändert. In jeder Zeile von $\overline{\mathfrak{D}}_r^{(i)}$ von der $(i+1)$-ten ab kommt aber nur ein einziges von 1 verschiedenes Element vor, so dass jede solche Zeile für jede Spalte der vorhergehenden Matrix höchstens eine Multiplikation und eine Addition erfordert. Beim Produkt von $\overline{\mathfrak{D}}_r^{(2)}$ mit $\overline{\mathfrak{D}}_r^{(1)}$ haben wir nur in der ersten Spalte des Produktes und von der dritten Zeile ab wesentliche Rechenoperationen auszuführen, im ganzen also $n-2$ Multiplikationen und ebenso viele Additionen.

Die folgende Matrix $\overline{\mathfrak{D}}_r^{(3)}$ lässt die drei ersten Zeilen des vorhergehenden Produktes unverändert. Jedes tiefere Element der ersten Spalte erfordert je eine Multiplikation und eine Addition. Dasselbe gilt für die zweite Spalte. Die übrigen Spalten kann man jedoch ohne Rechnung bilden. Wir haben somit $2(n-3)$ Multiplikationen und ebenso viele Additionen.

Danach liegen die ersten vier Zeilen von $\mathfrak{D}_r$ fest, da $\overline{\mathfrak{D}}_r^{(4)}$ erst von der fünften Zeile an wirkt. Es kostet demnach $3(n-4)$ Multiplikationen und Additionen. Usw.

Insgesamt erfordert das Produkt aller $\overline{\mathfrak{D}}_r^{(i)}$ an Multiplikationen und Additionen:

$$1\,(n-2) + 2\,(n-3) + 3\,(n-4) + \dots + (n-2)\,1, \text{ also}$$

$$\tfrac{1}{6}\,n\,(n-1)\,(n-2) \text{ Multipl. und ebenso viele Additionen.} \quad \text{(II, 15)}$$

b) Dieselbe Anzahl ist nötig um das Produkt aller $\mathfrak{D}_l^{(i)}$ zu finden.

c) Nunmehr müssen die beiden Dreiecksmatrizen $\overline{\mathfrak{D}}_l$ und $\overline{\mathfrak{D}}_r$ mit den Diagonalelementen 1 multipliziert werden. Das Element p_{11} des Produktes besteht nun formell aus einer Summe von n Produkten, so dass formell n Mult. und n—1 Additionen notwendig sind. Von den n Produkten haben aber zwei den Faktor 1, so dass in Wirklichkeit nur n—2 Produkte zu bilden sind. Ähnlich erfordert das Element p_{12}: n—3 Mult. und n—2 Additionen usw. Das Element $p_{1,n-1}$ erfordert noch eine Addition, jedoch keine Multiplikation mehr. Somit kostet die erste Zeile: des Produktes:

$(n-2) + (n-3) + \ldots + 1 + 0$ Multiplikationen und
$(n-1) + (n-2) + \ldots + 2 + 1$ Additionen.

Ähnlich erfordert die zweite Zeile der Produktmatrix:

$(n-3) + (n-4) + \ldots + 1 + 0$ Multiplikationen und
$(n-2) + (n-3) + \ldots + 2 + 1$ Additionen. Usw.

Das Produkt $\overline{\mathfrak{D}}_l \mathfrak{D}_r$ kostet somit insgesamt wieder $\frac{1}{6}n(n-1)(n-2)$ Multipl. und $\frac{1}{6}n(n+1)(n-1)$ Additionen.

d) Die Multiplikation mit einer Diagonalmatrix kostet schliesslich noch n^2 Multiplikationen, aber keine Additionen mehr.

Rechnen wir zu a, b, c, d hinzu die Operationen, welche $\mathfrak{A}$ zu einer Dreiecksmatrix machen, nämlich

$$\sum_1^n (n^2-n) = \tfrac{1}{3}n(n^2-1) \text{ Multiplikationen und}$$

$$\sum_1^n (n-1)^2 = \tfrac{1}{6}n(n-1)(2n-1) \text{ Additionen.}$$

Insgesamt haben wir das

Ergebnis. 1. Man kann die Reziproke $\mathfrak{A}^{-1}$ berechnen, indem man auf $\mathfrak{A}$ das GAUSSsche Verfahren anwendet, wodurch $\mathfrak{A}$ übergeht in $\mathfrak{D}_l$, daraufhin $\mathfrak{D}_l$ wieder mittels des GAUSSschen Verfahrens in eine Diagonalmatrix $\mathfrak{D}$ überführt, ferner die Matrix $\overline{\mathfrak{D}}_r$ berechnet und schliesslich das Produkt $\mathfrak{D}^{-1}\overline{\mathfrak{D}}_l\overline{\mathfrak{D}}_r = \mathfrak{A}^{-1}$ bestimmt.

2. Die Zahl der hierzu nötigen Operationen beträgt

$\frac{5}{6}n^3 - \frac{1}{2}n^2 + \frac{2}{3}n$ Multiplikationen und
$\frac{5}{6}n^3 - \frac{3}{2}n^2 + \frac{2}{3}n$ Additionen (II, 16)

3. Dies ist nicht nur viel weniger als nach dem Verfahren von SCHMIDT, sondern sogar weniger als nach der Relation von SCHUR.

4. Wenn man also schon orthogonalisiert, so ist die völlige Orthogonalisierung mit Benutzung des GAUSSschen Ansatzes merkwürdigerweise ökonomischer als die „teilweise" Orthogonalisierung nach SCHMIDT.

Dies führt uns zu den Schlussbemerkungen unseres Vorwortes zurück. Die Methode, die GAUSS zur Auflösung seiner Normalgleichungen empfahl,

war nicht neu und war GAUSS natürlich ebenso bekannt wie jedem andern. Jedenfalls wäre es wohl GAUSS ein Leichtes gewesen, eine andere Methode zu ersinnen, etwa die iterative von SEIDEL. Wenn er trotzdem die alte einfache Methode anführte, so hatte er vorher sicherlich alle möglichen Lösungsmethoden überdacht und war zu dem Ergebnis gekommen, dass sie die beste sei. Nur liess er bei der Veröffentlichung die Begründung weg. Dies lag ja auch ganz in der Linie von GAUSS' Charakter. Jedenfalls kann man sicher sein, dass er ganz genau wusste, was er tat, als er seine Gleichungen nicht anders aufzulösen wusste als nach jener alten Methode, und ist es verfehlt, an seiner Methode so einfache Kritik üben zu wollen.

III. Iterationsmethoden. Bestimmte Lösungen.

Der Bericht von MISES und POLLACZEK–GEIRINGER.

Als Verfechter iterativer Methoden haben sich vor allem MISES und POLLACZEK–GEIRINGER in ihrem bekannten Bericht vom Jahre 1929 bekannt. Wir geben ihren Gedankengang im Folgenden kurz wieder, bedienen uns aber dabei etwas anderer Bezeichnungen, wo diese den Gedankengang vereinfachen.

Das Prinzip der Methode.

Das System $\mathfrak{A}\,\mathfrak{x} = \mathfrak{r}$ habe die wahre Lösung $\mathfrak{x}$, die unbekannt ist. Bekannt ist nur ein Näherungsvektor $\mathfrak{x}'$, der das Gleichungssystem also nicht genau erfüllt, sondern einen Fehlervektor

$$\mathfrak{L}(\mathfrak{x}') = \mathfrak{A}\,\mathfrak{x}' - \mathfrak{r} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(III, 1)}$$

liefert. Mittels eines iterativen Verfahrens soll nunmehr aus $\mathfrak{x}'$ und $\mathfrak{L}(\mathfrak{x}')$ ein besserer Näherungswert $\mathfrak{x}''$ gefunden werden, so dass also

$$|\mathfrak{x}'' - \mathfrak{x}| < |\mathfrak{x}' - \mathfrak{x}|$$

Das sucht man auf die Weise zu erreichen, dass mittels der Zahlen $d_1, \ldots, d_n$:

$$x_i'' = x_i' + d_i L_i(\mathfrak{x}') \quad i = 1, \ldots, n$$

gesetzt wird, also in Matrizen:

$$\mathfrak{x}'' = \mathfrak{x}' + \mathfrak{D}\,\mathfrak{L}(\mathfrak{x}'),$$

wo $\mathfrak{D} = \mathfrak{D}(d_1, \ldots, d_n)$ eine Diagonalmatrix ist.

Allgemein muss die Folge $\mathfrak{x}', \mathfrak{x}'', \mathfrak{x}''', \ldots \to \mathfrak{x}$ konvergieren, wobei

$$\mathfrak{x}^{(k+1)} = \mathfrak{x}^{(k)} + \mathfrak{D}\,\mathfrak{L}(\mathfrak{x}^{(k)}) \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(III, 2)}$$

oder, was auf dasselbe hinauskommt, die Folge der Fehlervektoren

$$\left.\begin{array}{l} \mathfrak{z}', \mathfrak{z}'', \mathfrak{z}''', \ldots \to \mathfrak{o} \text{ gehen, wobei } \mathfrak{z}^{(k)} = \mathfrak{x}^{(k)} - \mathfrak{x} \text{ und} \\ \mathfrak{z}^{(k+1)} = \mathfrak{z}^{(k)} + \mathfrak{D}\,\mathfrak{A}\,\mathfrak{z}^{(k)} = (E + \mathfrak{D}\,\mathfrak{A})\,\mathfrak{z}^{(k)} = (E + \mathfrak{D}\,\mathfrak{A})^k\,\mathfrak{z}'. \end{array}\right\} \quad \text{(III, 2a)}$$

Die *Frage* ist: wie und unter welchen Bedingungen solche Zahlen d_i gefunden werden können, dass jene Folgen konvergieren.

Hinreichende Bedingungen. Wenn

$$\left.\begin{array}{l} |1+d_1 a_{11}|+|d_2 a_{21}|+\ldots+|d_n a_{n1}| \leqq m<1 \\ |d_1 a_{12}|+|1+d_2 a_{22}|+\ldots+|d_n a_{n2}| \leqq m<1 \\ \quad\ldots\ldots\ldots\ldots\ldots\ldots \\ |d_1 a_{1n}|+|d_2 a_{2n}|+\ldots+|1+d_n a_{nn}| \leqq m<1, \end{array}\right\} \quad \text{(III, 3)}$$

so erhält man leicht, dass das Verfahren konvergiert, weil

$$|z_1^{(k+1)}|+\ldots+|z_n^{(k+1)}| \leqq m\,(|z_1^{(k)}|+\ldots+|z_n^{(k)}|) \leqq m^k\,(|z_1'|+\ldots+|z_n'|)$$

also gegen 0 geht, wenn $k \to \infty$.

Nun ist es aber praktisch meist unmöglich, obiges System nach den d_i aufzulösen. Man geht daher umgekehrt vor: Man erteilt den d_i bestimmte Werte und sucht dann Bedingungen für die a_{ik}, so dass (3) erfüllt ist.

Die *einfachste Annahme der* d_i besteht darin, die Diagonalelemente der Matrix $E+\mathfrak{D}\mathfrak{A}$ in (2a) zum Verschwinden zu bringen, d.h.

$$d_i = -1/a_{ii} \quad \ldots\ldots\ldots \quad \text{(III, 4)}$$

zu setzen. Dann geht (3) über in die Bedingungen:

$$\sum_{i \neq j} \left|\frac{a_{ij}}{a_{ii}}\right| \leqq m<1 \text{ für alle } j \text{ von 1 bis } n \quad \ldots \quad \text{(III, 5)}$$

Die Iteration (2) wird dann zu

$$a_{ii}\, x_i^{(k+1)} = r_i - \sum_{j \neq i} a_{ij}\, x_j^{(k)}, \quad \ldots\ldots \quad \text{(III, 6)}$$

d.h. *die i-te Gleichung des Systems* $\mathfrak{A}\mathfrak{x}=\mathfrak{r}$ *wird nach* x_i *aufgelöst.* Und Bedingung (5) sagt, dass *in jeder Spalte von* $\mathfrak{A}$ *die Diagonalkoeffizienten „überwiegen" müssen.* Später werden wir diese Bedingung anders formulieren, wodurch erst ihr Kern sichtbar wird.

Bringt man das System $\mathfrak{A}\mathfrak{x}=\mathfrak{r}$ durch Dividieren jeder Zeile durch ihren Diagonalkoeffizienten a_{ii} oder, was dasselbe ist, durch Multiplikation mit der Diagonalmatrix $\mathfrak{a}$ mit den Elementen $1/a_{ii}$ auf ein System mit den Diagonalkoeffizienten 1:

$$\mathfrak{a}\mathfrak{A}\mathfrak{x} = \mathfrak{a}\mathfrak{r} \text{ oder } \mathfrak{B}\mathfrak{x} = \mathfrak{s} \text{ mit } b_{ii}=1,$$

so geht (4) über in

$$d_i = -1, \text{ also } \mathfrak{D} = -E. \quad \ldots\ldots \quad \text{(III, 4')}$$

und (III, 5) und (III, 6) werden

$$\sum_{i \neq j} |b_{ij}| < 1 \text{ für alle } j \text{ von 1 bis } n \quad \ldots \quad \text{(III, 5')}$$

und

$$x_i^{(k+1)} = s_i - \sum_{j \neq i} b_{ij}\, x_j^{(k)}. \quad \ldots\ldots \quad \text{(III, 6')}$$

Neben (5 und 6) bzw. (5' und 6') gibt es noch eine *zweite Konvergenzbedingung,* die im Kern auf E. SCHMIDT zurückgeht. Wir nehmen der

Einfachheit halber das System $\mathfrak{B}\,\mathfrak{x} = \mathfrak{s}$, wo also von jetzt an stets $b_{ii} = 1$. Wir wählen wieder $\mathfrak{D} = -E$. Dann wird (2a), zu

$$z^{(k+1)} = \mathfrak{M}\,\mathfrak{z}^{(k)} = \mathfrak{M}^k\,\mathfrak{z}', \text{ wo } \mathfrak{M} = E - \mathfrak{B}.$$

SCHMIDT konnte nun zeigen, dass $\mathfrak{M}^k$ auf jeden Fall gegen die Nullmatrix geht, wenn

$$C = \sum_i \sum_j m_{ij}^2 < 1, \text{ wo } m_{ij} \text{ die Elemente von } \mathfrak{M} \text{ sind.}$$

Zusammengefasst haben wir somit:

Satz. $\mathfrak{x}$ sei der unbekannte Lösungsvektor des Gleichungssystems

$$\mathfrak{L}(\mathfrak{x}) \equiv \mathfrak{B}\,\mathfrak{x} - \mathfrak{s} = 0, \text{ wo } b_{ii} = 1.$$

Damit dann bei beliebigem $\mathfrak{x}'$ die Vektorenfolge $\mathfrak{x}', \mathfrak{x}'', \mathfrak{x}''', \ldots$ wo

$$\mathfrak{x}^{(k+1)} = \mathfrak{x}^{(k)} - \mathfrak{L}\,(\mathfrak{x}^{(k)},$$

gegen $\mathfrak{x}$ konvergiert, ist hinreichend, dass *entweder* in $\mathfrak{B}$ die Diagonalelemente $b_{ii} = 1$ „überwiegen", worunter wir verstehen, dass in jeder Spalte von $\mathfrak{B}$ das Diagonalelement grösser ist als die Summe der Beträge aller übrigen Elemente, also

$$\sum_{i=1}^{n} |b_{ij}| < 2 \text{ für alle } j \text{ von 1 bis } n \ldots \quad \text{(III, 7)}$$

oder dass die Quadratsumme aller Elemente von $\mathfrak{B}$ kleiner als $n + 1$ ist:

$$\sum_i \sum_j b_{ij}^2 < n + 1 \ldots \quad \text{(III, 8)}$$

Die Bedingungen (7 und 8) überschneiden sich, denn beide sind verschiedene Abschwächungen der wahren, aber unbekannten Konvergenzbedingung. Es kann also sein, dass die eine der beiden hinreichenden Bedingungen die Konvergenz garantiert, während die andere dies nicht tut.

Der Fall von MISES. MISES selbst hat noch den Fall $\mathfrak{D} = c\,E$, wo c eine Zahl ist, betrachtet. (2a) geht dann über in

$$\mathfrak{z}^{(k+1)} = (E + c\,\mathfrak{A})^k\,\mathfrak{z}' \equiv \mathfrak{N}^k\,\mathfrak{z}'.$$

Damit nun $\mathfrak{N}^k$ gegen die Nullmatrix geht, ist bekanntlich notwendig und hinreichend, dass die Wurzeln von $\mathfrak{N}$ absolut kleiner als 1 sind. Dies gibt für die Wurzeln λ_i von $\mathfrak{A}$ die Bedingungen

$$\left|\lambda_i + \frac{1}{c}\right| < \left|\frac{1}{c}\right| \text{ für } i \text{ von 1 bis } n \ldots \quad \text{(III, 9)}$$

Für *positiv oder negativ definite* $\mathfrak{A}$ *ist diese Bedingung erfüllt*, also auch für alle Normalgleichungen, wie sie bei Ausgleichungsrechnungen auftreten. Denn für sie haben alle Wurzeln λ_i dasselbe (positive bzw. negative) Zeichen und sind reell. Man kann daher einen Kreis angeben mit dem Mittelpunkt $(p, 0)$, der die imaginäre Achse berührt und alle Wurzeln von $\mathfrak{A}$ enthält. Dann ist (9) erfüllt für $c = -1/p$.

Mathematics. — *Représentation d'éléments* $\mathfrak{p}$*-adiques et* $\mathfrak{q}$*-adiques.* By F. LOONSTRA. (Communicated by Prof. J. G. VAN DER CORPUT.)

(Communicated at the meeting of September 27, 1947.)

Introduction.

L'évaluation p-adique des nombres rationnels permet une généralisation: au moyen de certains idéaux premiers $\mathfrak{p}$ on peut valuer les éléments de corps qui seront définis plus tard, de la même manière que les nombres rationnels d'un nombre premier p. Quand on procède à l'extension complète minimale au moyen de cette évaluation, il résulte qu'on peut représenter les éléments de cette extension d'une manière analogue à celle des nombres p-adiques par des suites de classes d'équivalence ou par des séries infinies. Au cas spécial d'un idéal premier principal $\mathfrak{p} = (p)$ l'analogie est plus grande.

A cause de l'intérêt pour la théorie des corps fonctionnels et des corps de nombres nous basons toute la manipulation sur la théorie d'idéaux „classique" de DEDEKIND. Nous nous en servirons spécialement dans la seconde partie: on étend un anneau O à une extension complète minimale au moyen d'un idéal $\mathfrak{q}$. Il se trouve que cette extension complète minimale est la somme directe des extensions complètes minimales qu'on obtient à l'aide des facteurs d'idéaux premiers $\mathfrak{p}_i$ de $\mathfrak{q}$.

§ 1. *Représentation des éléments* $\mathfrak{p}$*-adiques.*

Nous supposons qu'O soit un anneau d'intégrité (anneau commutatif sans diviseur de zéro) avec élément d'unité, lorsque les trois conditions suivantes seront remplies:

I. Toute chaîne d'idéaux $\mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{a}_3, \ldots$ de O avec $\mathfrak{a}_i \subset \mathfrak{a}_{i+1}$ se termine après un nombre fini;

II. Tous les idéaux premiers différents de O sont des idéaux maximaux;

III. O est fermé totalement dans le corps des quotients K de O.

Alors tout idéal $\mathfrak{q}$ de O est d'une manière seulement un produit d'idéaux premiers-la succession et les facteurs O exceptés; en ce cas les idéaux primairs sont des puissances des idéaux premiers.

Supposons $\mathfrak{p} \neq O$ un idéal premier de O, on a:

A. Toutes les puissances $\mathfrak{p}, \mathfrak{p}^2, \ldots$ diffèrent et n'ont de commun que l'élément 0. Nous supposons en outre: B. Soit $a \subset O$ divisible précisément par $\mathfrak{p}^\alpha$, c'est à dire par $\mathfrak{p}^\alpha$, mais pas par $\mathfrak{p}^{\alpha+1}$, b précisément par $\mathfrak{p}^\beta$, alors $a \cdot b$ divise précisément $\mathfrak{p}^{\alpha+\beta}$; $\mathfrak{p}^\alpha$ est l'ensemble des sommes $\Sigma\, p_{\mu_1} p_{\mu_2} \ldots p_{\mu_\alpha}$.

Maintenant on définit, si a divise précisément $\mathfrak{p}^\alpha$, $\varphi(a) = e^{-\alpha}$ avec e un nombre réel < 1, et $\varphi(0) = 0$. Alors on a $\varphi(a + b) \leqq \max\,(\varphi(a), \varphi(b))$.

En supposant $\varphi\left(\frac{a}{b}\right) = \frac{\varphi(a)}{\varphi(b)}$, l'évaluation d'un anneau d'intégrité peut être étendue pour les éléments $\frac{a}{b}$ de K jusqu'à une évaluation du corps des quotients K de O. Comme on sait on peut construire à l'aide de cette évaluation $\mathfrak{p}$-adique une extension parfaite $K_\mathfrak{p}$ de K pour laquelle on a le théorème de convergence de CAUCHY.

Nous tenterons de représenter les éléments de $K_\mathfrak{p}$ par des suites de classes d'équivalence et par des séries infinies. Considérons alors K et pour $\lambda = 0, 1, 2, \ldots$ les modules M_λ, contenant ces éléments $a = \frac{m}{n}$ de K avec $\varphi(a) \leqq e^{-\lambda}$. Si $a = \frac{m}{n}$, pendant que m divise précisément $\mathfrak{p}^{\lambda 1}$ et n divise précisément $\mathfrak{p}^{\lambda 2}$, alors $\varphi\left(\frac{m}{n}\right) = e^{-\lambda}$, avec $\lambda = \lambda_1 - \lambda_2$; nous dirons que $a = \frac{m}{n}$ divise totalement $\mathfrak{p}^\lambda$ (maintenant λ peut être négatif).

Si $\varphi(a) \leqq e^{-\lambda}$ et $\varphi(b) \leqq e^{-\lambda}$, on a $\varphi(a-b) \leqq e^{-\lambda}$, alors avec a et b M_λ contient également $a-b$, de sorte que M_λ est un module.

Supposons maintenant $\{r_\mu\}$ une suite fondamentale $\mathfrak{p}$-adique d'éléments de K, alors il existe pour chaque $\lambda = 0, 1, 2, \ldots$ un $n = n(\lambda)$, de sorte que $\varphi(r_\mu - r_\nu) \leqq e^{-\lambda}$ pour $\mu, \nu > n(\lambda)$, c'est à dire $r_\mu \equiv r_\nu \pmod{\mathfrak{p}^\lambda}$.

Pour chaque $\lambda = 0, 1, 2, \ldots$ il existe alors un $n(\lambda)$ de sorte que tous les éléments r_μ pour $\mu > n(\lambda)$ appartiennent à une seule classe d'équivalence $R_\lambda \bmod M_\lambda$.

La suite fondamentale définit donc une suite de classes d'équivalence

$$R_0 \supset R_1 \supset R_2 \supset \ldots\ldots$$

qui sont enchaînées de la manière indiquée. Si inversement $\{r_1, r_2, \ldots\}$ présente une suite d'éléments de K, qui définit une suite $\{R_\mu\}$ de classes d'équivalence $R_\lambda \bmod M_\lambda$ enchaînées l'une à l'autre, de sorte que r_μ est contenu en R_λ pour tous les $\mu > n(\lambda)$, cette suite $\{r_\nu\}$ est une suite fondamentale. En effet: pour $\mu > n(\lambda)$ on a que r_μ est situé en R_λ, c'est à dire $r_{\mu_1} \equiv r_{\mu_2} \pmod{M_\lambda}$ pour $\mu_1, \mu_2 > n(\lambda)$, alors $\varphi(r_{\mu_1} - r_{\mu_2}) \leqq e^{-\lambda}$, en d'autre termes: la suite $\{r_\mu\}$ satisfait à la définition de suite fondamentale. Il faut donc noter que pour une suite fondamentale on détermine pour chaque λ une classe d'équivalence R_λ par M_λ, c'est à dire l'ensemble des éléments de la suite fondamentale $\{r_\mu\}$ avec $\mu > n(\lambda)$. Soit spécialement $\{r_\mu\}$ une suite de zéro $\mathfrak{p}$-adique, c'est à dire qu'il existe pour chaque $\varepsilon > 0$ un $n_0 = n_0(\varepsilon)$ de sorte que pour $n > n_0(\varepsilon)$ $\varphi(r_n) \leqq \varepsilon$, alors $R_\lambda = M_\lambda$ est la classe de zéro d'équivalence pour $\lambda = 0, 1, 2, \ldots$. En additionnant deux suites fondamentales $\{r_\mu\} + \{s_\mu\} = \{r_\mu + s_\mu\}$ les suites des classes d'équivalence correspondantes sont additionnées aussi: $\{R_\lambda + S_\lambda\}$. Si l'on additionne spécialement une suite de zéro à une suite fondamentale, la suite des classes d'équivalence correspondante ne change pas. Si inverse-

ment deux suites $\{r_\mu\}$ et $\{s_\mu\}$ appartiennent à la même suite des classes d'équivalence $\{R_\lambda\}$, leur différence est une suite de zéro.

A chaque élément $\mathfrak{p}$-adique $a = \lim_{\nu \to \infty} r_\nu$ correspond réversible uniquement une suite des classes d'équivalence $\{R_\lambda\}$ de la sorte mentionnée. En revenant de la suite des classes d'équivalence d'un élément $\mathfrak{p}$-adique a à une suite fondamentale spéciale, on n'a qu'à choisir de chaque classe d'équivalence R_λ un r'_λ comme représentant: alors $a = \lim_{\lambda \to \infty} r'_\lambda$.

Au cas de l'évaluation p-adique des nombres rationnels il est possible d'écrire chaque nombre p-adique comme une série; nous nous demandons, si l'on peut écrire aussi les éléments de $K_\mathfrak{p}$ comme des suites infinies.

Au cas $\mathfrak{p} = (p)$ soit un idéal premier principal engendré par un élément p, la possibilité est évidente.

Or, observons la suite des classes d'équivalence correspondant à l'élément $\mathfrak{p}$-adique: $\{R_0, R_1, R_2, \ldots\}$. Soit

$$r'_1 = s_0, \; r'_{\lambda+1} - r'_\lambda = s_\lambda \qquad (\lambda = 1, 2, \ldots),$$

on a

$$r'_{\lambda+1} = s_0 + s_1 + s_2 + \ldots + s_\lambda.$$

Il faut noter que $R_\lambda \supset R_{\lambda+1}$, c'est à dire r'_λ et $r'_{\lambda+1}$ appartiennent tous les deux à R_λ, c'est à dire leur différence fait partie de M_λ, or s_λ $(\lambda = 1, 2, \ldots)$ est un élément de K, qui est totalement divisible par $\mathfrak{p}^\lambda$. Nous avons alors:

$$a = \lim_{\nu=0}^{\lambda} \sum s_\nu = \sum_{\nu=0}^{\infty} s_\nu,$$

en d'autres mots: Chaque élément de $K_\mathfrak{p}$ peut être représenté par une somme infinie $a = s_0 + s_1 + s_2 + \ldots + s_\nu + s_{\nu+1} + \ldots$, où $s_\nu = \frac{m_\nu}{n_\nu}$ représente un élément de K $(\nu = 1, 2, \ldots)$ qui est totalement divisible par $\mathfrak{p}^\nu$.

Au cas spécial, où $\mathfrak{p} = (p)$ est en outre un idéal premier principal, chaque élément, divisible par $\mathfrak{p}$ a la forme $a = r \cdot p$ et par conséquent chaque élément divisible par $\mathfrak{p}^\lambda$ a la forme $t \cdot p^\lambda$; en ce cas la forme d'un élément de $K_\mathfrak{p}$ devient $a = s_0 + s_1 p + s_2 p^2 + \ldots + s_\nu p^\nu + \ldots$, avec s_ν $(\nu = 1, 2, \ldots)$ des éléments de K dont le dénominateur n_ν n'est pas divisible par $\mathfrak{p} = (p)$. Notez qu'il faut qu'au cas général les numérateurs des éléments $s_1, s_2, \ldots$ soient divisibles l'un après l'autre par $\mathfrak{p}, \mathfrak{p}^2, \ldots$, tandis qu'il est possible qu'ils divisent une puissance plus élevée.

Quand il s'agît d'une limite $\mathfrak{p}$-adique d'éléments de O, nous parlons d'un élément $\mathfrak{p}$-adique entier. Cela signifie qu'il existe en R_0 tous les éléments de la suite, en d'autres mots $R_0 = M_0$. R_λ contient les éléments de la suite à partir de certain $n(\lambda)$, c'est à dire chaque classe d'équivalence contient donc sûrement un élément de O.

Pour que l'élément $\mathfrak{p}$-adique soit une limite d'éléments de O, il faut et et il suffit que $R_0 = M_0$ pour $\mathfrak{p} = (p)$. Dans ce cas on peut démontrer que la congruence $m \cdot x \equiv n \pmod{\mathfrak{p}^\lambda}$ pour chaque λ entier a une solution

en O, c'est à dire chaque classe d'équivalence contient sûrement un élément de O de sorte que nous avons faire à un élément $\mathfrak{p}$-adique entier. Pour $\lambda = 0$ l'assertion est triviale, pour $\lambda = 1$ la solution existe parce que $\mathfrak{p}$ est un idéal premier maximal, c'est à dire $O/\mathfrak{p}$ est un corps et alors la congruence $m \cdot x \equiv n(\mathfrak{p})$ avec $m \not\equiv 0(\mathfrak{p})$ est soluble.

Supposons que $m \cdot x \equiv n(\mathfrak{p}^\lambda)$ ait une solution $x = x_0$, c'est à dire $mx_0 - n \equiv 0(\mathfrak{p}^\lambda)$; cela signifie donc $mx_0 - n = a \cdot p^\lambda$, tandis qu'avec $x_0\ x = x_0 - b \cdot p^\lambda$ aussi est une solution pour chaque b de O. Maintenant $mx \equiv n(\mathfrak{p}^{\lambda+1})$ a-t-il une solution? Nous déterminerons en $m(x_0 - b \cdot p^\lambda($ — n b de manière que $a \cdot p^\lambda - m \cdot b \cdot p^\lambda \equiv 0(\mathfrak{p}^{\lambda+1})$, c'est à dire $a - m \cdot b \equiv 0(\mathfrak{p})$ et ceci est possible parce que $m \not\equiv 0(\mathfrak{p})$ et $\mathfrak{p}$ est un idéal premier maximal. Avec $mx \equiv n(\mathfrak{p}^\lambda)$ $mx \equiv n(\mathfrak{p}^{\lambda+1})$ possède de même une solution. Soit α un élément $\mathfrak{p}$-adique entier, on peut donc dans le cas $\mathfrak{p} = (p)$ choisir des éléments de O pour tous les r'_λ et de même pour tous les s_λ.

Si $\mathfrak{p} = (p)$ chaque élément $\mathfrak{p}$-adique entier est une série en puissances de p avec des coëfficients de O et inversement. En ce cas on peut transformer immédiatement chaque élément $\mathfrak{p}$-adique α aux classes d'équivalence $\{R_0, R_1, \ldots\}$ en un élément $\mathfrak{p}$-adique entier. En effet, soit $r'_0 \subset R_0$ on peut arriver en multipliant r'_0 avec p^m à ce que le dénumérateur de $p^m r_0$ ne contienne plus de facteur p, de sorte que $p^m \cdot \alpha$ est entier.

En développant $p^m \cdot \alpha$ en une série aux coëfficients de O, nous trouvons pour

$$\alpha = a_{-m} p^{-m} + a_{-m+1} p^{-m+1} + \ldots + a_0 + a_1 p + a_2 p^2 + \ldots$$

avec $a_{-m}, a_{-m+1}, \ldots, a_0, a_1, \ldots$ de O.

Au lieu d'étendre le corps des quotients K de O au corps $K_\mathfrak{p}$, nous aurions pu étendre seulement l'anneau O à l'extension complète $O_\mathfrak{p}$ de O à l'aide de l'évaluation $\mathfrak{p}$-adique. Il est évident que $O_\mathfrak{p}$ est précisément l'anneau des éléments entiers $\mathfrak{p}$-adiques de $K_\mathfrak{p}$.

§ 2. *Relation entre l'anneau $O_\mathfrak{q}$ et les anneaux $O_{\mathfrak{p}_i}$.*

A cause des propositions de § 1 chaque idéal $\mathfrak{q}$ de O peut être représenté comme un produit d'idéaux premiers. Supposons donc: $\mathfrak{q} = \mathfrak{p}_1 \cdot \mathfrak{p}_2 \cdot \ldots \cdot \mathfrak{p}_n$ et en outre que les idéaux premiers $\mathfrak{p}_i$ satisfassent aux conditions de § 1. De l'unicité de la représentation d'un idéal comme produit des idéaux premiers s'ensuit que les puissances de $\mathfrak{q}$ différent et que leur intersection ne contient que l'élément 0. La propriété des idéaux premiers B. est: „Si a divise précisément $\mathfrak{p}^\alpha$, b divise précisément $\mathfrak{p}^\beta$, alors $a \cdot b$ divise précisément $\mathfrak{p}^{\alpha+\beta}$.“; cette propriété ne vaut pas pour un idéal $\mathfrak{q}$ quelconque: soit $\mathfrak{q} = \mathfrak{p}_1 \cdot \mathfrak{p}_2 \cdot \ldots \cdot \mathfrak{p}_n$ et a divise précisément $\mathfrak{p}_1^{\alpha_1}, \mathfrak{p}_2^{\alpha_2}, \ldots, \mathfrak{p}_n^{\alpha_n}$, alors, si $\alpha = \min(\alpha_1, \alpha_2, \ldots, \alpha_n)$, a divise présisément $\mathfrak{q}^\alpha$. Si b divise précisément $\mathfrak{q}^\beta$, on a que $a \cdot b$ divise une puissance de $\mathfrak{q}$. dont l'exposant est au moins $\alpha + \beta$.

Soit $\mathfrak{q}$ un idéal, représenté comme un produit d'idéaux $\mathfrak{p}_i$, satisfaisant aux propriétés de § 1; si a divise précisément $\mathfrak{q}^\alpha$, nous définissons

$$\varphi_{\mathfrak{q}}^{(a)} = e^{-\alpha},\ \varphi_{\mathfrak{q}}^{(1)} = 1,\ \varphi_{\mathfrak{q}}^{(0)} = 0$$

et on a $\varphi_{\mathfrak{q}}(a-b) \leqq \max(\varphi_{\mathfrak{q}}(a), \varphi_{\mathfrak{q}}(b))$ et puis $\varphi_{\mathfrak{q}}(a \cdot b) \leqq \varphi_{\mathfrak{q}}(a) \cdot \varphi_{\mathfrak{q}}(b)$.

Les définitions d'une suite fondamentale, de la convergence se font de la manière usuelle. Les suites fondamentales $\mathfrak{q}$-adiques déterminent un anneau commutatif $O'_{\mathfrak{q}}$. Comme chaque suite $\{a, a, a, \ldots\}$ est certainement une suite fondamentale de $O'_{\mathfrak{q}}$, $O'_{\mathfrak{q}}$ contient un sous-anneau isomorphe à O. Nous évaluons aussi les éléments de $O'_{\mathfrak{q}}$ de la manière usuelle: soit $\alpha = \{a_1, a_2, \ldots\}$ une suite fondamentale, nous définissons $\varphi_{\mathfrak{q}}^{(\alpha)} = \lim_{n \to \infty} \varphi_{\mathfrak{q}}(a_n)$.

Il s'ensuit

$$\varphi_{\mathfrak{q}}(\alpha \cdot \beta) = \lim_{n \to \infty} \varphi_{\mathfrak{q}}(a_n \cdot b_n) \leqq \lim_{n \to \infty} \varphi_{\mathfrak{q}}(a_n) \cdot \lim_{n \to \infty} \varphi_{\mathfrak{q}}(b_n) = \varphi_{\mathfrak{q}}(\alpha) \cdot \varphi_{\mathfrak{q}}(\beta)$$

et de même

$$\varphi_{\mathfrak{q}}(\alpha + \beta) \leqq \max(\varphi_{\mathfrak{q}}(\alpha), \varphi_{\mathfrak{q}}(\beta)).$$

Les suites de zéro (relatives à l'évaluation $\mathfrak{q}$-adique) déterminent un idéal N de $O'_{\mathfrak{q}}$; l'anneau complet relatif à l'évaluation $\mathfrak{q}$-adique de O c'est l'anneau des classes d'équivalence $O'_{\mathfrak{q}}/N = O_{\mathfrak{q}}$ Il était possible de représenter les éléments de $K_{\mathfrak{p}}$ par des suites de classes d'équivalence et par des séries infinies; cette possibilité existe de même pour les éléments de $O_{\mathfrak{q}}$. Par M_λ on étend le module des éléments a de O avec $\varphi(a) \leqq e^{-\lambda}$. Chaque $\varphi_{\mathfrak{q}}$-suite fondamentale définit uniquement une suite des classes d'équivalence $\{R_0, R_1, R_2, \ldots\}$ et inversément on détermine uniquement de chaque suite des classes d'équivalence un élément $\mathfrak{q}$-adique α de $O_{\mathfrak{q}}$ en choisissant un représentant r'_λ de chaque classe d'équivalence; on a $\alpha = \lim_{\lambda \to \infty} r'_\lambda$. En définissant $r'_1 = s_0$ et $r'_{\lambda+1} - r'_\lambda = s_\lambda$ ($s_\lambda \subset M$; $\lambda = 1, 2, \ldots$), on représente α comme une suite infinie. Alors

$$r_{\lambda+1} = s_0 + s_1 + s_2 + \ldots + s_{\lambda-1} + s_\lambda \text{ et } \alpha = \lim_{\lambda \to \infty} \sum_{\nu=0}^{\lambda} s_\nu = \sum_{\nu=0}^{\infty} s_\nu.$$

Chaque élément $\mathfrak{q}$-adique est représenté alors comme une suite infinie d'éléments de O, s_ν divisant $\mathfrak{q}_\nu$ ($\nu = 1, 2, \ldots$).

En particulier si les facteurs $\mathfrak{p}_i$ sont des idéaux premiers principaux engendrés par les éléments p_i, l'idéal $\mathfrak{q} = (q) = (p_1 \cdot p_2 \cdot \ldots \cdot p_n)$ est un idéal principal engendré par $p_1 \cdot p_2 \cdot \ldots \cdot p_n$ et chaque élément $\mathfrak{q}$-adique est représenté comme une série

$$\alpha = s_0 + s_1 q + s_2 q^2 + \ldots + s_\lambda q^\lambda + \ldots .$$

Nous déterminerons maintenant le rapport de l'anneau $O_{\mathfrak{q}}$ et les anneaux $O_{\mathfrak{p}_1}, O_{\mathfrak{p}_2}, \ldots, O_{\mathfrak{p}_n}$. D'abord à chaque élément $\mathfrak{q}$-adique de $O_{\mathfrak{q}}$ correspond uniquement un élément $\mathfrak{p}_1$-adique α_1 de $O_{\mathfrak{p}_1}$, ..., un élément $\mathfrak{p}_n$-adique α_n de $O_{\mathfrak{p}_n}$. C'est évident au cas où $\mathfrak{p}_i$ sont des idéaux premiers principaux; en effet, on considère la série infinie correspondant à α comme une série

infinie en p_1 (resp. $p_2, \ldots, p_n$). Si d'autre part $\mathfrak{q}$ est un produit des idéaux premiers $\mathfrak{p}_1, \mathfrak{p}_2, \ldots \mathfrak{p}_n$, on considère la représentation

$$\alpha = s_0 + s_1 + s_2 + \ldots + s_k + \ldots,$$

dans laquelle s_k divise $\mathfrak{q}^k$. Si s_k divise $\mathfrak{q}^k$, de toute évidence s_k divise $\mathfrak{p}_i^k$. A chaque élément $\mathfrak{q}$-adique α correspond alors uniquement un élément $\mathfrak{p}_1$-adique α_1, un élément $\mathfrak{p}_2$-adique $\alpha_2, \ldots$, un élément $\mathfrak{p}_n$-adique α_n.

Avant de continuer nous voulons remarquer d'abord que si $\mathfrak{q} = \mathfrak{p}^k$, donc la puissance k-ième d'un idéal premier maximal $\mathfrak{p}$, chaque élément $\mathfrak{q}$-adique peut être considéré comme élément $\mathfrak{p}$-adique. Soit inversément α un élément $\mathfrak{q}$-adique, on peut considérer α de même comme élément $\mathfrak{q}^k$-adique. En effet, nous écrivons:

$$\begin{aligned}
\alpha &= s_0 + s_1 + s_2 + \ldots + s_{k-1} + s_k + \ldots \\
\alpha &= (s_0 + s_1 + s_2 + \ldots + s_{k-1}) + (s_k + s_{k+1} + \ldots + s_{2k-1}) + \ldots \\
\alpha &= \quad r_0 \quad + \quad r_1 \quad + \ldots,
\end{aligned}$$

r_1 divisant $\mathfrak{q}^k$, r_2 divisant $\mathfrak{q}^{2k}, \ldots$ etc., en d'autre mots: α est représenté comme élément $\mathfrak{q}^k$-adique. Supposons en outre que $\mathfrak{q}$ et $\mathfrak{q}'$ soient deux idéaux contenant les mêmes facteurs premiers $\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_n$, qui n'ont que des exposants différents, alors il y a sans doute une puissance minimale $\mathfrak{q}^k$ de $\mathfrak{q}$, divisant $\mathfrak{q}'$, et de même une puissance minimale $\mathfrak{q}'^{k'}$ de $\mathfrak{q}'$, divisant $\mathfrak{q}$. Maintenant on peut voir immédiatement que chaque élément $\mathfrak{q}$-adique peut être représenté aussi comme élément $\mathfrak{q}'$-adique et inversément que chaque élément $\mathfrak{q}'$-adique peut être représenté comme un élément $\mathfrak{q}$-adique.

L'investigation des anneaux $\mathfrak{q}$-adiques peut donc se limiter à ces idéaux $\mathfrak{q}$, qui sont le produit de différents idéaux premiers, alors à ces idéaux $\mathfrak{q}$. ne contenant ces facteurs premiers différents qu'à la puissance première.

En vertu de ces considérations nous donnons la définition suivante de l'égalité de deux éléments α de $O_\mathfrak{q}$ et α' de $O_{\mathfrak{q}'}$ aux idéaux différents $\mathfrak{q}$ et $\mathfrak{q}'$. Soit

$$\alpha = s_0 + s_1 + s_2 + \ldots \text{ et } \alpha' = s_0' + s_1' + s_2' + \ldots,$$

nous définissons la valeur approximée k-ième de α: $\alpha^{(k)} = s_0 + s_1 + \ldots + s_k$ et la valeur approximée k-ième de α': $\alpha'^{(k)} = s_0' + \ldots + s_k'$.

Nous appelons α et α' identique pour l'idéal $\mathfrak{q}'$, si leurs valeurs approximées d'un ordre suffisament élevé sont congruentes entre elles pour chaque puissance de $\mathfrak{q}'$ d'un ordre quelconque. De même on appelle α et α' identiques pour l'idéal $\mathfrak{q}$, si les congruences analogues sont réalisées pour chaque puissance de $\mathfrak{q}$, d'un ordre quelconque.

Nous appelons $O_\mathfrak{q}$ un sous-ensemble de $O_{\mathfrak{q}'}$, si l'on peut déterminer à chaque α de $O_\mathfrak{q}$ un élément α' de $O_{\mathfrak{q}'}$, avec $\alpha' = \alpha$, relatif à l'idéal $\mathfrak{q}'$. Soit $O_\mathfrak{q}$ sous-ensemble de $O_{\mathfrak{q}'}$ et $O_{\mathfrak{q}'}$ sous-ensemble de $O_\mathfrak{q}$, on définit $O_\mathfrak{q} = O_{\mathfrak{q}'}$. En vertu de l'explication précédente, $O_\mathfrak{q} = O_{\mathfrak{q}'}$ si les idéaux $\mathfrak{q}$ et $\mathfrak{q}'$ contiennent les mêmes facteurs premiers. D'autre part $O_\mathfrak{p}$ est un sous-ensemble propre de $O_\mathfrak{q}$, si $\mathfrak{p}$ est un facteur premier de $\mathfrak{q}$.

Maintenant nous cherchons la relation qui existe entre les éléments de O_q et ceux des sous-ensembles propres $O_{\mathfrak{p}_i}$ avec $\mathfrak{p}_i$ $(i = 1, 2, \ldots, n)$ les facteurs d'idéaux premiers différents.

Nous supposons que q ne soit divisible que par les premières puissances d'idéaux premiers (suffisant aux conditions du § 1).

A chaque élément α de O_q correspond un élément $\mathfrak{p}_1$-adique α_1 déterminé uniquement, de même un élément $\mathfrak{p}_2$-adique α_2, déterminé uniquement, ... etc.

Indiquons la correspondance de la manière suivante:

$$\alpha \rightarrow (\alpha_1, \alpha_2, \ldots, \alpha_n) \text{ et } \beta \rightarrow (\beta_1, \beta_2, \ldots, \beta_n)$$

alors

$$\alpha + \beta \rightarrow (\alpha_1 + \beta_1, \alpha_2 + \beta_2, \ldots, \alpha_n + \beta_n) \text{ et } \alpha \cdot \beta \rightarrow (\alpha_1 \beta_1, \alpha_2, \beta_2, \ldots, \alpha_n \beta_n).$$

Inversément nous allons démontrer, si α_1 est un élément $\mathfrak{p}_1$-adique quelconque, α_2 un élément $\mathfrak{p}_2$-adique quelconque, ... etc., qu'il y a toujours un élément q-adique α de manière que $\alpha \rightarrow (\alpha_1, \alpha_2, \ldots, \alpha_n)$ selon la correspondance mentionnée ci-dessus.

Pour cela nous nous servons de:

Théorème I: Supposons q un idéal aux facteurs premiers différents $\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_n$, de sorte que $\mathfrak{q} = \mathfrak{p}_1 \cdot \mathfrak{p}_2 \cdot \ldots \cdot \mathfrak{p}_n$. Nous appelons les anneaux rendus complets et évalués à l'aide des idéaux $\mathfrak{p}_i$, q: $O_{\mathfrak{p}_1}, O_{\mathfrak{p}_2}, \ldots, O_{\mathfrak{p}_n}, O_q$. Appelons $a^{(1)}, a^{(2)}, \ldots, a^{(n)}$ n éléments de O, alors il existe une suite infinie d'éléments $\{a_1, a_2, a_3, \ldots\}$ de O, de sorte que cette suite s'approche de $a^{(1)}$ comme suite fondamentale $\mathfrak{p}_1$-adique, de $a^{(2)}$ comme suite fondamentale $\mathfrak{p}_2$-adique, ..., de $a^{(n)}$ comme suite fondamentale $\mathfrak{p}_n$-adique.

Démonstration: Il est toujours possible de déterminer x des congruences $x \equiv \alpha_i (\mathfrak{a}_i)$ $(i = 1, 2, \ldots, r)$, si $\mathfrak{a}_1, \mathfrak{a}_2, \ldots, \mathfrak{a}_r$ sont des idéaux avec $(\mathfrak{a}_i, \mathfrak{a}_k) = 0$ $(i \neq k)$. C'est pourquoi nous déterminerons un élément x_1 de O, satisfaisant aux congruences

$$\begin{aligned} x_1 &\equiv a^{(1)} \pmod{\mathfrak{p}_1} \\ x_1 &\equiv a^{(2)} \pmod{\mathfrak{p}_2} \\ &\ldots \\ x_1 &\equiv a^{(n)} \pmod{\mathfrak{p}_n} \end{aligned}$$

D'après le théorème cité cette possibilité existe, parce que les idéaux $\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_n$ sont des idéaux premiers maximaux; il s'ensuit que $(\mathfrak{p}_i, \mathfrak{p}_k) = 0$ $(i, k = 1, 2, \ldots, n;\ i \neq k)$.

D'après $(\mathfrak{p}_i, \mathfrak{p}_k) = 0$ $(i \neq k)$ il s'ensuit $(\mathfrak{p}_i^{\varrho}, \mathfrak{p}_k^{\varrho}) = 0$ $(i \neq k)$. En effet, si $(\mathfrak{p}_i, \mathfrak{p}_k) = 0$, il y a deux éléments p_i et k_k avec $1 = p_i + p_k$, alors

$$1 = p_i^{2\varrho - 1} + \ldots + p_k^{2\varrho - 1}.$$

Maintenant $p_i^{\varrho} \subset \mathfrak{p}_i^{\varrho}, p_k^{\varrho} \subset \mathfrak{p}_i^{\varrho}$, alors la partie droite contient des termes contenus en $\mathfrak{p}_i^{\varrho}$ ou en $\mathfrak{p}_k^{\varrho}$ et on a $1 = \pi_i + \pi_k$, ou $\pi_i \subset \mathfrak{p}_i^{\varrho}, \pi_k \subset \mathfrak{p}_k^{\varrho}$ et cette condition n'est pas seulement nécessaire pourque $(\mathfrak{p}, \mathfrak{p}) = 0$, mais elle suffit pourque $(\mathfrak{p}_i^{\varrho}, \mathfrak{p}_k) = 0$ $(i \neq k)$.

Si nous avons déterminé un élément x_1, nous construirons un élément x_2 des congruences

$$x_2 \equiv a^{(1)} \pmod{\mathfrak{p}_1^2}$$
$$x_2 \equiv a^{2)} \pmod{\mathfrak{p}_2^2}$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$x_2 \equiv a^{(n)} \pmod{\mathfrak{p}_n^2}$$

dont la possibilité existe à cause du fait, que $(\mathfrak{p}_i^2, \mathfrak{p}_k^2) = 0$ $(i \neq k)$.

En général, nous déterminerons un élément x_ϱ de O, de sorte que

$$x_\varrho \equiv a^{(1)} \pmod{\mathfrak{p}_1^\varrho}$$
$$x_\varrho \equiv a^{(2)} \pmod{\mathfrak{p}_2^\varrho}$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$x_\varrho \equiv a^{(n)} \pmod{\mathfrak{p}_n^\varrho}$$

La suite $\{x_1, x_2, x_3, \ldots\}$ possède la propriété qu'elle est une suite fondamentale $\mathfrak{p}_1$-adique, $\mathfrak{p}_2$-adique, ..., $\mathfrak{p}_n$-adique, tandis qu'elle approxime comme suite $\mathfrak{p}_k$-adique à $a^{(k)}$ $(k = 1, 2, \ldots, n)$.

Il s'ensuit de la construction:

$$\varphi_k(x_\varrho - a^{(k)}) \leqq e^{-\varrho}, \varphi_k(x_\sigma - a^{(k)}) \leqq e^{-\sigma}, \text{ c'est à dire pour } \sigma > \varrho:$$
$$\varphi_k(x_\varrho - x_\sigma) \leqq \max(e^{-\varrho}, e^{-\sigma}) = e^{-\varrho},$$

et alors que la suite $\{x_i\}$ est une suite fondamentale $\mathfrak{p}_k$-adique: en effet en choisissant un nombre réel $\varepsilon > 0$, il y a un ϱ_0, de sorte que l'inégalité dernière est satisfaite pour $\varrho \geqq \varrho_0$, $\sigma \geqq \varrho_0$. C'est valable pour tous les idéaux premiers $\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_n$.

Alors la suite construite est une suite fondamentale $\mathfrak{p}_1$-adique, $\mathfrak{p}_2$-adique, ..., $\mathfrak{p}_n$-adique. En outre il s'ensuit que la suite $\{x_i\}$ comme suite fondamentale $\mathfrak{p}_k$-adique se rapproche de $a^{(k)}$ $(k = 1, 2, \ldots, n)$.

Théorème II: Soit $a_1^{(1)}, a_2^{(1)}, a_3^{(1)}, \ldots$ une suite fondamentale $\mathfrak{p}_1$-adique,
$a_1^{(2)}, a_2^{(2)}, a_3^{(2)}, \ldots$ une suite fondamentale $\mathfrak{p}_2$-adique,
$\ldots$
$a_1^{(n)}, a_2^{(n)}, a_3^{(n)}, \ldots$ une suite fondamentale $\mathfrak{p}_n$-adique,

tandis que les éléments de $O_{\mathfrak{p}_1}, O_{\mathfrak{p}_2}, \ldots, O_{\mathfrak{p}_n}$ définis par ces suites sont représentés par $\alpha_{\mathfrak{p}_1}, \alpha_{\mathfrak{p}_2}, \ldots, \alpha_{\mathfrak{p}_n}$. Alors il y a une suite d'éléments $a_1, a_2, \ldots$ de O, étant une suite fondamentale $\mathfrak{p}_1$-adique, $\mathfrak{p}_2$-adique, ..., $\mathfrak{p}_n$-adique et approximant $\alpha_{\mathfrak{p}_1}, \alpha_{\mathfrak{p}_2}, \ldots, \alpha_{\mathfrak{p}_n}$; $\alpha_{\mathfrak{p}_k} = \lim_{m \to \infty} a_m$ $(\mathfrak{p}_k)$ $(k = 1, 2, \ldots, n)$.

Demonstration: Nous considérons $\alpha_{\mathfrak{p}_i}$ comme la limite d'une suite fondamentale $c_i^{(1)}, c_i^{(2)}, c_i^{(3)}, \ldots$ (pour $i = 1, 2, \ldots, n$).

D'abord nous faisons attention aux n éléments $c_1^{(1)}, c_2^{(1)}, \ldots, c_n^{(1)}$ et en vertu du théorème précédent nous construirons une suite d'éléments de O convergeant pour l'évaluation $\mathfrak{p}_k$-adique à $c_k^{(1)}$ $(k = 1, 2, \ldots, n)$. Et puis nous construirons une suite d'éléments de O convergeant pour l'évaluation $\mathfrak{p}_k$-

adique à $c_k^{(2)}$ $(k = 1, 2, \ldots, n)$ etc., de sorte que nous obtenons le schème suivant:

$$a_1^{(1)}, a_2^{(1)}, a_3^{(1)}, \ldots \to c_1^{(1)}(\mathfrak{p}_1);\ a_1^{(1)}, a_2^{(1)}, a_3^{(1)}, \ldots \to c_2^{(1)}(\mathfrak{p}_2); \ldots;\ a_1^{(1)}, a_2^{(1)}, a_3^{(1)}, \ldots \to c_n^{(1)}(\mathfrak{p}_n)$$

$$a_1^{(2)}, a_2^{(2)}, a_3^{(2)}, \ldots \to c_1^{(2)}(\mathfrak{p}_1);\ a_1^{(2)}, a_2^{(2)}, a_3^{(2)}, \ldots \to c_2^{(2)}(\mathfrak{p}_2); \ldots;\ a_1^{(2)}, a_2^{(2)}, a_3^{(2)}, \ldots \to c_n^{(2)}(\mathfrak{p}_n)$$

$$a_1^{(3)}, a_2^{(3)}, a_3^{(3)}, \ldots \to c_1^{(3)}(\mathfrak{p}_1);\ a_1^{(3)}, a_2^{(3)}, a_3^{(3)}, \ldots \to c_2^{(3)}(\mathfrak{p}_2); \ldots;\ a_1^{(3)}, a_2^{(3)}, a_3^{(3)}, \ldots \to c_n^{(3)}(\mathfrak{p}_n)$$

. . . . . ; . . . . ; . . . .

Nous considérons maintenant la suite $a_1^{(1)}, a_2^{(2)}, \ldots, a_k^{(k)}, \ldots$. Cette suite possède la propriété, qu'elle converge comme une suite $\mathfrak{p}_k$-adique à $\alpha_{\mathfrak{p}_k}$ $(k = 1, 2, \ldots, n)$; nous démontrerons cette assertion pour $\mathfrak{p}_k$:

$$\varphi_k(a_m^{(m)} - \alpha_{\mathfrak{p}_k}) = \varphi_k(a_m^{(m)} - c_k^{(m)} + c_k^{(m)} - \alpha_{\mathfrak{p}_k}) \leqq \max(\varphi_k(a_m^{(m)} - c_k^{(m)});\ \varphi_k(c_k^{(m)} - \alpha_{\mathfrak{p}_k})),$$

tandis que en vertu de la construction $\varphi_k(a_m^{(m)} - c_k^{(m)})$ et $\varphi_k(c_k^{(m)} - \alpha_{\mathfrak{p}_k})$ pourront devenir arbitrairement petits.

Par cela l'existence de la suite cherchée $\{a_1, a_2, \ldots\}$, c'est à dire la suite $\{a_i^{(i)}\}$, $a_i^{(i)} = a_i$ est démontrée. Alors: soit $\alpha_{\mathfrak{p}_1} \subset O_{\mathfrak{p}_1}$, $\alpha_{\mathfrak{p}_2} \subset O_{\mathfrak{p}_2}, \ldots\ldots$, $\alpha_{\mathfrak{p}_n} \subset O_{\mathfrak{p}_n}$, il y a une suite $\{a_1, a_2, \ldots\}$ d'éléments de O, qui est une suite fondamentale $\mathfrak{p}_k$-adique $(k = 1, 2, \ldots, n)$ convergeant à $\alpha_{\mathfrak{p}_k} = \lim_{r \to \infty} a_r(\mathfrak{p}_k)$.

Cette suite $\{a_1, a_2, \ldots\}$ est une suite fondamentale $\mathfrak{p}_k$-adique

$$(k = 1, 2, \ldots, n)$$

alors en outre une suite fondamentale $\mathfrak{q}$-adique. En effet, la divisibilité par $\mathfrak{p}_k^e$ $(k = 1, 2, \ldots, n)$ implique toujours une divisibilité par $\mathfrak{q}^e$.

Alors à $\alpha_{\mathfrak{p}_1}, \alpha_{\mathfrak{p}_2}, \ldots, \alpha_{\mathfrak{p}_n}$ correspond aussi un élément $\alpha \subset O_{\mathfrak{q}}$. Cet élément α est déterminé uniquement par $\alpha_{\mathfrak{p}_k}$ $(k = 1, 2, \ldots, n)$; en effet soit $\{a_1', a_2', \ldots\}$ une seconde suite de O avec $\alpha_{\mathfrak{p}_h} = \lim_{m \to \infty} a_m'(\mathfrak{p}_h)$, $(h = 1, 2, \ldots, n)$, on a

$$\lim_{m \to \infty} \varphi_h(a_m - a_m') = 0 \text{ alors } \lim_{m \to \infty} a_m = \lim_{m \to \infty} a_m'(\mathfrak{y}).$$

Directement il s'ensuit de la construction que pour

$$(\alpha_{\mathfrak{p}_1}, \alpha_{\mathfrak{p}_2}, \ldots, \alpha_{\mathfrak{p}_n}) \to \alpha \text{ et } (\beta_{\mathfrak{p}_1}, \beta_{\mathfrak{p}_2}, \ldots, \beta_{\mathfrak{p}_n}) \to \beta$$

on a

$$(\alpha_{\mathfrak{p}_1} + \beta_{\mathfrak{p}_1}, \ldots, \alpha_{\mathfrak{p}_n} + \beta_{\mathfrak{p}_n}) \to \alpha + \beta \text{ et } (\alpha_{\mathfrak{p}_1} \cdot \beta_{\mathfrak{p}_1}, \ldots, \alpha_{\mathfrak{p}_n} \cdot \beta_{\mathfrak{p}_n}) \to \alpha \cdot \beta.$$

Par cela la relation unique et inverse

$$\alpha \longleftrightarrow (\alpha_{\mathfrak{p}_1}, \alpha_{\mathfrak{p}_2}, \ldots, \alpha_{\mathfrak{p}_n})$$

entre les éléments de $O_{\mathfrak{q}}$ et les systèmes de tout élément $\alpha_{\mathfrak{p}_1}$ de $O_{\mathfrak{p}_1}$, $\alpha_{\mathfrak{p}_2}$ de $O_{\mathfrak{p}_2}, \ldots, \alpha_{\mathfrak{p}_n}$ de $O_{\mathfrak{p}_n}$ est déterminée. Si en outre $\beta \longleftrightarrow (\beta_{\mathfrak{p}_1}, \beta_{\mathfrak{p}_2}, \ldots, \beta_{\mathfrak{p}_n})$, la relation satisfait en outre à

$$\alpha + \beta \longleftrightarrow (\alpha_{\mathfrak{p}_1} + \beta_{\mathfrak{p}_1}, \ldots, \alpha_{\mathfrak{p}_n} + \beta_{\mathfrak{p}_n}),$$
$$\alpha \cdot \beta \longleftrightarrow (\alpha_{\mathfrak{p}_1} \cdot \beta_{\mathfrak{p}_1}, \ldots, \alpha_{\mathfrak{p}_n} \cdot \beta_{\mathfrak{p}_n}).$$

Par là nous avons démontré le

Théorème III: $O_{\mathfrak{q}}$ est la somme directe des anneaux $O_{\mathfrak{p}_1}, O_{\mathfrak{p}_2}, \ldots, O_{\mathfrak{p}_n}$, si l'idéal $\mathfrak{q} = \mathfrak{p}_1 \cdot \mathfrak{p}_2 \cdot \ldots \cdot \mathfrak{p}_n$.

Mathematics. — *On the figure of four projective spaces* $[n_1—1]$, $[n_2—1]$, $[n_3—1]$ *and* $[n_4—1]$ *in a* $[n—1]$, *where* $n_1 + n_2 + n_3 + n_4 = 2n$. I. By G. H. A. GROSHEIDE F.WZN. (Communicated by Prof. J. A. SCHOUTEN.)

(Communicated at the meeting of September 27, 1947.)

1. We consider four projective spaces A, B, C and D respectively of $n_1—1$, $n_2—1$, $n_3—1$ and $n_4—1$ dimensions, all belonging to a projective $(n—1)$-dimensional space G_n. Thereto we assume the existence of the relation

$$n_1 + n_2 + n_3 + n_4 = 2n \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

on account of which our spaces possess at least one line transversal.

The first part of the present investigation is concerned with the composition of an integrity basis for all projective invariants of the figure.

The second part discusses the presence of line transversals and examines the cross ratios of the four points of intersection upon these lines. This leads to the construction of a system of concomitants by means of which the projective geometrical classification of all figures of the described type can be completed in the case that the values of the absolute invariants don't give decisive informations. It appears that the integrity basis obtained in the first part contains (and in general is) a smallest one.

On taking $n = 2k$; $n_1 = n_2 = n_3 = n_4 = k$ the spaces A, B, C, D are "medials" in an odd-dimensional projective space [1]), also for $k = 1$ points on a straight line, for $k = 2$ straight lines in a space of three dimensions, and so on. Consequently many of the following results are extensions of theorems referring to the well-known figure of four medials. From recent papers dealing with this special case must be mentioned next the works of TURNBULL [2]), those of WEITZENBÖCK [3]), BOTTEMA [4]) and BRUINS [5]). After replacing the medials through spaces of an arbitrary number of dimensions, it is true, the self-duality of the figure is lost (at least in general); but on the other hand the limitation to even values of n is also taken off.

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[1]) The denomination "medial" was proposed by TURNBULL (1941).

[2]) H. W. TURNBULL, I. The geometry of matrices. Philos. Trans. Royal Soc. London, Ser. A. No. 805. Vol. 239 pp. 233—267, 1942. — II. The projective invariants of four medials. Proc. Edinburgh Math. Soc. Ser. 2. Vol. 7 pp. 55—72, 1942.

[3]) R. WEITZENBÖCK, Die projektiven Invarianten von vier Ebenen im R_5. Proc. Kon. Akad. v. Wetensch., Amsterdam, **35**, 1026—1029 (1932).

[4]) O. BOTTEMA, De figuur van vier vlakken in R_5. Verslagen Ned. Akad. v. Wetensch., Amsterdam, **53**, 30—37, 53—57 (1944).

[5]) E. M. BRUINS, De projectieve invarianten van vier G_d in G_{2d}. Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, **49**, 738—743 (1946).

2. Without limiting our problem we may assume

$$n_1 \leqq n_2 \leqq n_3 \leqq n_4 < n \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and after that we get from (1)

$$n_1 + n_2 \leqq n;\ n_1 + n_3 \leqq n;\ n_2 + n_4 \geqq n;\ n_3 + n_4 \geqq n \ldots \quad . \quad (3)$$

Evidently invariants consisting of a single bracket factor exist if and only if two of the four subspaces are dual with another in G_n. On account of (1) in this case the remaining two subspaces are likewise dual with another in G_n. So we arrive at the following review:

I. Duality of D with C: $n_3 + n_4 = n$.

Since $n_1 + n_2 = n$ and thus

$$(n_3 — n_1) + (n_4 — n_2) = (n_3 — n_2) + (n_4 — n_1) = 0,$$

this happens after the assumption (2) only in the case of four medials.

II. Duality of D with B, but not with C: $n_2 + n_4 = n < n_3 + n_4$.

Now we obtain from (1) $(n_2 — n_1) + (n_4 — n_3) = 0$ and hence from (2)

$$n_1 = n_2 = n - n_4 < n_3 = n_4 = n - n_1.$$

III. Duality of D with A, but not with B: $n_1 + n_4 = n < n_2 + n_4$.

It is not necessary that there exists any other equality between n_1, n_2, n_3, n_4 save $n_2 + n_3 = n$ (duality of B and C).

IV. D is not dual in G_n with one of the spaces A, B or C.

3. Let A be defined by the n_1 linearly independent points

$$\{y^{(1)}\},\quad \{y^{(2)}\}, \ldots, \{y^{(n_1)}\}$$

and B by the n_2 linearly independent points

$$\{z^{(1)}\},\quad \{z^{(2)}\}, \ldots, \{z^{(n_2)}\}.$$

If we put

$$a_{i_1 i_2 \ldots i_{n_1}} = a_{i_1}\, a_{i_2} \ldots a_{i_{n_1}} = (y^{(1)}\, y^{(2)} \ldots y^{(n_1)})_{i_1 i_2 \ldots i_{n_1}},$$
$$b_{j_1 j_2 \ldots j_{n_2}} = b_{j_1}\, b_{j_2} \ldots b_{j_{n_2}} = (z^{(1)}\, z^{(2)} \ldots z^{(n_2)})_{j_1 j_2 \ldots j_{n_2}},$$

where $a_{i_1}, a_{i_2}, \ldots, a_{i_{n_1}}$ and $b_{j_1}, b_{j_2}, \ldots, b_{j_{n_2}}$ are "complex symbols", as introduced by WEITZENBÖCK [6]), then A and B have the equations

$$(a^{n_1}\, \pi^{n-n_1}) = (-1)^{n_1(n-n_1)}\, (n-n_1)!\, (a\, \pi')^{n_1} = n_1!\, (y^{(1)}\, y^{(2)} \ldots y^{(n_1)}\, \pi^{n-n_1}) = 0$$

and

$$(b^{n_2}\, \pi^{n-n_2}) = (-1)^{n_2(n-n_2)}\, (n-n_2)!\, (b\, \pi')^{n_2} = n_2!\, (z^{(1)}\, z^{(2)} \ldots z^{(n_2)}\, \pi^{n-n_2}) = 0$$

respectively. The space C is determined by $n — n_2$ linearly independent primes

$$\{v_{(1)}\},\quad \{v_{(2)}\}, \ldots, \{v_{(n-n_2)}\}$$

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[6]) R. WEITZENBÖCK, Invariantentheorie. Groningen 1923.

that contain it. Putting

$$c'_{i_1 i_2 \ldots i_{n-n_3}} = c'_{i_1} c'_{i_2} \ldots c'_{i_{n-n_3}} = (v'_{(1)} v'_{(2)} \ldots v'_{(n-n_3)})_{i_1 i_2 \ldots i_{n-n_3}}$$

we find as the equation of C

$$(c'^{\,n-n_3} \pi'^{\,n_3}) = (c^{n_3} \pi^{n-n_3}) = n_3 ! \, (c' \pi)^{n-n_3} = (n-n_3) ! \, (v'_{(1)} v'_{(2)} \ldots v'_{(n-n_3)} \pi'^{\,n_3}) = 0,$$

where

$$c_{j_1 j_2 \ldots j_{n_3}} = \operatorname{sign} \begin{pmatrix} 1\ 2 \ \ldots\ n_3 & n_3+1 \ldots & n \\ j_1\ j_2 \ldots j_{n_3} & i_1 \ \ldots & i_{n-n_3} \end{pmatrix} c'_{i_1 i_2 \ldots i_{n-n_3}}.$$

If D is the intersection of the linearly independent primes

$$\{w'_{(1)}\}, \ \{w'_{(2)}\}, \ldots, \ \{w'_{(n-n_4)}\}$$

and

$$d'_{i_1 i_2 \ldots i_{n-n_4}} = d'_{i_1} d'_{i_2} \ldots d'_{i_{n-n_4}} = (w'_{(1)} w'_{(2)} \ldots w'_{(n-n_4)})_{i_1 i_2 \ldots i_{n-n_4}},$$

then D has the equation

$$(d'^{\,n-n_4} \pi'^{\,n_4}) = (d^{n_4} \pi^{n-n_4}) = n_4 ! \, (d' \pi)^{n-n_4} = (n-n_4) ! \, (w'_{(1)} w'_{(2)} \ldots w'_{(n-n_4)} \pi'^{\,n_4}) = 0.$$

4. Occupied with the composition of an integrity basis we need only give attention to invariants that can be expressed symbolically as products of bracket factors built up by symbols a_i, b_i, c_i and d_i. If such an invariant J is of a degree α with respect to $a_{i_1 i_2 \ldots i_{n_1}}$, we introduce the equivalent symbols $\underset{(1)}{a_i}, \underset{(2)}{a_i}, \ldots, \underset{(\alpha)}{a_i}$ by means of

$$a_{i_1 i_2 \ldots i_{n_1}} = \underset{(j)}{a_{i_1}} \underset{(j)}{a_{i_2}} \ldots \underset{(j)}{a_{i_{n_1}}} \qquad (j = 1, 2, \ldots, \alpha).$$

Also for degrees β in $b_{i_1 i_2 \ldots i_{n_2}}$, γ in $c_{i_1 i_2 \ldots i_{n_3}}$ and δ in $d_{i_1 i_2 \ldots i_{n_4}}$ we use the equivalent symbols

$$\underset{(1)}{b_i}, \underset{(2)}{b_i}, \ldots, \underset{(\beta)}{b_i} \ ; \ \underset{(1)}{c_i}, \underset{(2)}{c_i}, \ldots, \underset{(\gamma)}{c_i} \text{ and } \underset{(1)}{d_i}, \underset{(2)}{d_i}, \ldots, \underset{(\delta)}{d_i}.$$

Supposing $\delta > 0$ we draw all symbols $\underset{(1)}{d}$ in the same bracket factor, which leads to

$$J \equiv \Sigma \varrho \, (\underset{(1)}{d^{n_4}} \ldots) J^*,$$

or J can be expressed as a sum (Σ) of invariants of the form $(\underset{(1)}{d^{n_4}} \ldots) J^*$. In this and in other formulas ϱ always is an indication for a coefficient that is an ordinary rational number. Because of

$$(\underset{(1)}{d^{n_4}} \underset{(j)}{d} \ldots) J^* \equiv 0 \qquad (j = 2, 3, \ldots, \delta) \quad . \quad . \quad . \quad . \quad . \quad (4)$$

the number of bracket factors in the originally product surpasses δ—1 and so it is possible to repeat this process δ-times and to arrive at least at

$$J \equiv \Sigma \varrho \, (\underset{(1)}{d^{n_4}} \ldots) (\underset{(2)}{d^{n_4}} \ldots) \ldots (\underset{(\delta)}{d^{n_4}} \ldots) J_1 (a, b, c) \qquad (\delta > 0).$$

This result shows that it is not necessary to take in consideration other invariants than those of the form

$$J = (d^{n_1}\ldots)_{(1)}(d^{n_2}\ldots)_{(2)}\ldots(d^{n_\delta}\ldots)_{(\delta)} J_1(a, b, c) \qquad (\delta \geqq 0).$$

5. For $\gamma > 0$ and $J_1 \neq 1$ we draw all symbols $c_{(1)}$ in the first bracket factor of J_1, all symbols $c_{(2)}$ in the second bracket factor of J_1 and so on. If J_1 is the product of N factors, then this process ends with the gathering of the symbols $c_{(N)}$ for $\gamma > N \geq 1$ and with the gathering of the symbols $c_{(\gamma)}$ for $\gamma \leqq N$. In the first case, on account of

$$(c^{n_3} c\ldots)_{(1)\,(i)} J^* \equiv 0 \qquad (i = 2, 3, \ldots, \gamma) \quad . \quad . \quad . \quad . \quad . \quad (5)$$

all n_3 symbols $c_{(j)}$ $(N+1 \leqq j \leqq \gamma)$ occur in d-factors of J, which in the same manner happens for $\gamma > 0$, $J_1 = 1$, $\delta > 0$. Thus for $0 \leqq N < \gamma$ we may write J as a sum

$$J \equiv \Sigma_\varrho (d^{n_4} c^{n-n_4})_{(1)\,(j)} (d^{n_4} c^{n_3+n_4-n}\ldots)_{(2)\,(j)}\ldots(d^{n_\delta}\ldots)_{(\delta)} J_1 \qquad (N+1 \leqq j \leqq \gamma)$$

and it follows that

J contains a bracket factor $(d^{n_4} c^{n_3})$ for $n_3 + n_4 = n$ . . . (I),
J vanishes identically for $n_3 + n_4 > n$.

After dividing the invariant of the four medials through $[(d^{n_4} c^{n_3})]^{\gamma - N}$ there remains an invariant with $0 \leqq \gamma \leqq N$. Hence we have permission to consider in future only invariants of the form

$$J = (d^{n_1}\ldots)_{(1)}\ldots(d^{n_\delta}\ldots)_{(\delta)}(c^{n_3}\ldots)_{(1)}\ldots(c^{n_\gamma}\ldots)_{(\gamma)} J_2(a, b) \qquad (\gamma, \delta \geqq 0).$$

6. Unless $J = 1$, at least one of the numbers α and β differs from zero. For $J_2 \neq 1$ we assume $\beta > 0$ and gather all n_2 symbols $b_{(1)}$ in the first bracket factor of J_2. After that on account of

$$(b^{n_2} b\ldots)_{(1)\,(i)} J^* \equiv 0 \qquad (i = 2, 3, \ldots, \beta) \quad . \quad . \quad . \quad . \quad . \quad (6)$$

the other places in this factor must be occupied by $n - n_2$ symbols a. Now from

$$(a^{n_1} a\ldots)_{(1)\,(i)} J^* \equiv 0 \qquad (i = 2, 3, \ldots, \alpha)\ldots \quad . \quad . \quad . \quad . \quad (7)$$

it follows

J contains a bracket factor $(b^{n_2} a^{n_1})$ for $n - n_2 = n_1$ . . (I),
J vanishes identically for $n - n_2 > n_1$.

As the same conclusion can be drawn after the assumption $J_2 \neq 1$, $\alpha > 0$, we may put in the following $J_2 = 1$. It stands to reason that in the

case of two medials A and B the invariants $(a^{n_1} b^{n_2})$ and $(c^{n_3} d^{n_4})$ must be kept in remembrance. Because of $J_2 = 1$ there exist the relation

$$\alpha n_1 + \beta n_2 + \gamma n_3 + \delta n_4 = (\gamma + \delta) n . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

7. In this section we prove that equalization of β and γ is permitted. For $\beta > \gamma \geqq 0$ we have first

$$J \equiv \Sigma \varrho \underset{(1)}{(d^{n_4} \ldots)} \ldots \underset{(\delta)}{(d^{n_4} \ldots)} \underset{(1)}{(b^{n_2} \ldots)} \ldots \underset{(\gamma}{(b^{n_2} \ldots)}$$

and then, since on account of (6) all n_2 symbols $\underset{(j)}{b}$ $(\gamma + 1 \leqq j \leqq \beta)$ occur in d-factors

$$J \equiv \Sigma \varrho \underset{(1)\ (\gamma+1)}{(d^{n_4} b^{n-n_4})} \underset{(2)\ (\gamma+1)}{(d^{n_4} b^{n_2+n_1-n} \ldots)} \ldots \underset{(\gamma)}{(b^{n_2} \ldots)}.$$

This shows that

$$\begin{aligned} &J \text{ contains a bracket factor } (d^{n_4} b^{n_2}) \text{ for } n_2 + n_4 = n \quad . \quad . \quad \text{(II)}, \\ &J \text{ vanishes identically} \qquad\qquad\quad \text{for } n_2 + n_4 > n . \end{aligned}$$

For $\beta < \gamma$ we draw all symbols $\underset{(i)}{b}$ that occur in c-factors of J in the bracket factor $\underset{(i)}{(c^{n_3} \ldots)}$ and arrive at the identity

$$\underset{(1)}{(c^{n_3} \ldots)} \ldots \underset{(\gamma)}{(c^{n_3} \ldots)} \equiv \Sigma \varrho \underset{(1)\ (1)}{(c^{n_3} b^{p} \ldots)} \ldots \underset{(\beta)\ (\beta)}{(c^{n_3} b^{q} \ldots)} \underset{(\beta+1)}{(c^{n_3} \ldots)} \ldots \underset{(\gamma)}{(c^{n_3} \ldots)}.$$

Hence, having regard to (3) and (7), we may conclude

$$\begin{aligned} &J \text{ contains a bracket factor } (c^{n_3} a^{n_1}) \text{ for } n_1 + n_3 = n \quad . \quad . \quad \text{(II)}, \\ &J \text{ vanishes identically} \qquad\qquad\quad \text{for } n_1 + n_3 < n . \end{aligned}$$

8. The proof delivered in the preceding section was based upon (6) and (7) and upon the relations

$$n_2 + n_4 \geqq n \text{ and } n_1 + n_3 \leqq n.$$

Therefore we can demonstrate on the same manner that it is allowed to assume the equality of

$$\alpha \text{ and } \gamma \ (\alpha = \gamma) \text{ if } n_1 + n_4 \geqq n,\ n_2 + n_3 \leqq n$$

and of

$$\beta \text{ and } \delta \ (\beta = \delta) \text{ if } n_2 + n_3 \geqq n,\ n_1 + n_4 \leqq n.$$

Now with $\beta = \gamma$ we deduce from (1) and (8)

$$\alpha n_1 + \delta (n_4 - n) = \gamma (n - n_2 - n_3) = \gamma (n_1 + n_4 - n)$$

or

$$(\alpha - \gamma) n_1 = (\delta - \gamma)(n - n_4) = (\delta - \beta)(n - n_4).$$

Hence both from $\alpha = \gamma$ and $\beta = \delta$ it follows

$$\alpha = \beta = \gamma = \delta > 0.$$

Each invariant that contains no invariant of two medials as a factor, is of the same degree with respect to the coordinates of all four spaces.

9. We suppose that there are i_j symbols $\underset{(j}{b}$ contained in d-factors of J and thus $n_2 - i_j$ symbols $\underset{(j)}{b}$ in c-factors $(j = 1, 2, \ldots, \alpha)$. From

$$(\underset{(1)}{d^{n_4}}\ \underset{(j)}{b^{n-n_4}})\,(\underset{(2)}{d^{n_4}}\ \underset{(j)}{b}\ldots)\,J^* \equiv 0$$

it follows

$$n_4 + i_j \leqq n$$

and if $n_2 + n_3 > n$ we have on account of

$$(\underset{(1)}{c^{n_3}}\ \underset{(j)}{b^{n-n_3}})\,(\underset{(2)}{c^{n_3}}\ \underset{(j)}{b}\ldots)\,J^* \equiv 0$$

that holds

$$n_2 + n_3 - i_j \leqq n.$$

Thus it is possible to draw all i_j mentioned symbols $\underset{(j)}{b}$ in the bracket factor $(\underset{(j)}{d^{n_4}}\ldots)$ and all $n_2 - i_j$ remaining symbols $\underset{(j)}{b}$ in the bracket factor $(\underset{(j)}{c^{n_3}}\ldots)$, after which J becomes a sum of products of the appearance

$$J_3 = (\underset{(1)}{d^{n_4}}\ \underset{(1)}{b^{i_1}}\ldots)\ldots(\underset{(\alpha)}{d^{n_4}}\ \underset{(\alpha)}{b^{i_\alpha}}\ldots)\,(\underset{(1)}{c^{n_3}}\ \underset{(1)}{b^{n_2-i_1}}\ldots)\ldots(\underset{(\alpha)}{c^{n_3}}\ \underset{(\alpha)}{b^{n_2-i_\alpha}}\ldots),$$

where the places indicated by dots are occupied by symbols a. Evidently it is sufficient to consider only invariants of the type of J_3. We will show that J_3 can be expressed as a polynomial of invariants that are products of two bracket factors only.

10. The number f_1 which indicates how many symbols contained in the first two bracket factors of

$$J_3 = (\underset{(1)}{d^{n_4}}\ \underset{(1)}{b^{i_1}}\ \underset{(1)}{a^h}\ldots)\,(\underset{(1)}{c^{n_3}}\ \underset{(1)}{b^{n_2-i_1}}\ \underset{(1)}{a^k}\ldots)\,J_d\,J_c$$

are different from $\underset{(1)}{a}$, $\underset{(1)}{b}$, $\underset{(1)}{c}$ and $\underset{(1)}{d}$ will be called the "deficit" of J_3. If $f_1 = 0$, there is divisibility of J_3 by the invariant

$$(\underset{(1)}{d^{n_4}}\ \underset{(1)}{b^{i_1}}\ a^{n-n_4-i_1})\,(\underset{(1)}{c^{n_3}}\ \underset{(1)}{b^{n_2-i_1}}\ a^{n-n_2-n_3+i_1})$$

and we are ready. For $f_1 = n_1 - h - k > 0$ we collect in $(\underset{(1)}{d^{n_4}}\ldots)\,J_d$ and in $(\underset{(1)}{c^{n_3}}\ldots)\,J_c$ all symbols $\underset{(1)}{b}$, $\underset{(2)}{a}$, $\underset{(3)}{a}$, $\ldots$, $\underset{(\alpha)}{a}$ and arrive at

$$J_3 \equiv \Sigma\varrho\,(\underset{(1)}{d^{n_4}}\ \underset{(1)}{b^{i_1}}\ \underset{(1)}{a^{h_1}}\ldots)\,(\underset{(1)}{c^{n_3}}\ \underset{(1)}{b^{n_2-i_1}}\ \underset{(1)}{a^{k_1}}\ldots)\prod_{s=2}^{\alpha}(\underset{(s)}{d^{n_4}}\ \underset{(s)}{a^{j_s}}\ \underset{(1)}{a^{h_s}}\ldots)\,(\underset{(s)}{c^{n_3}}\ \underset{(1)}{a^{n_1-j_s}}\ \underset{(1)}{a^{k_s}}\ldots)$$

$$(h_1 \geqq h;\ k_1 \geqq k;\ \sum_{s=2}^{\alpha}(h_s + k_s) = n_1 - h_1 - k_1 \leqq f_1).$$

From

$$(d^{n_4}_{(2)}\, a^{n-n_4}_{(2)})\, d^{n_4}_{(3)}\, a_{(2)} \ldots) J^* \equiv 0$$

it follows that this is also possible if $n_1 + n_4 > n$.

Now we make use of the identities

$$(d^{n_4}_{(s)}\, a^{j_s}_{(s)}\, a^{h_s}_{(1)} \ldots)\, (c^{n_3}_{(s)}\, a^{n_1-j_s}_{(s)}\, a^{k_s}_{(1)} \ldots) \equiv \Sigma \varrho\, (d^{n_4-\sigma}_{(s)}\, a^{n_1}_{(s)} \ldots)\, (c^{n_3}_{(s)}\, d^{\sigma}_{(s)}\, a^{h_s+k_s}_{(1)} \ldots) \equiv$$

$$\equiv \Sigma \varrho\, (d^{n_4}_{(s)}\, a^{n_1-l_s}_{(s)} \ldots)\, (c^{n_3}_{(s)}\, a^{l_s}_{(s)}\, a^{h_s+k_s}_{(1)} \ldots)$$

for $s = 2, 3, \ldots, \alpha$, and after that draw all symbols $a_{(1)}$ contained in c-factors in $(c^{n_3}_{(1)} \ldots)$, which leads to

$$J_3 \equiv \Sigma \varrho\, (d^{n_4}_{(1)}\, b^{i_1}_{(1)}\, a^{h_1}_{(1)} \ldots)\, (c^{n_3}_{(1)}\, b^{n_2-i_1-g_1}_{(1)}\, a^{n_1-h_1}_{(1)} \ldots)\, J^*_c\, J^*_d,$$

where $g_1 \leqq \sum_{s=2}^{\alpha} (h_s + k_s) \leqq f_1$ and $h_1 \geqq h$.

The terms of this sum with $g_1 = 0$ are invariants that contain $(d^{n_4}\, b^{i_1}\, a^{h_1})\, (c^{n_3}\, b^{n_2-i_1}\, a^{n_1-h_1})$ as a factor and can put out of court in future; those with $g_1 > 0$ again must be considered. Each of them equals a sum of invariants of the form

$$J_4 = (d^{n_4}_{(1)}\, b^{i_1}_{(1)}\, a^{h_1}_{(1)} \ldots)\, (c^{n_3}_{(1)}\, b^{n_2-i_1-g_2}_{(1)}\, a^{n_1-h_1}_{(1)} \ldots) \prod_{s=2}^{\alpha} (d^{n_3}_{(s)}\, b^{m_s}_{(s)} \ldots)\, (c^{n_3}_{(s)}\, b^{n_2-m_s}_{(s)}\, b^{p_s}_{(1)} \ldots)$$

$\left(g_2 = \sum_{s=2}^{\alpha} p_s \leqq g_1\right)$. For $g_2 = 0$ the invariant J_4 contains a factor

$$(d^{n_4}\, b^{i_1}\, a^{h_1})\, (c^{n_3}\, b^{n_2-i_1}\, a^{n_1-h_1}).$$

For $g_2 > 0$ we transform J_4 by means of the identities

$$(d^{n_4}_{(s)}\, b^{m_s}_{(s)} \ldots)\, (c^{n_3}_{(s)}\, b^{n_2-m_s}_{(s)}\, b^{p_s}_{(1)} \ldots) \equiv \Sigma \varrho\, (d^{n_4}_{(s)}\, c^{\tau}_{(s)}\, b^{p_s}_{(1)} \ldots)\, (c^{n_3-\tau}_{(s)}\, b^{n_2}_{(s)} \ldots) \equiv$$

$$\equiv \Sigma \varrho\, (d^{n_4}_{(s)}\, b^{q_s}_{(s)}\, b^{p_s}_{(1)} \ldots)\, (c^{n_3}_{(s)}\, b^{n_2-q_s}_{(s)} \ldots)$$

$(s = 2, 3, \ldots, \alpha)$ and after that gather as many as possible symbols $b_{(1)}$ occurring in d-factors in $(d^{n_4}_{(1)} \ldots)$. Now we have $J_4 \equiv 0$ or

$$J_4 \equiv \Sigma \varrho\, (d^{n_4}_{(1)}\, b^{i_1+g_2}_{(1)}\, a^{h_1-f_2}_{(1)} \ldots)\, (c^{n_3}_{(1)}\, b^{n_2-i_1-g_2}_{(1)}\, a^{n_1-h_1}_{(1)} \ldots)\, \bar{J}_c\, \bar{J}_d,$$

where $f_2 \leqq g_2 \leqq g_1 \leqq f_1$, $n - n_4 - i_1 \geqq g_2 > 0$.

The terms of this sum with deficit $f_2 = 0$ contain a factor

$$(d^{n_4}\, b^{i_1+g_2}\, a^{h_1})\, (c^{n_3}\, b^{n_2-i_1-g_2}\, a^{n_1-h_1})$$

and are reducible invariants. If there exist other terms, they have a deficit f_2 that does not differ from the deficit f_1 of J_3, it is true, but the number of symbols $b_{(1)}$ contained in the bracket factor $(d^{n_4}_{(1)} \ldots)$ of such a term surpasses the corresponding number for J_3. After one or more repetitions of

the process described in this section the non vanishing invariants with a deficit different from zero, that then appear, are necessarily of the form

$$(\underset{(1)}{d^{n_4}}\ \underset{(1)}{b^{n-n_4}})\,(\underset{(1)}{c^{n_3}}\ \underset{(1)}{b^{n_2+n_4-n}}\ \underset{(1)}{a^{n_1-f}}\ \ldots)\,J_c\,J_d \qquad (f>0)$$

and gathering of the symbols $\underset{(1)}{a}$ in $(\underset{(1)}{c^{n_3}}\ldots)$ on the prescribed manner furnishes exclusively vanishing and reducible invariants.

Thus the proof of the contention at the end of the preceding section is delivered. In general (IV) an integrity basis is composed by the invariants

$$I_p=(d^{n_4}\,a^p\,b^{n-n_4-p})\,(c^{n_3}\,a^{n_1-p}\,b^{n-n_1-n_3+p}) \qquad (p=0,1,\ldots,T),$$

where T is an integer defined as follows

$$T=n_1 \qquad \text{for } n_1\leqq n-n_4$$
$$T=n-n_4 \qquad \text{for } n_1\geqq n-n_4.$$

If there are pairs of with another dual spaces, we must add two (III), four (II) or six (I) of the invariants

$$(d^{n_4}\,a^{n_1}),\ (c^{n_3}\,b^{n_2});\ (d^{n_4}\,b^{n_2}),\ (c^{n_3}\,a^{n_1});\ (d^{n_4}\,c^{n_3}),\ (a^{n_1}\,b^{n_2}).$$

Aerodynamics. — *Un amplificateur pour l'étude de la turbulence d'un écoulement d'air.* By R. BETCHOV and E. KUYPER. (Mededeling no. 52 uit het Laboratorium voor Aero- en Hydrodynamica der Technische Hogeschool.) (Communicated by Prof. J. M. BURGERS.)

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Nous donnons ici la description d'un amplificateur permettant d'étudier les tensions électriques alternatives produites sur un anémomètre à fil chaud par les fluctuations d'un écoulement d'air. Notre dispositif permet la mesure de la turbulence, du facteur de corrélation ainsi que le contrôle visuel du degré de corrélation.

1. *Principes généraux.*

On sait que la résistance d'un fil de platine dépend fortement de sa température, ce qui permet d'étudier la vitesse d'un vent en mesurant la résistance d'un fil chauffé par un courant continu et exposé au vent. Le vent refroidit le fil et fait tomber sa résistance. La sensibilité du système croit avec la température du fil, mais l'étalonnage devient instable si celle ci est trop élevée.

On travaille généralement avec une température constante (une résistance constante) en ajustant le courant de chauffage. Le fil chaud est monté dans un pont de WHEATSTONE et étalonné en mesurant la vitesse du vent avec un tube de PITOT. Une grandeur importante est la dérivée du courant par rapport à la résistance pour une vitesse constante et un état donné du fil; cette grandeur est facilement mesurée avec le pont.

Lorsque l'écoulement de l'air se fait avec turbulence, la vitesse fluctue autour d'une valeur moyenne et le refroidissement irrégulier produit des fluctuations de la résistance du fil. Si l'intensité de chauffage est indépendante de la résistance du fil, on obtient donc des fluctuations de tension aux bornes du fil chaud.

La fluctuation de la vitesse du vent peut être décomposée en un spectre d'oscillations, partant de basses fréquences. L'inertie thermique du fil réduit l'amplitude des variations de résistance dès que la fréquence dépasse 10 à 30 cycles, et produit également un déphasage. Il est donc désirable d'utiliser un amplificateur qui puisse compenser ces effets.

Pour comparer les fluctuations du vent en deux points différents du champ d'écoulement, on utilise deux fils chauds, aussi petits que possibles (diamètre 3 à 15 microns, longueur 1 à 2 mm), placés dans des ponts semblables, et on étudie les tensions produites. C'est pour cette raison que nous avons construit un amplificateur symétrique, qui a l'avantage d'assurer également une meilleure réponse avec un seul fil chaud.

On peut filtrer le signal du fil chaud pour étudier le spectre de FOURIER de la turbulence.

2. *Théorie du fil chaud.*

Le fil chaud de résistance R traversé par un courant I reçoit par secondes RI^2 joules; l'énergie emportée par le vent est donnée par la formule de KING, soit avec V vitesse du vent, T la différence entre la température du fil et celle de l'air:

$$\text{énergie emportée par seconde} = (a + b\sqrt{V})\,T,$$

avec a et b deux constantes.

Si ces deux transports d'énergie ne sont pas égaux, le fil s'échauffe. Avec m = masse du fil, s = chaleur spécifique et compte tenu du passage des calories aux joules, on obtient:

$$RI^2 = 4{,}2\,ms\,(dT/dt) + (a + b\sqrt{V})\,T \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Si la vitesse du vent est très faible (inférieure à 1 mètre par seconde) la convection intervient et la relation (1) n'est plus valable. Nous négligeons ici la conduction qui transmet des calories au support du fil chaud et abaisse la température aux extrémités. Cet effet peut être important lorsque le fil est court (quelques millimètres).

La relation entre la résistance du fil et sa température est pratiquement linéaire et nous poserons:

$$R = R_0\,(1 + \alpha\,T) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

avec R_0 et α deux constantes si T ne varie pas trop. Enfin le courant I est lié à la résistance R, mais si le fil est alimenté à travers une résistance assez grande, le courant est pratiquement constant. Nous écrirons cependant que toute variation de R est accompagnée par une variation du courant selon:

$$\delta I = -h\,\delta R. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

où la constante h dépend du circuit de chauffage du fil. Nous supposons que dans cette relation n'interviennent ni effets d'inductions, ni effets de capacités.

En l'absence de turbulence, le fil prend un état d'équilibre avec $dT/dt = 0$. On a alors:

$$RI^2 = \frac{a + b\sqrt{V}}{\alpha R_0}\,(R - R_0) \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Nous introduisons ici deux grandeurs que l'on peut déduire de l'étalonnage du fil:

$$\left.\begin{aligned} I_0^2 &= \frac{a}{\alpha R_0}\left(1 - \frac{R_0}{R}\right) \\ \left(\frac{\partial I}{\partial R}\right)_V &= \frac{a + b\sqrt{V}}{2\alpha R_0 R I} - \frac{I}{2R}. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (5)$$

Il convient de remarquer que I_0 n'est pas l'intensité du courant lorsque la vitesse du vent est nulle (effet de convection), mais une grandeur obtenue par extrapolation. Pour la déterminer on fait un diagramme avec en abcisse la racine de V et en ordonnée le carré de I; la droite ainsi obtenue donne I_0^2.

Lorsque la vitesse du vent fluctue, l'inertie thermique intervient et avec une fluctuation δV on obtient l'équation:

$$I^2\delta R + 2RI\delta I = \frac{4{,}2\,ms}{\alpha R_0}\frac{d\delta R}{dt} + \frac{1}{2}\frac{\delta V}{V}R(I^2-I_0^2) + \left\{\left(\frac{\partial I}{\partial R}\right)_V + \frac{I}{2R}\right\}2RI\delta R \quad . \quad (6)$$

En utilisant (3) on a:

$$2RI\left\{\left(\frac{\partial I}{\partial R}\right)_V + h\right\}\delta R + \frac{4{,}2\,ms}{\alpha R_0}\frac{d\delta R}{dt} = -\frac{1}{2}\frac{\delta V}{V}R(I^2-I_0^2).$$

Considérons δV comme un oscillation harmonique et écrivons $\varepsilon^{j\omega t}$ au lieu de δV, avec $r\,\varepsilon^{j\omega t}$ au lieu de δR et $j = \sqrt{-1}$. On obtient:

$$r\left(1 + j\frac{\omega}{\omega^*}\right) = -\frac{1}{4}\frac{v}{V}\frac{I^2-I_0^2}{I\{(\partial I/\partial R)_V + h\}} \quad . \quad . \quad . \quad . \quad . \quad (7)$$

avec

$$\omega^* = \frac{2\alpha R_0}{4{,}2\,ms}\left\{\left(\frac{\partial I}{\partial R}\right)_V + h\right\}RI \quad . \quad . \quad . \quad . \quad . \quad (8)$$

La tension alternative aux bornes du fil chaud sera $\delta e = R\delta I + I\delta R$ et en posant $\delta e = e\varepsilon^{j\omega t}$ nous obtenons:

$$e = Ir\left(1 - \frac{hR}{I}\right) = -\frac{1}{4}\frac{v}{V}\frac{I^2-I_0^2}{(\partial I/\partial R)_V + h}\frac{1-hR/I}{1+j\omega/\omega^*} \quad . \quad . \quad (9)$$

On voit que la tension alternative reproduit les fluctuations avec déphasage et réduction des amplitudes dès que ω est voisine ou supérieure à la fréquence propre du fil chaud ω^*. En général ω^* est de l'ordre de 50 cycles par seconde. Dans la littérature on emploie souvent au lieu de ω^* la grandeur:

$$M = 2\pi/\omega^* \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

3. *Ponts et préamplificateurs.*

La figure 1 montre les deux ponts alimentés par rapport à la terre par des batteries d'accumulateurs. Les tensions alternatives provoquées dans les fils chauds sont transmises aux grilles des CF 50 par des condensateurs de 20 mf, qui servent également à mettre à la terre les tensions de souffle des résistances des grilles à travers la faible résistance des fils chauds. Ce dispositif d'entrée est nécessaire car la tension des fils est variable et il est préférable de contrôler la polarisation de ces tubes.

On peut ajuster ces polarisations de manière à avoir des amplifications

égales. Il n'est pas nécessaire de blinder les lignes reliant les fils chauds aux condensateurs d'entrée.

La compensation de l'inertie du fil chaud se fait dans le circuit d'anode des mêmes CF 50 où l'intensité commandée par la grille traverse une self de $L = 10$ hy et une résistance r variable de 400 à 10000 ohms.

La tension de sortie ainsi produite, avec une pente k de la lampe donne:

$$e_{\text{sortie}} = k\,(r + j\,L\,\omega)\,e_{\text{grille}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

En combinant avec (9) on obtient pour une fluctuation v du vent de fréquence $\omega/2\pi$:

$$e_{\text{sortie}} = -\frac{kr}{4}\,\frac{1 + j\omega/\omega_0}{1 + j\omega/\omega^*}\,\frac{I^2 - I_0^2}{(\partial I/\partial R)_V + h}\,\frac{v}{V}\left(1 - \frac{hR}{I}\right) \quad . \quad . \quad (12)$$

avec $\omega_0 = r/L$.

Il faut donc ajuster le circuit r; L de manière à ce que $\omega_0 = \omega^*$, mais on ne peut modifier que la résistance r et il convient de l'éliminer en écrivant:

$$e_{\text{sortie}} = \frac{-kL\omega^*}{4}\,\frac{\omega_0 + j\omega}{\omega^* + j\omega}\,\frac{I^2 - I_0^2}{(\partial I/\partial R)_V + h}\,\frac{v}{V}\left(1 - \frac{hR}{I}\right) \quad . \quad . \quad (13)$$

Si la fluctuation du vent est composée d'un spectre de composantes de fréquences différentes, la compensation est parfaite en amplitude et en phase pour autant que $\omega_0 = \omega^*$ et que les capacités parasites soient négligeables. Pratiquement on aura un écart entre ω_0 et ω^*, mais si la plus grande part du spectre de la fluctuation de la vitesse est formée de fréquences supérieures à ω^*, la formule (13) exprime une relation presque constante entre la turbulence et la tension de sortie. L'erreur sur ω_0 agit sur les composantes de fréquence inférieure à ω^*.

Il est donc plus important de connaitre la valeur de ω^* que d'ajuster la résistance r à sa bonne valeur.

On a donc, avec $\tau = v/V =$ turbulence, et compte tenu de (8), en première approximation:

$$e_{\text{sortie}} = -\frac{kL}{4}\left\{\frac{2\,a\,R_0}{4{,}2\ ms}\right\} R\,I\,(I^2 - I_0^2)\,\tau \quad . \quad . \quad . \quad . \quad . \quad (14)$$

Le transformateur de sortie élimine les fréquences inférieures à 10 cycles. Au delà de 2000 cycles la capacité des selfs produit une résonnance autour de environ 3500 cycles, la réponse du préampli est indiquée sur la figure 2. Un atténuateur termine cet étage.

Pour atténuer le flux du champ ambiant à 50 périodes, nous avons placé les selfs avec les lampes dans un blindage magnétique épais de 5 mm et pesant environ 3 kgs. Les selfs sont placées dans le prolongement l'une de l'autre et leurs fers se touchent. En orientant convenablement leur direction on peut ainsi obtenir un signal perturbateur minimum et symétrique.

Les sorties du préampli sont reliées à deux amplificateurs dont l'un fait la différence et l'autre la somme (amplificateurs D et S).

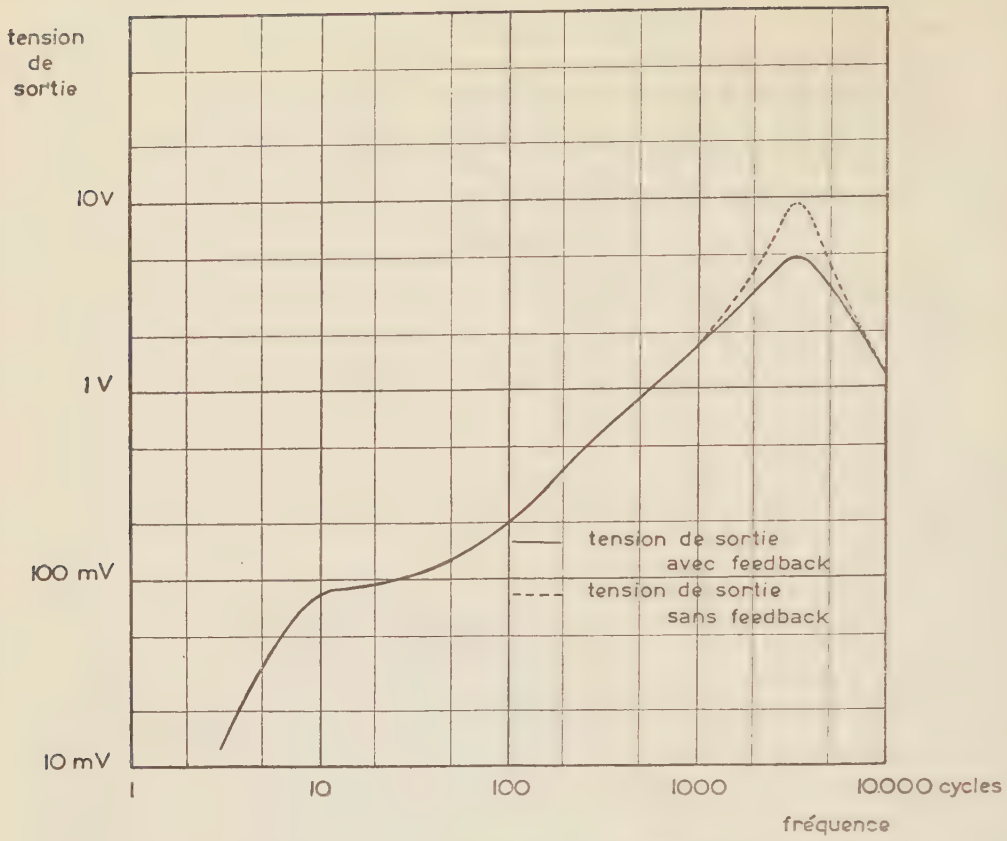


Fig. 2. *Réponse en fréquence des amplificateurs.*
La tension de sortie est mesurée au secondaire du transformateur, lorsque l'une des grilles des *CF* 50 reçoit 1 mV (bruit de souffle à la sortie: environ 7 mV).
Valeur de ω^* : environ 50 cycles.

4. *L'amplificateur D.*

Cet amplificateur est monté en push-pull et il donne la différence entre les deux tensions de sortie des préamplis. Il supprime également les perturbations symétriques. Deux étages montés selon la figure 1 mènent soit à un transformateur de sortie soit à la paire de plaques symétriques (déviation verticale) d'un oscillographe. La réponse avec un seul fil chaud est donc améliorée par la présence de la deuxième ligne d'amplification, car l'accroissement du bruit de souffle est négligeable à côté de la suppression du bruit de fond.

Le secondaire du transformateur est relié à un thermocouple qui donne une indication proportionnelle à la moyenne du carré de la fluctuation.

Pour améliorer la réponse en fréquence du thermocouple il faut compenser la résonnance des selfs du préampli autour de 3000 cycles et un circuit self-capacité reliant le secondaire du transfo à l'une des grilles *EL* 3 permet d'obtenir la réponse de la figure 2 (feedback). Ce circuit n'est pas utilisé avec l'oscillographe car le contrôle est alors essentiellement qualitatif.

Lorsque les deux ponts sont alimentés avec des tensions de mêmes signes, le thermocouple enregistre la moyenne du carré de la différence des deux fluctuations de la vitesse du vent; si les tensions d'alimentations des ponts sont de signes opposés il indique la moyenne du carré de la somme, soit:

$$\left.\begin{aligned} E_{th} &= c_{ste}\,\overline{(v'-v'')^2} \\ E^*_{th} &= c_{ste}\,\overline{(v'+v'')^2} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (15)$$

avec E_{th} la tension fournie par le thermocouple, v' et v'' les fluctuations aux deux points observés. On peut ainsi déduire la corrélation:

$$K = \frac{E^*_{th}-E_{th}}{E^*_{th}+E_{th}} = \overline{\left(\frac{v'\,v''}{\frac{1}{2}(v'^2+v''^2)}\right)} \quad . \quad . \quad . \quad . \quad . \quad (16)$$

5. *L'amplificateur S.*

La mesure d'une corrélation est compliquée et nous avons cherché à rendre la corrélation visible à l'oscillographe. Pour cela un amplificateur effectue la somme des tensions de sortie des préamplis (voir figure 1) et son signal est appliqué aux plaques asymétriques (déviation horizontale) du tube cathodique. Le contrôle de la bonne marche des amplificateurs se fait en appliquant aux grilles *CF* 50 un signal symétrique ou push-pull et l'on peut obtenir une réponse en phase satisfaisante. Si l'une des pentodes *EF* 6 de l'ampli *S* amplifie plus que l'autre, on peut réduire sa pente en plaçant une petite résistance dans son circuit d'écran.

Deux fils chauds placés dans un écoulement turbulent produisent alors une figure en forme de rosace, à symétrie circulaire s'il n'y a pas de corrélation, en forme d'ellipse dans le cas contraire. Tout se passe comme si l'on étudiait directement les tensions des fils chauds dans un système d'axes inclinés de 45 degrés et orthogonaux.

Ce dispositif signale également la présence de corrélations intermittentes.

6. *La protection du thermocouple.*

Cet instrument est délicat et une rupture de fil chaud, une panne dans l'alimentation du secteur, une variation brusque du régime d'écoulement de l'air etc. peuvent suffire à le brûler. Nous avons donc utilisé un thyratron et un relais pour le mettre hors circuit dès qu'un signal trop fort est appliqué au transfo.

La grille du thyratron est rendue négative par une polarisation ajustable et un signal trop fort provoque la décharge, la fermeture du relais et l'allumage d'une lampe témoin.

Le retour à l'état normal se fait en interrompant le courant de la cathode. La constante de temps du relais est améliorée par une résistance, parallèle au thyratron, produisant un courant trop faible pour fermer le relais.

Enfin on peut remplacer le thermocouple par une résistance équivalente de 20 ohms, ceci pour régler la polarisation de la grille du thyratron et contrôler la bonne marche de l'amplificateur.

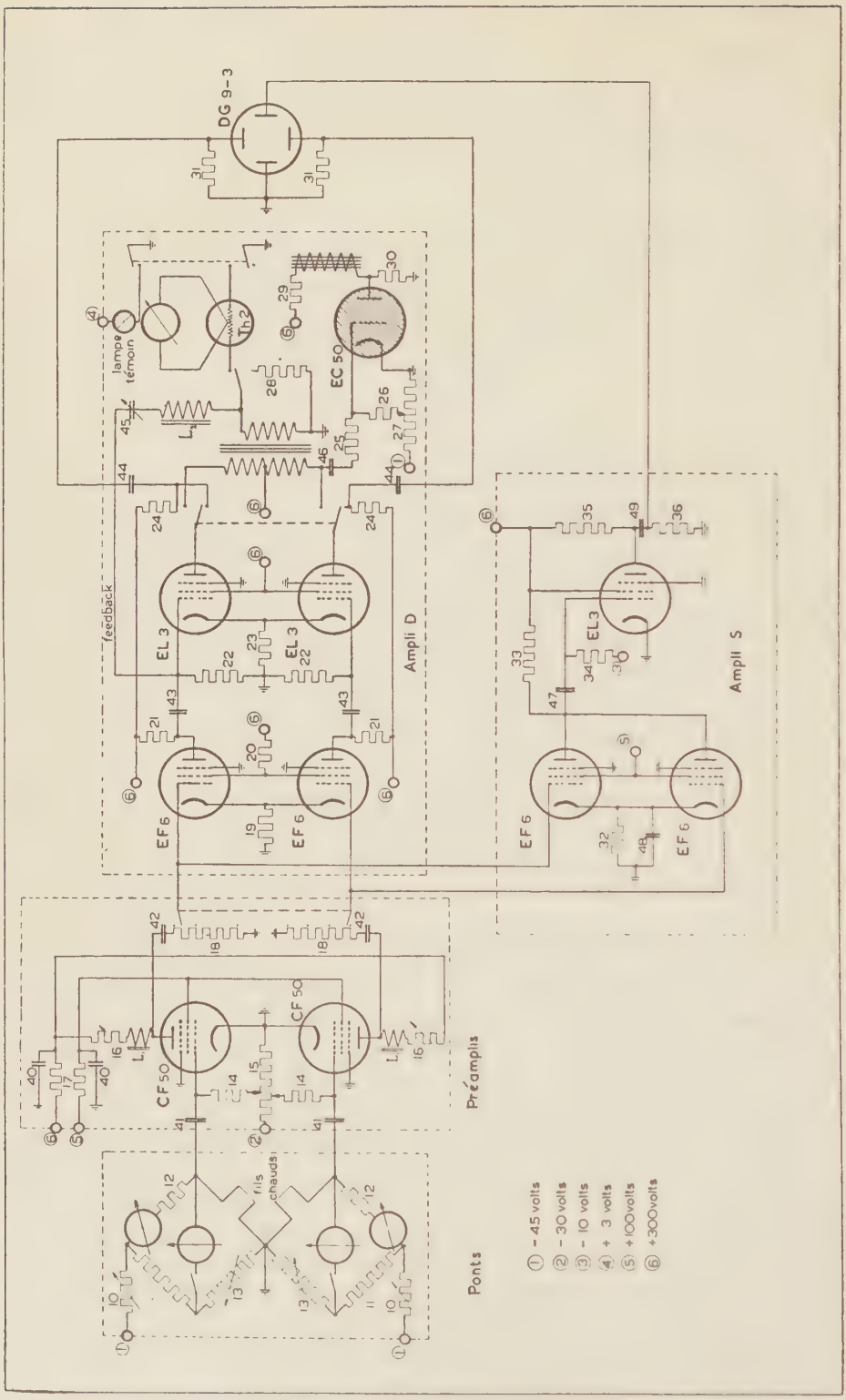


Fig. 1. *Schéma général des amplificateurs.*

7. *La détermination de ω^*.*

Nous avons vu que la grandeur ω^*, qui dépend du régime du fil chaud, doit être connue avec précision. Nous étudions actuellement un pont de mesure, en courant alternatif, permettant d'étudier la réponse en fréquence du fil chaud. Nous espérons publier sous peu la solution adoptée.

Valeurs des éléments de la figure 1.

10	600 ohms ; 100 watts	30	50 K Ω
11	1 K Ω (kilo-ohm)	31	1 M Ω
12	100 Ω 20 watts	32	1500 Ω
13	0 à 1 K Ω	33	75 K Ω
14	20 K Ω	34	1 M Ω
15	10 K Ω	35	15 K Ω
16	0 à 10 K Ω	36	1 M Ω
17	30 K Ω	40	100 mf (microfarad)
18	1 M Ω (10^6 ohm)	41	20 mf
19	700 Ω	42	0,25 mf
20	220 K Ω	43	0,25 mf
21	100 K Ω	44	0,25 mf
22	1 M Ω	45	env. 300 mmf (10^{-12} farad)
23	150 Ω	46	0,1 mf
24	5 K Ω ; 3 watts	47	0,2 mf
25	1 M Ω	48	2000 mf
26	0,5 M Ω	49	0,1 mf
27	25 K Ω		
28	20 Ω	L_1	10 hy et 400 Ω
29	30 K Ω	L_2	5 hy et 800 Ω

Botany. — *Researches on plant growth regulators.* XIII. *Leaf growth factors.* I. By W. KRUYT and H. VELDSTRA. (Communicated by Prof. V. J. KONINGSBERGER.)

(Communicated at the meeting of September 27, 1947.)

Introduction.

The growth which may be observed in different parts of the plant, is a very complicated process, in the regulation of which several phytohormones play a part.

The growth of the leaf is also influenced by these substances, whereby one must distinguish between the growth of the veins which is influenced by auxin (AVERY (1), WENT and THIMANN (2), WENT (3)) and the growth of the mesophyl, which is independent of auxin. WENT showed that in etiolated pea-seedlings the leaf growth is dependent on the growth factors accumulated in the cotyledons. BONNER, HAAGEN–SMIT and WENT (4) in 1939 elaborated a test to be able to compare by means of pieces of leaf blades of *Raphanus* and *Nicotiana* the activity of leaf growth factors.

In 1939 BONNER and HAAGEN–SMIT (5) published a comprehensive article on the activity of several synthetic substances as leaf growth factors. As might be expected the test object also plays an important part. Of the examined amino acids arginine for example proved to be the most active one for the growth of leaf-discs of *Nicotiana sylvestris,* whilst this substance was inactive in *Raphanus*. Therefore caution is a first necessity especially when different authors are comparing their results. Even if experiments are carried out on a similar species there is always a possibility of difference in variety or race.

Of the purine derivatives *adenine* was especially active even in solutions as dilute as 20 μg per litre. BONNER and HAAGEN–SMIT therefore used several other methods to study this exceptional activity. Whole leaves of ten days old etiolated pea-seedlings were cut off and put into different solutions under sterile conditions. After five weeks the growth in "pea-diffusate" proved to give the best results; 2 mg l adenine added to a mixture of 1 % cane-sugar and inorganic salts caused an inhibition, whereas the addition of 0.2 mg/l adenine caused an important increase in growth with regard to the control. Here the effect of adenine was comparable with that in the leaf growth test.

In the cultivation of pea-roots a stimulation of growth was also noticed though this was not apparent till after six transferences. The effect was less than the increase of growth caused by vitamin B_1 and nicotinic acid.

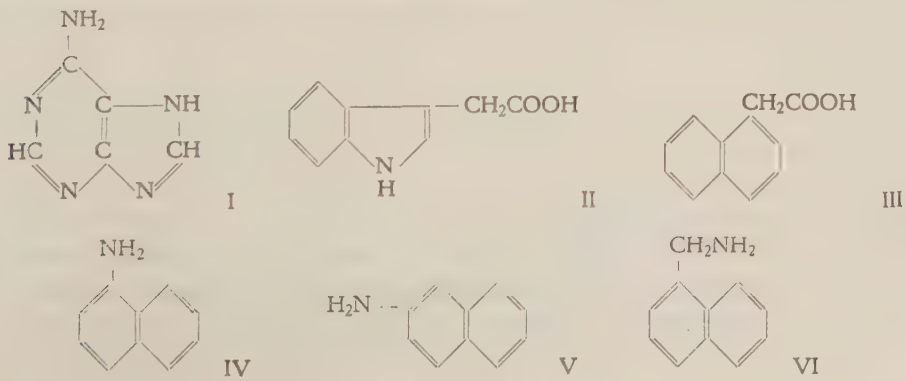
Finally BONNER and HAAGEN–SMIT examined the influence of adenine on the growth of Cosmos plants, cultivated in a hothouse in washed sand,

regularly watered with nutrient-solution. The cultivation took place in daylight, both short and long photoperiods being applied. The first two or three weeks no differences were observed but after that period the plants treated with 0.1 mg l adenine became steadily larger than the controls, whereby the leaves in particular showed an increase in size. A concentration of 0.5 mg/l already caused inhibition. Under long-day condition the influence of adenine was especially noticeable on the longitudinal growth, whilst under short-day condition the measurements of the leaves were chiefly affected. Thus BONNER and HAAGEN–SMIT stated that the addition of adenine to plants may cause an increase of leaf-surface under certain conditions. A picture showed a very clear difference between leaves of control plants and treated specimens (adenine 0.1 mg/l), five weeks after germination.

As a result of the increase of leaf-surface after addition of adenine there will be an increase in the production of e.g. auxin and vitamin B_1 in these leaves. Therefore the addition of adenine will indirectly influence the growth of stem and root and a better developed plant may result. D. M. BONNER and J. BONNER (6) report that by adding 0.1 mg/l adenine to the nutrient solution in *Cosmos sulphureus* an important increase in the dry weight of the seedling (up to as much as three times) may be the result; the roots would show only a slight increase.

Uric acid, the structure of which is closely related to adenine, showed in cultivation experiments with Cosmos an effect similar to adenine, although it had little effect in experiments on leaves (BONNER and HAAGEN–SMIT). Identical results were obtained with *Brassica alba*. The combination of adenine and uric acid caused a result corresponding with the effect of twice the amount of each component separately.

In connection with our investigations on the practical applications of plant growth regulators a further study of the above-mentioned effects of adenine was considered desirable. On account of the high growth substance activity of α-naphthalene acetic acid (III), to be considered as a structural imitation of hetero-auxin (II), we thought it attractive to include some easily accessible substances the structure of which reminds to a certain extent of that of adenine (I).



For this reason α-naphthylamine (IV) was examined and in comparison to it also its β-isomer (V). Because of the typical function of the CH_2-group in the side-chain of the acids, derived from indole or naphthalene and active as growth-substances (see VELDSTRA (7)), α-(aminomethyl-) naphthalene (VI) was added as a counterpart of these acids.

The differences in properties between the ring-systems of indole and naphthalene being certainly less than those between the purine- and naphthalene nuclei, the comparison between adenine and α-naphthylamine may seem to be a risky one. This choice, however, was also deemed justifiable on account of the fact that of the naphthylamines and closely connected compounds typical reactions in other aspects are already known (compare (8)).

Material and methods.

In general we decided to follow the methods used by BONNER and HAAGEN–SMIT (5), i.e. the cultivation in sand, to which nutrient solution is added regularly. The above-named authors, however, have given no further details as to the Cosmos used; from the published photographs it can be concluded only that it was a fine-leaved type. In the article by D. M. BONNER and J. BONNER (6) *Cosmos sulphureus* is mentioned as an object that would react to adenine as a leaf growth factor. Therefore we started with the use of seeds of *Cosmos sulphureus* (harvest 1942) obtained from the Hortus at Leyden. During cultivation, however, it proved not to be a fine-leaved type. Comparison of our material with plants from the State-herbarium at Leyden and with reproductions in Cavanilles Icones (9) in which this plant is described for the first time, shows that *Cosmos sulphureus* Cav. is indeed a broad-leaved type so that we must conclude that BONNER and HAAGEN–SMIT did not use the genuine *Cosmos sulphureus* in their experiments.

On account of the war we had no opportunity of receiving further information from the authors.

After our first experiment with *Cosmos sulphureus* Cav. the work was continued with the seed of the fine-leaved type *Cosmos bipinnatus* Cav. called "Sensation Innocene" (seed Nr 896 of C. G. van Tubergen's Bulb and Seed Trade Ltd, Nursery Zwanenburg — Haarlem), a white flowering Cosmos.

BONNER and HAAGEN–SMIT (5) do not give a detailed specification as to the nutrient-solution used. They speak of a Shive's solution but as there exist several Shive's solutions (10) this description is insufficient. The nutrient-solution which we used (according to Shive) had the following composition: $Ca(NO_3)_2 \cdot 4\,H_2O$, 1.2 g/l; $MgSO_4$. ca $3\,H_2O$, 2.6 g/l; KH_2PO_4, 2.5 g/l; $Fe_2(PO_4)_2$, trace. The cultivation took place in coarse sand which had been well washed with tapwater three times. The Cosmos seeds were first made to germinate in sowing-pans (inside measurements $26 \times 26 \times 6.5$ cm) or in boxes at regular distances (three cm square). It

proved to be of importance not to sow too deep because otherwise the seedlings had too much difficulty in penetrating the sand-layer, which might of course cause differences in development. The various groups of one experiment were watered immediately after the sowing with nutrient-solution with or without the addition of substances in different concentrations, which were to be tested on their leaf growth activity. To begin with cultivation was done under double glass (sometimes with soil-heating), but soon after the germination regular aeration took place and finally the plants were placed on a table in the centre of the hothouse. After some seven days the plants have sufficiently developed to be transplanted. We transferred them into pots (top diameter 9 cm, bottom diam. 5 cm, height 8 cm) with washed coarse sand. Of every group 20 to 40 best specimens were selected. These pots, each containing only one plant, were dug into sand on the cultivation table to prevent evaporation as much as possible. At first we kept every group apart but later on we randomized all the pots of one experiment-series so that only the order and the presence of a coloured stick showed to which group the plant belonged in order that in watering the right nutrient-solutions should be given. Only in this way it is possible to distribute favourable or unfavourable position-influences evenly over all the plants of one series.

The pots were watered from time to time with equal quantities of nutrient-solution and now and then with distilled water only to prevent accumulation of salts. After five or six weeks the plants of each group were measured and finally the fresh and the dry weight of aerial and subterranean parts were determined separately. Usually a selection of the material took place before the ending of the experiment in order that aberrant badly grown or abnormal specimens could be separated.

Beside in the hothouses of the "De Proeftuin" at Boskoop the experiments were also made in our own hothouse at Lunteren.

Experimental results.

1. *Experiment with Cosmos sulphureus Cav. at Boskoop.*

This experiment was started on March 10th 1943 and ended on April 22nd 1943. In addition to a control group which was watered exclusively with Shive's nutrient-solution there were five other groups to the nutrient-solutions of which adenine 0.1 mg/l and 0.5 mg/l, α-naphthylamine 0.1, 0.5 and 1.0 mg/l was added respectively as a leaf growth factor. This time the pots were not randomized.

At the end of the experiment the control plants did not yet show a flower-bud whereas in the other groups there were 3, 4, 1, 2 and 4 specimens respectively with a visible flower-bud. The results are summarized in table I.

On the whole the addition of α-naphthylamine in a concentration of 0.5 mg/l gave the best results. The longitudinal growth is not greatly influenced, more so, however, is the weight of the shoots and the roots. With the addition of adenine the best figures were noted at the highest concentration, viz. 0.5 mg/l which evidently here caused no inhibition yet.

However, we strongly suspect that differences in position have in-

TABLE I.

Group	Number of plants	Solution applied (March 10th 1943— April 22nd 1943)	Average length (mm)	Fresh weight (g) of:			Dry weight (g) of:	
				shoots	roots	10 leaves of the 10 best specimes	shoots	roots
1	20	Shive (= S)	84	48.3	26.8	4.1	4.97	3.10
2	20	S + adenine 0.1 mg/l	88	51.8	31.2	4.6	5.53	3.24
3	20	S + adenine 0.5 mg/l	90	59.4	36.3	5.2	6.32	3.92
4	20	S + α–N.A. *) 0.1 mg/l	81	52.4	32.4	4.8	5.78	3.83
5	20	S + α–N.A. 0.5 mg/l	80	61.1	40.2	5.4	6.75	4.83
6	20	S + α–N.A. 1.0 mg/l	83	58.0	37.4	5.6	6.42	3.84

*) N.A. = naphthylamine

fluenced the result in favour of the effect of the substances investigated. The six groups had been placed in order of the numbers behind each other whereby group 1 stood at the end of the table, rather close to a glass side-wall facing south, so that during sunshine it was practically all the time in the shade. In this way the other groups received more light as their distance to the wall increased.

2. *Experiment with Cosmos bipinnatus Cav. "Sensation Innocence" at Boskoop.*

The experiment with the fine-leaved Cosmos bipinnatus lasted from May 10th 1943 till June 21st 1943. The treatment of six different groups was quite similar to that of the former experiment. Watering was practically exclusively carried out on Monday, Wednesday and Saturday; each group alternately with the nutrient-solution in question and distilled water. A harmful accumulation of salts, as very likely occurred in a measure in our first experiment (some cases of curled leaves) is well-nigh excluded in this way. Each pot received 50 ml every time. The temperature of the air during the experiment was minimal 11.4° C and maximal 16.1 while the soil temperature was minimal 13.9 and maximal 22.1. The position of the groups was the same as in the preceding test.

At the breaking up of the experiment six weeks after sowing the flower-buds were not yet visible. Of each group the 25 best plants were selected and of these 25 the 10 very best specimens were kept apart. Beside this, one leaf of every third leafpair counting from the bottom (ignoring the seedleaves) was torn off. These leaves have been weighed and photographed separately.

We find the results obtained collected in table II.

The influence of adenine as a leaf growth factor may be practically neglected here and a dry-weight increase of 3 times as was stated by BONNER and BONNER (6) on *Cosmos sulphureus* was not obtained at all in this case. A possible influence of the use of Shive's solution or variety of Cosmos different from those in our experiments, should be taken into consideration, though it is doubtful whether this would have any such great influence.

An increase of rooting after addition of adenine is not found either. α-Naphthylamine reacts favourably on the development of the aerial

TABLE II.

Number of plants	Solution applied (May 10th 1943—June 21st 1943)	Average length (cm)	Fresh weight (g) of:						Dry weight (g) of:	
			10 leaves of the 10 best specimens	25 leaves	shoots of the 10 best specimens	25 shoots	roots of the 10 best specimens	25 roots	25 shoots	25 roots
25	Shive (= S)	27	2.6	5.6	39.5	78.7	8.9	18.7	6.2	1.6
25	S + adenine 0.1 mg/l	27	3.1	6.8	42.6	90.2	9.5	21.0	6.9	1.7
25	S + adenine 0.5 mg/l	26	3.0	6.4	42.7	91.2	10.2	21.6	6.9	1.7
25	S + α–N.A. *) 0.1 mg/l	26	3.3	7.4	42.4	93.6	9.9	22.1	7.4	1.9
25	S + α–N.A. 0.5 mg/l	27	3.3	7.7	46.8	101.3	11.1	24.6	8.0	1.9
25	S + α–N.A. 1.0 mg/l	28	3.9	9.3	54.1	122.4	12.6	29.6	9.7	2.4

*) N.A. = naphthylamine.

parts as well as on the rooting. There is no influence on the longitudinal growth; the treated plants are only more sturdy. In contrast with the results obtained in the experiment with *Cosmos sulphureus* (exp. 1) 0.5 mg/l proves to have not yet an optimal effect, so that the optimum may be at 1 mg/l or more.

The unfavourable influence of the position in the hothouse with respect to the development of the control group is here less obvious than in our first experiment. This may be explained if it is taken into consideration that during this second period the sun was already much higher so that the difference in exposure for the various groups was much less pronounced. We must however, take it into account that in consequence of the position the results may again be flattering for α-naphthylamine.

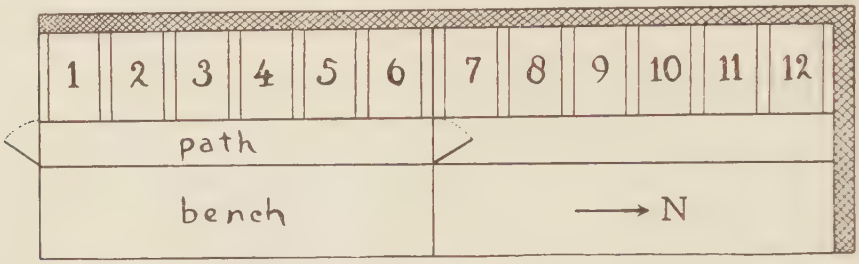
3. *Experiment with Cosmos bipinnatus Cav. "Sensation Innocence" at Lunteren.*

In this experiment which lasted from May 21st 1943 till July 1st 1943, the number of groups was twelve. Beside the control group and the groups treated with adenine in concentrations of 0.1 and 0.5 mg/l, α- and β-naphthylamine and α-(aminomethyl-) naphthalene were tested as leaf growth factors in concentrations of 0.1, 0.5 and 1.0 mg/l. The pots received 50 ml of liquid on Monday, Wednesday and Friday; alternately nutrient-solution and distilled water.

As we shall see from the results the placing of the plants in this experiment was not an ideal one either, so that differences in position should certainly be taken into consideration when judging the results. The place of cultivation was a hothouse situated in a N—S direction which was shut off by a stone wall on the W and the N sides and divided in two by a glass partition. Beside this, part of the roof of the rear half was frosted. The bed of sand lay alongside the western wall so that plants placed in it only received direct sunlight for part of the day. The placing of the different groups in the sand bed was done as is shown in the sketch (see p. 1148). Undoubtedly the glass partition together with the two walls have influenced the development of various groups unfavourably.

It should be noted that some treatments (especially those with α-naphthylamine 0.5 mg/l) had caused, as early as the seedling stage, a development of a stronger (and more branched) root system. This was observed when the plants were transferred from the sowing trays into the pots. A closer investigation as to the action of α-naphthylamine on the development of the root system of seedlings of Cosmos and other plants might

perhaps produce interesting information. With some experiments which we performed in this direction with Cosmos bipinnatus "Sensation Innocence" and tomato "Potentaat" these results did not always prove to be reproducable. In this matter evidently a number of yet practically unknown factors plays a part and therefore it is not always possible to reach



Arrangement of the groups of experiment 3 on the bench along the side-wall of the greenhouse at Lunteren.

the same starting-point. Meanwhile the reaction of this substance is also studied in the cultivation *in vitro* of pea- and tomato-roots and of pea-embryos.

At the end of the test on July 1st 1943, nearly six weeks after the sowing, the 30 best plants of the 35 specimens of every group were selected. Of these the 10 very best ones were kept apart. Only on one specimen (of group 10) a flower-bud was visible. From each plant one leaf of every third real leaf pair counting from the bottom (ignoring the seed leaves) was torn off. These leaves were again weighed separately while fresh. From each group six of the best plants were photographed and photos were also taken of the picked leaves from the 10 very best plants and of the roots. The results of this experiment are collected in table III.

TABLE III.

The first number in each column refers to the 10 best plants; the second number refers to the whole group of 30 plants.

Group	Solution applied (May 21st 1943— July 1st 1943)		Average length in cm	Fresh weight (g) of:			Dry weight (g) of:	
				leaves	shoots	roots	shoots	roots
1	Shive (= S)		30/28	4.3 /12.0	55.0 /142.5	19.6/51.9	4.8 /12.3	1.6 /4.8
2	S + adenine	0.1 mg/l	31/27	5.15/12.75	66.7 /155.9	27.7/60.2	5.9 /13.7	2.1 /5.3
3	S + „	0.5 „	31/29	5.1 /12.4	67.35/156.45	26.0/58.9	6.0 /13.6	2.4 /6.05
4	S + α–N.A. *)	0.1 mg/l	30/28	4.2 /10.7	59.2 /149.85	22.3/51.8	5.0 /12.35	1.8 /4.0
5	S + „	0.5 „	33/31	4.4 /11.8	64.4 /164.6	20.6/50.1	5.35/13.45	1.6 /4.3
6	S + „	1.0 „	33/30	4.9 /11.2	67.7 /148.2	21.5/43.2	5.5 /11.6	1.85/4.0
7	S + β–N.A.	0.1 mg/l	31/28	3.2 / 7.55	46.3 /109.9	12.9/26.5	3.6 / 8.25	0.95/2.35
8	S + „	0.5 „	31/29	4.8 /11.4	65.3 /154.4	19.8/48.9	5.4 /12.6	1.8 /4.35
9	S + „	1.0 „	33/30	4.3 /11.85	66.5 /167.0	21.6/52.3	5.45/13.55	1.7 /4.5
10	S + α–A.M.N. **)	0.1 mg/l	31/27	4.6 /11.3	61.7 /150.5	20.1/46.6	5.05/12.0	1.8 /4.15
11	S + „	0.5 „	30/28	3.7 / 9.9	50.8 /134.9	14.8/39.0	4.3 /11.15	1.3 /3.75
12	S + „	1.0 „	26/25	3.5 / 8.6	43.6 /105.6	12.2/28.6	3.5 / 8.35	1.1 /2.45

*) N.A. = naphthylamine.
**) A.M.N. = (aminomethyl-)naphthalene.

The influence of adenine and naphthylamine is here less than it proved to be in the first experiment. At an earlier stage (about a week before the ending of this experiment) the mutual differences between the groups

appeared greater. For a correct judgment of the differences between the activity of α- and β-naphthylamine and α-(aminomethyl-)naphthalene the acquired results are of little use. It is remarkable, however, that 0.5 mg/l α-naphthylamine produces nearly the same effect as 1.0 mg/l β-naphthylamine.

In what degree position differences here have had an influence is very difficult to ascertain. The low figures concerning the fresh- and dry-weights of the aerial parts of group 7 might be ascribed to the influence of the partition in the hothouse. Since, however, the average length of the plants of group 7 corresponds with that of the control group we must certainly be careful with this conclusion. It is quite possible that plants of group 7 have stretched more in length as a result of bad exposure to light than would have been the case under favourable conditions. We believe that the reason for the low figures in group 12 must rather be found in the unfavourable influence of its position in the corner of the hothouse than in too high a concentration of α-(aminomethyl-)naphthalene.

Mineralogy. — *Sur le réalgar de Matra, Corse* [1]. By L. P. G. KONING. (Communicated by Prof. H. A. BROUWER.)

(Communicated at the meeting of September 27, 1947.)

Le gisement de réalgar de Matra, Moita, Corte, Corse, a été découvert par E. NENTIEN (1). Depuis, la cristallographie du minéral principal dans ce gisement, le réalgar, a été décrite par plusieurs investigateurs.

A. LACROIX (2) donnait dans sa Minéralogie de la France et de ses Colonies en 1893 une description brève de ce minéral. En 1921 J. ORCEL (3) étudiait la cristallographie de réalgar un peu plus détaillée et décidait en outre, que le réalgar de Matra, Corse, est caractérisé par des faces certaines.

Cependant nous devons l'étude la plus détaillée à H. UNGEMACH (4), qui a découvert un grand nombre des faces nouvelles au réalgar de Matra.

En 1933 H. BUTTGENBACH (5) publiait le résultat d'un examen de réalgar de Matra.

De toutes ces recherches de réalgar de Matra il résultait que les cristaux de ce gisement sont caractérisés par un nombre défini de formes, se trouvant à presque tous les individus, c'est-à-dire: a (100), b (010), l (110), m (120), v (130), c (001), r (011), q (021), x ($\bar{1}01$), z ($\bar{2}01$), n ($\bar{1}11$) et e ($\bar{1}21$).

Les notations de ces faces se rapportent à l'orientation cristallographique, proposée par V. GOLDSCHMIDT (6).

Dans cette publication les résultats d'une recherche cristallographique de cristaux de réalgar de Matra, Corse seront données. Aux cristaux mesurés j'ai trouvé quelques formes nouvelles, qui sont marquées * dans le Table II. Les résultats des calculations sont donnés d'une manière, proposée par TERPSTRA (7) et en dernier lieu j'ai déterminé les angles caractéristiques d'après le systématique cristallographique de BARKER (7).

Les cristaux de réalgar de Matra, étudiés par moi, se trouvent dans quelques géodes et fentes dans une pièce de dolomie et ils se présentent souvent en cristaux nets. Ils ont tous une couleur splendide et une transparence parfaite, pendant que la longueur varie jusqu'à 2 mm. Un cristal de réalgar, pas appartenant à la pièce de dolomie, atteint une longueur de 8 mm; c'est le cristal no. 1.

Les cristaux mesurés sont tous détachés, de sorte qu'ils ne présentent qu'une seule extrémité avec des faces terminales. Afin de donner toutes les

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[1]) Je veux remercier sincèrement M. DOMINIQUE VECCHINI à Bastia, Corse, qui a bien voulu mettre les cristaux à ma disposition.

faces dans une même figure j'ai dessiné les cristaux terminés aux deux extrémités (fig. 1 et 2).



Fig. 1.

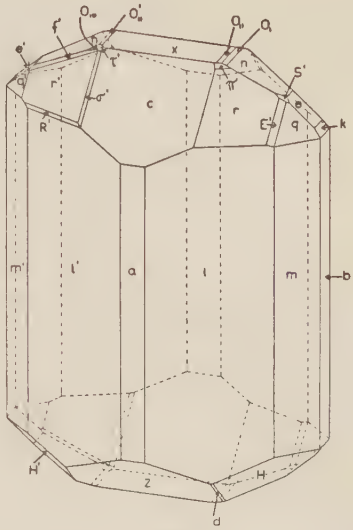


Fig. 2.

Les mesures cristallographiques ont été faits avec une goniomètre à deux cercles. La plupart des faces, étant irréprochables et d'un poli parfait, donnaient des réflections excellentes.

Il apparaît de l'examen cristallographique que les cristaux ne comprennent que les formes mentionnées si-dessus, mais aussi quelques formes accessoires. Les combinaisons des faces principales des cristaux examinés sont résumées dans le Table I.

Parmi les faces prismiques la forme *l* (110) est mieux développée que la forme *m* (120). La forme *a* (100) paraît aux lamelles minces, pendant que la forme *b* (010) est développée un peu plus large. Les prismes montrent souvent une striation verticale, surtout entre *l* (110) et *v*(130). Parmi les images originées dans les parties striées se trouvent de telles distinctement à observer qui correspondent aux formes: γ ($\bar{3}50$), *C* (450), *g* ($\bar{5}40$), β_1 ($\bar{8}.11.0$), μ (140), *w* ($\bar{1}60$), *A* ($\bar{2}70$), *B* ($\bar{5}80$), β ($\bar{3}40$) et χ ($\bar{3}10$), appartenantes aux cristaux no. 1 et 6.

Il est nécessaire de ne donner à ces formes qu'une valeur peu importante, puisqu'elles jouent une rôle de faces accidentelles ou accessoires. Elles sont présentes comme lammelles microscopiques entre les prismes principaux. UNGEMACH était aussi sceptique envers la présence de tant de faces avec une notation plus ou moins compliquée, car il écrivait:

„J'appuie tout particulièrement sur ce fait de l'excellence de la très grande majorité des faces, car l'étrangeté de beaucoup des notations que j'apporte pourrait faire croire qu'il s'agit de formes accidentelles, vicinales ou autres. Moi-même, j'ai dû mesurer plusieurs cristaux de combinaison identique devant de me rendre à l'évidence que les faces en question appar-

TABLE I.

Faces	*l* (110)	*w* (230)	*m* (120)	*v* (130)	*b* (010)	*a* (100)	*D* (560)	*ξ* (250)	*i* (210)	*α* (320)
no des crist.										
1	*l*	*w*	*m*	*v*	*b*	—	*D*	*ξ*	*i*	—
2	*l*	—	*m*	*v*	*b*	*a*	—	—	—	—
3	*l*	—	*m*	*v*	*b*	—	—	—	—	—
4	*l*	—	*m*	—	*b*	—	—	—	—	—
5	*l*	*w*	*m*	—	*b*	*a*	—	—	—	—
6	*l*	—	*m*	*v*	*b*	—	*D*	—	—	*α*
7	*l*	—	*m*	*v*	*b*	*a*	—	—	—	—
8	*l*	—	*m*	*v*	*b*	—	—	—	—	—

Faces	*r* (011)	*q* (021)	*c* (001)	*z* ($\bar{2}$01)	*n* ($\bar{1}$11)	*e* (121)	*H* ($\bar{2}$21)	O^{1} ($\bar{5}$45)	*x* ($\bar{1}$01)	*k* ($\bar{1}$31)	*d* ($\bar{2}$11)
no des crist.											
1	*r*	*q*	*c*	*z*	*n*	*e*	*H*	0_1	*x*	*k*	*d*
2	*r*	—	*c*	*z*	*n*	—	—	—	—	—	—
3	*r*	—	*c*	*z*	*n*	—	—	—	—	—	—
4	—	—	—	*z*	*n*	—	—	—	—	—	—
5	*r*	—	—	*z*	*n*	—	—	0_1	—	—	—
6	*r*	—	*c*	*z*	*n*	*e*	—	—	—	—	—
7	—	—	*c*	*z*	*n*	—	—	—	—	—	—
8	*r*	—	*c*	*z*	*n*	*e*	*H*	—	—	—	—

TABLE II.

Forme	Notation	*ϱ*	*φ*	Forme	Notation	*ϱ*	*φ*
a	(100)	90°	90°	*c*	(001)	23°51′	90°
b	(010)	,,	0	*r*	(011)	33 23	42 14′
i	(210)	,,	71 50′	*q*	(021)	47 02	24 24
α	(320)	,,	66 10	*o′*	(0$\bar{1}$2)*	26 46	118 57
l	(110)	,,	56 36	*E′*	(0.12.7)*	43 27	27 53
D	(560)	,,	51 30	*x*	($\bar{1}$01)	16 37	270
w	(230)	,,	45 10	*n*	($\bar{1}$11)	29 46	328 36
m	(120)	,,	37 11	*e*	($\bar{1}$21)	45 36	343 02
ξ	(250)	,,	31 13	*k*	($\bar{1}$31)	56 13	348 30
v	(130)	,,	26 46	*τ′*	($\bar{1}$12)*	14 21	162 38
μ	($\bar{1}$40)	,,	20 41	*π′*	($\bar{1}$45)*	26 04	37 20
W	($\bar{1}$60)	,,	345 56	*G*	($\bar{1}$12)	40 23	73 19
A	($\bar{2}$70)	,,	336 40	*f′*	($\bar{1}$44)*	29 00	152 15
γ	($\bar{3}$50)	,,	317 22	*R′*	($\bar{1}$32)*	47 36	132 15
B	($\bar{5}$80)	,,	316 07	*z*	($\bar{2}$01)	46 05	270
β_1	($\bar{8}$.11.0)*	,,	312 30	*d*	($\bar{2}$11)	48 56	295 11
β	($\bar{3}$40)	,,	311 21	*H*	($\bar{2}$21)	54 56	313 14
C	($\bar{4}$50)	,,	309 15	*S′*	($\bar{2}$43)*	33 26	355 30
χ	($\bar{3}$10)	,,	257 27	0_1	($\bar{5}$45)*	26 10	322 40
g	($\bar{5}$40)	,,	241 55	0_{11}	($\bar{5}$35)*	22 41	314 31
				0_{111}	($\bar{4}$45)*	22 38	200 28

tiennent à des formes, compliquées certes, mais de position parfaitement fixée et ne donnant lieu à aucune ambiguité de notation."

Concernant les faces terminales je pouvais constater que les formes c (001) et n ($\bar{1}11$) sont les faces principales, contrairement à la forme z ($\bar{2}01$), toujours présente, mais dans la plupart des cas décidément inférieure. En deux cas la forme z ($\bar{2}01$) est développée assez grande.

Le résultat des mesures cristallographiques est résumé dans le Table II.

Les résultats des calculations des cristaux de réalgar de Matra, Corse, investiqués par moi, sont donnés dans le Table III.

TABLE III.

$X'_0 = 0.4434$		$p'_0 = 0.7414$
$\varrho_0 = 23°55'$		$q'_0 = 0.4883$
$\beta = 113°55'$		$d = 63°59'$
$\mu = 66°05'$		$f = 40°10'$
	$a = 0.7205$	
	$b = 1$	
	$c = 0.4883$	

Le clivage de réalgar n'est pas parfait. Il existe un clivage assez distinct suivant b (010), tandis que c (001) montre un clivage moins distinct.

Pour la densité du réalgar de Matra, je trouvais $d = 3{,}54$.

D'après le systématique cristallographique de BARKER (7), le réalgar est caractérisé par les angles suivants:

$cr = 29°\ 28'$
am = 52° 49'
$bq = 46°\ 53'$
$ra = 43°\ 55'$

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Physical Geography. — *Theory on central rectilinear recession of slopes.* II. By J. P. BAKKER and J. W. N. LE HEUX. (Communicated by Prof. F. A. VENING MEINESZ.)

(Communicated at the meeting of May 31, 1947.)

1. *Influence of the c-value upon the shape of our curves.*

As in the first part of our theory has been said, the ratio between the solid rock volume and the corresponding screes volume is marked by $\frac{1-c}{1}$.

For $c = \frac{1}{2}$, formula (14*a*) acquires the form

$$x = by$$

In Nature, however, c will seldom exceed the value $\frac{1}{3}$ [7]). Smaller values of c can be the result of transport of part of the debris by secundary factors.

From formula 14*a* in the form

$$x = ay - (a-b)\, y\, [1 + (1-2c)\, y^2]^{\frac{1-\frac{1}{c}}{\frac{1}{c}-2}}$$

it follows that for $c = -\infty$ the last exponent becomes $-\frac{1}{2}$ (see further tabel I).

TABLE 1.

c	$1-2c$	$\frac{c-1}{1-2c}$
$\frac{1}{3}$	$\frac{1}{3}$	-2
$\frac{1}{4}$	$\frac{1}{2}$	$-\frac{3}{2}$
$\frac{1}{5}$	$\frac{3}{5}$	$-\frac{4}{3}$
0	1	-1
$-\frac{1}{5}$	$\frac{7}{5}$	$-\frac{6}{7}$
$-\frac{1}{4}$	$\frac{6}{4}$	$-\frac{5}{6}$
-1	3	$-\frac{2}{3}$
-2	5	$-\frac{3}{5}$
-5	11	$-\frac{6}{11}$
-10	21	$-\frac{11}{21}$
-50	101	$-\frac{51}{101}$
$-\infty$	∞	$-\frac{1}{2}$

For maximum values of α in Nature (in the Alps $26° - 43°$) small changes in the positive value of c have relatively little influence upon the

[7]) The complicated morphological development of slopes and screes, owing to climatological periodicities in the Quaternary Age implies that in most cases in Nature the present screes volume will not correspond with the theoretical c-value.

shape of the curve (fig. 6). For high negative values of c however, the point of intersection of the curves with the topplateau tend to lie further

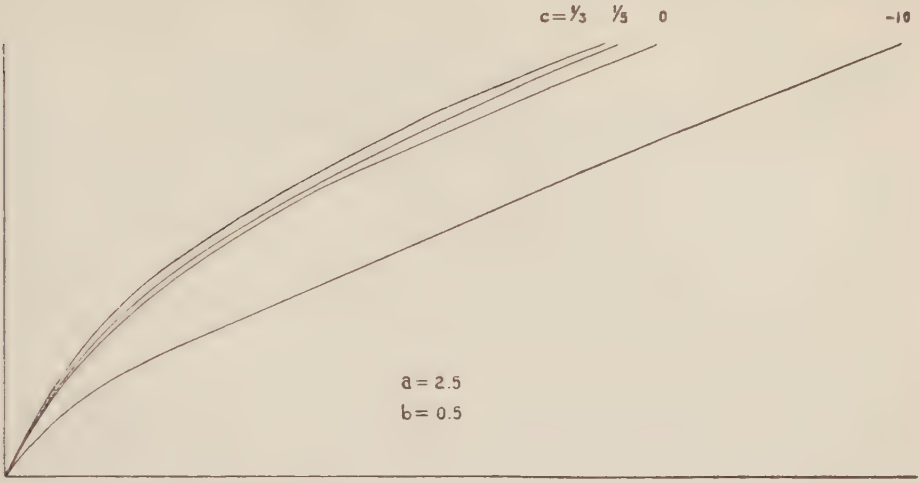


Fig. 6. The influence of the constant c on the form of our curves for $\alpha = 21° 48'$ and $\beta = 63° 34'$. For $c = -\infty$, see text (part I).

apart. We here touch upon a question, essential to the problem of the rounding of plateaus to which we shall return later.

The case $c = 0$ was already mentioned (a cubic curve [formula 15*a*]).

More attention should be paid to a special question for high negative c-values. As the reader will be aware, LEHMANN held the opinion that his theory applies for values between $c = \frac{1}{3}$ and $c = -\infty$. The author justified his point of view by demonstrating that for $c = -\infty$ RICHTER's denudation slope can be derived. Strictly speaking, this argument is inadequate. For as soon as c has a high negative value, it means that screes is very thin. It is then out of question that the rock cannot remain unchanged under this thin boulder mantle. One of LEHMANN's most important conditions — the remaining unchanged of the rock underneath the screes — is no longer fulfilled.

In LEHMANN's theory the above-mentioned contradiction is less evident, as, owing to the parallel recession, the forms which approximate RICHTER's denudation slope are forced somewhat further back than the theory would lead us to suppose, but for the rest they practically retain their shape.

Our theory gives rise to another complication, if we assume high negative values for c. For, if we start from the premiss, that in the case of bare mountain slopes the weathering intensity increases rectilinearly with the height, it is possible that this premiss remains when only a thin mantle of debris covers the slope. In the latter case recession will take place somewhat more slowly than with bare slopes, but assuming a little corrasion effect of soil-creep only, there is no reason to suppose that the recession-type will change.

Such a thin debris mantle does not occur for high negative values of c only, but also, for small negative and positive values of c, in the higher part PQ (fig. 7) of the nucleus.

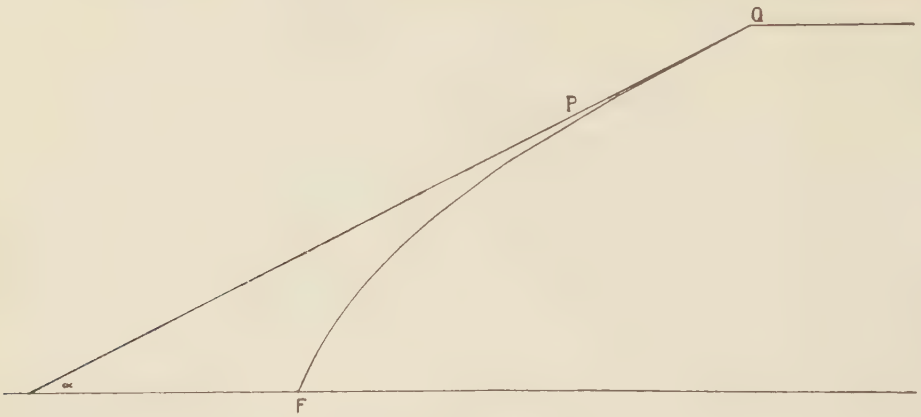


Fig. 7. See text.

For humid climates in the temperate zones we may approximately assume, that the screes from F to P (fig. 7) protects the underlying rock, which remains unchanged. For part PQ this premiss however, is improbable. In LEHMANN's theory this implies that part PQ in Nature will recede further parallel to the right, owing to which also point P will change somewhat its position.

In our theory, however, part PQ will undergo a further softening. This is only possible when we do not hold exclusively to the maximum value of α. We shall return to the subject later. Moreover, it is clear that part PQ (with very thin debris mantle) is greater in proportion as c becomes smaller.

2. *The influence of the β-value on the shape of our curves.*

For demonstrating the influence of β on the shape of the curves we choose the formula

$$x = ay - (a-b)\frac{y}{(1+\frac{1}{3}y^2)^2}$$

for $c = \frac{1}{3}$

$\alpha = 21°48'$ $(a = 2{,}5)$

and β is $90°$ $(b = 0)$; $63°34'$ $(b = 0{,}5)$; $45°$ $(b = 1)$ and $24°27'$ $(b = 2{,}2)$ respectively.

The β-value has a similar influence on the shape of the curve as in LEHMANN's theory (fig. 8).

We must bear in mind that many mountains, for instance, in the eastern Alps and European "Mittelgebirge", have an approximately straight-lined

profile with rounded tops. This fact implies, that for the convexity problem of such forms small differences between β and α must have our particular attention.

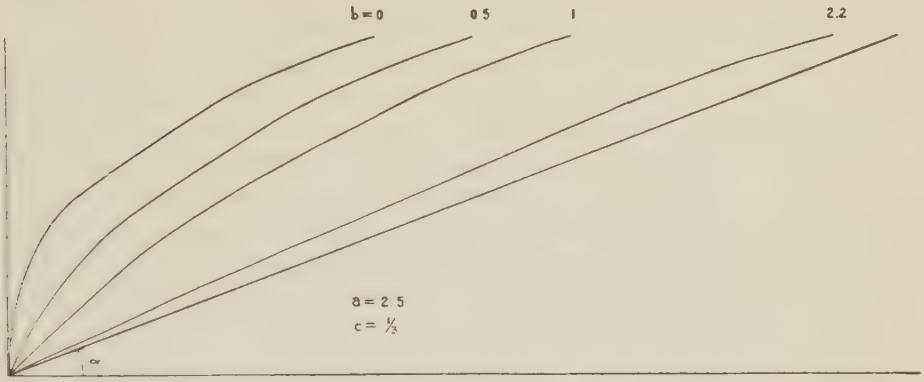


Fig. 8. The influence of the initial slope angle β on the curvature for $a = 2{,}5$ ($\alpha = 21° 48'$), $c = 1/3$ and $\beta = 90°$; 63° 34'; 45° and 24° 27' respectively.

3. *Influence of the α-value on the shape of our curves.*

LEHMANN asserted his theory for maximum values of the slope angles of screes in Nature, undoubtedly being inspired by the slope angles of RICHTER's rectilinear denudation slope in the Alps. This contains the valuable paradox:

It is not the slope angle in solid rock that determines the slope angle of the debris mantle, but inversely, the slope angle of the debris mantle that ultimately determines the slope angle of the underlying rock.

Rounding of drainage divides till they are nearly horizontal at the top, as found in GILBERT (13) is only possible if at the summit α does not differ too much from zero. The nature of slope development in every region is determined by the frequency of the slope angles of the alluvial cones and screes ad hoc. It is known that this frequency of slopes of debris mantles and solid rocks is generally not distributed in a regular way between the maximum and minimum values, but set values predominate lithologically and climatologically. It is also a well-known fact, that the minimum values of slope angles of rockfans on pediments can diminish to a few degrees, with which, naturally, the frequently occurring slope angles of 3° — 7° of pediments are intimately connected.

According to the investigations of PIWOWAR and STINY the maximum values of α in the Alps are found between 26° (Bündnerschiefer) and 43° (granites).

Valuable data for the Papago country, Arizona and the Sacaton Mountains are communicated by BRYAN (14) and HOWARD (8). In the Papago country K. BRYAN found the greatest frequency of boulder controlled slopes between 20° and 45° from the horizontal. These slope angles are characteristic of most granites, granite gneisses and horizontally bedded lava flows (usual angle of granite slopes 30° — 35°).

Mountain slopes at angles less than 20° from the horizontal are rare in the Papago country. These "rain-washed" slopes are developed on the least resistant rocks (closely jointed gneiss, schist, phyllite and felsite, tuff and shale). In the Papago country slopes below 15° seldom occur, except the dip slopes of lava lateau and pediment slopes.

In humid regions the pure weathering-removal slopes are generally less steep than in the snow-covered high mountains and the arid and semi-arid regions. This can be accentuated if the rock is marly or clayish. For such conditions De Martonne remarks (14, p. 635): "Dans un pareil terrain (Lias marls), la pente des versants ne peut dépasser certaines limites, d'autant plus étroites que le climat est plus humide, car la surface est délayée par l'eau, en formant une sorte de pâte qui glisse même sur des pentes de moins de 15°." In the Basin of Mainz we observed at debris cones slope angles of 5 — 10°.

Anyhow, a theory as ours, in its most general form, should hold for all α-values between the maximum and minimum values. When used in field research the α- and β-values must be adapted to the regional frequencies of these values.

In order to examine the influence of α we choose formula

$$x = ay - (a-b)\frac{y}{(1+y^2)^2}$$

for $c = 0$

$\beta = 63°34'$ $(b = 0{,}5)$

$\alpha = 42°16'$ $(a = 1{,}1)$; $33°41'$ $(a = 1{,}5)$; $26°34'$ $(a = 2)$; $15°31'$ $(a = 3{,}6)$ and $7°7'$ $(a = 8)$, respectively.

Fig. 9 gives a reproduction of the curves under these conditions. Just as in LEHMANN's theory the curve moves more to the right, as α becomes smaller.



Fig. 9. The influence of the slope angle α of the screes on the curvature for $b = 0{,}5$, $c = 0$ and $\alpha = 42°\ 16'$; $33°\ 41'$; $26°\ 34'$; $15°\ 31'$ and $7°\ 7'$ respectively ($a = 1{,}1$; $1{,}5$; 2; 3,6 and 8).

Without going too far into the matter, we shall especially devote our attention to the small values of α.

a being 8, the profile FKT would, as it were, suggest a two-cyclic development or, anyhow, a development in two stages. Certainly for kindred

phenomena in Nature, investigators repeatedly have given such an interpretation [8]) (1. plateau *KT*, owing to weathering-removal, slightly reduced at the borders; 2. *FK* formed by undercutting or dissection).

From our theory it follows that such a profile can be explained in a monocyclic way. Whether, nevertheless, interpretations with the aid of two-cycles or stages be preferred or at least may rank with the monocyclic conception has to be considered on its own merits in every region. In our opinion, however, polycyclic interpretation of "Gipfelfluren" and plateau-like phenomena has been overstressed in recent morphology. The theoretical conditions of the present-day views on geomorphological cycles, are rather uncertain. We do not wish to deny the existence of uplifted peneplains, the number of which, in our opinion, however, is more limited than is almost generally accepted.

Our profile *FKT* shows some relationship with the profile, found in GÖTZINGER (7, p. 113) published as early as 1907, which is reproduced in our fig. 10. GÖTZINGER

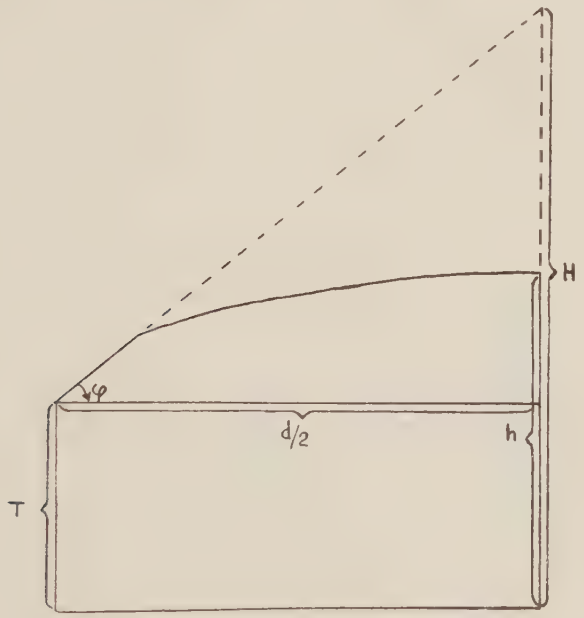


Fig. 10. (After G. GÖTZINGER). See text.

interprets this profile from the Wiener Wald monocyclically as well. The rounded plateau-like ridges, lying 388—645 m high, are held to be the direct descendants of a sharp crest-region at 1000—1500 m. The gully in its present stage has a depth *T*; *d* is the distance between these gullies (density of the valleys); *H* represents the original height of the present plateau-like form above the local baselevel.

Without having a theoretical basis, GÖTZINGER, partly from motives of tectonic nature, strongly objects to the two-cyclic interpretation that the ridges should be the remains of an uplifted peneplain. Bearing in mind, that particularly for higher negative *c*-values, a further softening of the upper part of our slope for $\alpha = 8$, is probable, theoretically, his point of view is quite possible.

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[8]) HOWARD, A. D. Op. cit. (8), fig. 32, p. 126.

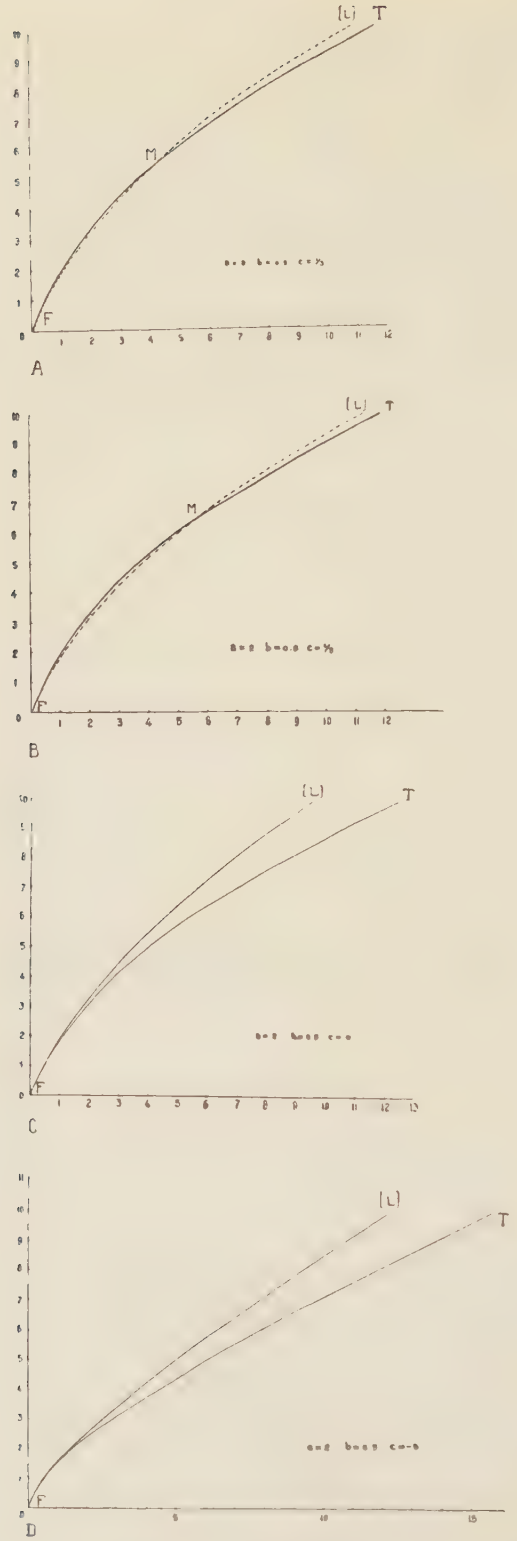


Fig. 11. *A, B, C* and *D*. Comparison of LEHMANN's curves (*L*) with ours (*FT*) for $\beta = 63° 34'$, $\alpha = 26° 43'$ and $c = 1/3, 1/5, 0, -5$ respectively.

As a result of what has been said concerning our fig. 7 for part PQ of our profiles with positive and small negative c-values, it follows that for the upper part of the curves our formulas (11) and (16) must be changed to

$$\frac{dx}{dy} = a' \ldots (11a) \quad \text{and} \quad x = a'y \ldots (16a) \quad \text{for} \quad a' < a.$$

4. *Comparison of the theories of parallel and central rectilinear recession.*

We shall restrict the comparison of our curves with those of LEHMANN to the case $a = 2$, $b = 0{,}5$ (fig. 11 *A*, *B*, *C*, *D*). In this case, for higher positive values of c ($c = \frac{1}{3}, \frac{1}{5}$) our curves only slightly differ from those of LEHMANN, which has been constructed with the aid of our projective geometric method (10). The curves intersect. Part FM of our curves proves to be somewhat to the left of those of LEHMANN; part MT is slightly to the right of LEHMANN's curve. The latter can be accentuated in Nature, because, in accordance to what has been said regarding fig. 7, part MT will probably recede further under the thin debris-mantle.

On the other hand, the irregularities which without doubt will occasionally occur in the process of weathering and accumulation in Nature, will make it difficult in many cases to distinguish LEHMANN's curve as such from ours. But mostly, there will be other morphological phenomena in the crest region, which may decide the question as to whether we have to do with parallel or with central straight-lined recession.

For $c = 0$ and for negative values of c our curves are clearly distinguishable from those of LEHMANN. Near the basic point the curves still coincide, but higher up they diverge widely. Moreover, the upper part of our curves approach, along a longer distance, a more rectilinear shape than in LEHMANN's curves. For reasons already mentioned (continuing weathering-removal underneath the thin debris mantle) our curve in Nature will, in the vicinity of the top-plateau, diverge still more to the right than the theoretical curve shows.

Final remarks.

Our theory forms a necessary extension of the recession-problem for crest regions etc., in which central rectilinear recession must be taken into consideration. This theory offers points of contact for a theoretical treatment of the convexity problem, which will be discussed in a following publication.

Finally, we wish to express our gratitude to Mrs. ROMP–MARSHALL and Mr. W. F. HEINEMEYER, for their help in the translation of the text and to Dr. E. M. BRUINS, for the indications he gave us.

We are indebted to Mr. W. VAN DIJK, Mr. D. NIJHOFF and Mr. E. VAN STAA, assistants for physical Geography at the University of Amsterdam, for their kindness in preparing the figures for the press.

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Chemotherapy. — *L'action inhibitrice des métaux sur la croissance du B. tuberculeux. V. Zinc, cadmium et mercure.* By ONG SIAN GWAN. (Communicated by Prof. G. G. J. RADEMAKER.)

(Communicated at the meeting of October 25, 1947.)

1. Les éléments du deuxième groupe, sous-groupe *b*, du tableau périodique: zinc, cadmium et mercure ont tous plus ou moins une action inhibitrice sur la croissance du B. tuberculeux. Parmi ces éléments le cadmium est d'une importance capitale au point de vue recherches expérimentales et cliniques. En effet, WALBUM (1926) a montré, le premier, l'action favorable de sels de cadmium sur la tuberculose expérimentale des animaux de laboratoire. Ce résultat fut confirmé dans la clinique par LUNDE (1926, 1927), MAIGRE et REYNIER (1932) et HEAF (1937). Ils ont constaté dans plusieurs cas la disparition de B. tuberculeux dans les crachats de 52 à 63 p. 100, la diminution de la fièvre et l'augmentation du poids. D'après HEAF l'action de sulfide ou de glycine de cadmium est comparable à celle obtenue par les sels d'or. WALBUM (1928, 1929) a cependant montré que l'action de chlorure de cadmium est supérieure à celle de sels d'or.

2. *Action inhibitrice de zinc, cadmium et mercure sur la croissance du B. tuberculeux.*

Les métaux utilisés dans toutes les expériences sont les suivants: **1.** Zinc, pro analyse, electrolytique, **2.** Cadmium Merck en poudre. **3.** Mercure bidestillé. Pour peser une petite quantité de mercure, par exemple 10 mg, on fait condenser le mercure dans le vide contre une paroi de verre et on enlève ensuite l'excès de mercure.

Le B. tuberculeux utilisé est une souche bovine (souche Vallée) qui a subi plusieurs passages dans le milieu de Sauton et que nous avons employé dans les recherches précédentes (1944, 1945 et 1946).

a. Expérience réalisée avec 10 mg de métal par 100 cc de milieu synthétique de Sauton.

La souche utilisée a subi neuf passages sur milieu de Sauton et qu'on désigne par V_9, elle est agée de 19 jours et l'expérience a duré 21 jours. Les observations obtenues sont reproduites dans le tableau 1.

Dans cette expérience deux observations ont été perdues, elles sont remplacées par des valeurs calculées d'après la formule de ALLAN et WISHART (1930) et améliorée par YATES (1933).

$$x = \frac{nN + tT - S}{(n-1)(t-1)}.$$

où n = nombre d'observations parallèles

TABLEAU 1.

Action inhibitrice de zinc, de cadmium et de mercure sur la croissance de B. tuberculeux. 10 mg de métal/100 cc.

	Témoins mg	Zinc mg	Cadmium mg	Mercure mg
	594,0	379,0	206,0	637,0
	609,0	185,4	42,0	535,0 [1])
	615,0	379,4	25,0	598,0 [1])
	539,5	385,0	37,0	594,0
Total	2357,5	1328,8	310,0	2364,0
Moyennes	589,4	332,2	77,5	591,0

Analyse de la variance.

Sources de variation	Somme des carrés	Degrés de liberté n	Variances moyennes
Métaux	723 610	3	241 203 **
Entre rangées	25 121	3	8 374
Résiduelle	34 665	7	4 952
Total	783 396	13	

[1]) valeur calculée. ** probabilité inférieure à 1 p. 100.

N = somme de t—1 observations situées sur la même rangée.
t = nombre de traitements
T = somme de n—1 observations de même traitement
S = somme totale des observations

Dans le cas de deux observations perdues on obtient de bons résultats par une méthode d'itération. Soient x_1 et x_2 deux observations perdues. On calcule d'abord x_1 en substituant x_2 par la valeur moyenne expérimentale. Ensuite on trouve x_2 en utilisant la valeur caculée de x_1 et on répète le calcul pour obtenir une meilleure approximation de x_1 ou de x_2.

L'analyse de la variance (analysis of variance) montre qu'il existe une différence très significative entre les traitements par les métaux. Une différence entre ces moyennes est considérée significative si elle est supérieure à

$$\sqrt{\frac{4952 \times 2}{4}} \times 2{,}365 \text{ mg} = 117{,}7 \text{ mg}.$$

On constate que le cadmium a donné la plus forte action inhibitrice, que l'action de zinc est supérieure à celle de mercure et que ce dernier n'a aucune action inhibitrice.

b. Expérience réalisée avec 20 mg de métal par 100 cc de milieu synthétique de Sauton.

L'expérience précédente nous montre q'avec 10 mg de mercure on ne constate pas de différence significative entre les poids moyens de la

culture à mercure et de la culture témoin. En répétant cette expérience avec 20 mg de mercure on obtient le même résultat. Cependant NOVY et SOULE (1925) ont constaté que la vapeur de mercure empêche la croissance de B. tuberculeux. Nous avons comparé l'action de la vapeur de mercure avec celle du mercure introduit dans le liquide. Pour cela on introduit une goutte de mercure bidestillé dans un tube conique qui se trouve dans le flacon contenant 100 cc de milieu de Sauton (fig. 1).

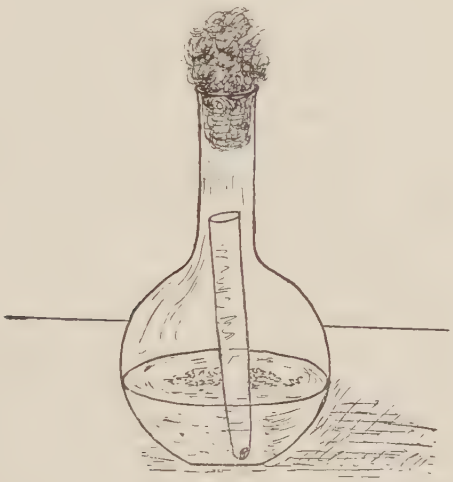


Fig. 1. Action inhibitrice de vapeur de mercure sur la croissance de B. tuberculeux.

TABLEAU 2.

Action inhibitrice de zinc, de cadmium et de mercure sur la croissance de B. tuberculeux. 20 mg de métal/100 cc.

	Témoins mg	Zinc mg	Cadmium mg	Mercure mg	Vapeur de mercure mg
	257,1	61,9	57,2	219,5	31,0
	189,0	92,0	233,6	135,0	67,7
	267,4	201,6	4,0	215,8	49,8
	184,1 [1])	80,8	4,8	60,2	95,3
Total	897,6	436,3	299,6	630,5	243,8
Moyennes	224,4	109,1	74,9	157,6	61,0

Analyse de la variance.

Sources de variation	Somme des carrés	Degrés de liberté *n*	Variances moyennes
Métaux	71 240	4	17 810 *
Résiduelle	72 576	14	5 184
Total	143 816	18	

* probabilité entre 5 p. 100 et 1 p. 100.

Le tableau 2 montre le résultat de cette expérience réalisée avec la souche V_S âgée de 32 jours. La durée de l'expérience est de 35 jours. Comme dans la première expérience on constate également que la différence entre les traitements est significative. La différence entre ces moyennes est significative si elle est supérieure à

$$\sqrt{\frac{5184 \times 2}{4}} \times 2145 \text{ mg} = 109{,}2 \text{ mg}.$$

On constate dans le tableau 2 que les moyennes obtenues par la vapeur de mercure: 61,0 mg, le cadmium: 74,9 mg et le zinc: 109,1 mg ont tous donné une différence significative avec le témoin: 224,4 mg.

Pour expliquer l'action plus élevée de vapeur de mercure sur celle de mercure introduit dans le liquide on compare la concentration de mercure dans les deux cas. D'après LANDOLT–BORNSTEIN (physikalisch-chemische Tabelle 1935, 5. Auflage, 3. Eg I p. 497) la concentration de mercure dans l'eau à 38° C. est égale à 0,04 γ/cc (1 γ = 0,001 mg). La tension de vapeur de mercure à la même temperature est égale à 0,005219 mm Hg (International Critical Tables 1928, vol. 3, p. 206), elle correspond à une concentration de mercure de 0,054 γ/cc d'air. On voit que la concentration de mercure dans l'air est légèrement supérieure à celle dans l'eau et le rapport est égal à 0,054/0,04 = 1,35. Ce rapport ne suffirait pas à expliquer la différence d'action de la vapeur de mercure et du mercure dissout dans le liquide. On pourrait expliquer cette différence par une plus grande absorption de mercure pendant la respiration de sorte que le B. tuberculeux ou le ferment de respiration est détruit.

On a placé dans cette expérience les quatre flacons contenant le mercure en vapeur à côté de quatre flacons contenant le mercure dans le liquide

TABLEAU 3.

Vapeur de cadmium.

	Témoins mg	Vapeur de cadmium mg
	588,5	667,0
	595,4	514,3
	657,6	585,4
	534,5	
Total	2376,0	1766,7
Moyennes	594,0	588,9

Analyse de la variance.

Sources de variation	Somme des carrés	Degrés de liberté n	Variances moyennes
Métaux	44,6	1	44,6
Résiduelle	19 294,4	5	3858,9
Total	19 339,0	6	

dans une étuve séparée. La différence entre les moyennes obtenues est égale à 96,6 mg, elle est presque significative puisqu'il faudrait cinq observations dans chaque groupe, pour rendre la différence significatif. Il semble donc que la diffusion de la vapeur de mercure dans l'air n'est pas assez vite pour influencer les autres cultures.

On peut se demander si la vapeur de cadmium à la température de l'étuve, 38° C, montre également une action inhibitrice. L'expérience a été réalisée comme plus haut (fig. 1) en introduisant 40 mg de Cd dans le tube conique et en utilisant une souche V_{10} de 77 jours. L'expérience a duré 30 jours, elle a donné un résultat négatif (tableau 3). Ceci est à prévoir si l'on compare les points d'ébullition des trois métaux: Zn: 920°, Cd: 767° et Hg: 357° C. A la température de 38° C. la tension de vapeur de cadmium est pratiquement nulle, elle est de l'ordre de 10^{-10} mm Hg, calculée à l'aide de la formule: log p (mm Hg) $=$ —5910/T—1,234 log T —0,000 156 T + 12,467, FOGLER et RODEBUSH (1923).

c. Expérience réalisée avec 2,5 mg/100 cc de sulfate de zinc ou de sulfate de cadmium.

Les sels employés sont $ZnSO_4$. 7 H_2O pro analyse et CdSO. 8/3 H_2O puriss. KAHLBAUM. Dans le premier cas la concentration de zinc est égale à 0,57 mg/100 cc ou environ 1: 175.000 et dans le second cas on a 1,09 mg de Cd/100 cc ou environ 1: 90.000.

La souche utilisée V_{17} est âgée de 19 jours, la durée de l'expérience est de 26 jours. Le tableau 4 montre que l'addition de sulfate de cadmium a donné une différence significative avec le témoin. Pour obtenir une

TABLEAU 4.

Action inhibitrice de sulfate de zinc et de sulfate de cadmium sur la croissance de B. tuberculeux.
2,5 mg de $ZnSO_4$. 7 H_2O ou de $CdSO_4$. 8/3 H_2O par 100 cc de liquide.

	Témoins mg	$ZnSO_4$. 7 H_2O mg	$CdSO_4$. 8/3 H_2O mg
	743,0	719,5	715,5
	741,5	713,0	727,5
	735,0	736,0	696,0
	763,0	743,5	704,0
Total	2982,5	2912,0	2843,0
Moyennes	745,6	728,0	710,8

Analyse de la variance.

Sources de variation	Somme des carrés	Degrés de liberté n	Variances moyennes
Traitements	2432,6	2	1216,3 *
Résiduelle	1606,5	9	177,4
Total	4039,1	11	

différence significative entre ces moyennes il faut qu'elle soit plus grande que

$$\sqrt{\frac{177,4 \times 2}{4}} \times 2,262 \text{ mg} = 21,3 \text{ mg}.$$

La différence entre les moyennes obtenues par le zinc et le témoin est égale à 17,6 mg, elle n'est pas significative. Pour que cette différence devienne significative il faudrait faire 7 cultures parallèles de chaque groupe.

3. *Action de cadmium en présence de sérum sanguin.*

Après stérilisation du milieu de Sauton contenant 20 mg de Cd par 100 cc on ajoute 5 cc de sérum de chèvre non chauffé. La souche V_{12} utilisée est âgée de 30 jours et la durée de l'expérience est de 42 jours. Le résultat (tableau 5) montre qu'il n'y a pas de différence entre les cultures témoins et les cultures en présence de Cd et de sérum.

TABLEAU 5.

Action de cadmium en présence de sérum sanguin.
20 mg de Cd et 5 cc de sérum par 100 cc de liquide.

	Témoins mg	Cadmium en présence de sérum mg
	757,5	385,0
	717,0	888,0
	772,5	911,1
	646,0	771,5
		958,0
Total	2893,0	3913,6
Moyennes	723,3	782,7

Analyse de la variance.

Sources de variation	Somme des carrés	Degrés de liberté n	Variances moyennes
Entre rangées	7 859	1	7 859
Résiduelle	132 581	4	33 145
Résiduelle	93 620	3	31 207
Total	234 060	8	

On pourrait expliquer ce résultat négatif comme suit. A mesure que le cadmium métallique se dissout il se combine avec les protéines du sérum. En effet, on obtient après un certain temps une solution colloïdale jaunâtre. Après stérilisation de la culture la solution devient plus opaque et on constate au fond du flacon un précipité qui doit être formé par le sulfide de cadmium. On sait que les sels de cadmium comme par exemple le sulfate de cadmium est un précipitant énergique de protéines. L'absence d'action de cadmium dans cette expérience s'explique par l'absorption de cadmium par les protéines.

4. *Culture de B. tuberculeux ayant été en contact avec le zinc ou le cadmium.*

a. Expérience réalisée avec 40 mg de Zn ou de Cd par 100 cc de milieu synthétique de Sauton.

Le milieu de SAUTON contenant 40 mg de métal est ensemencé avec une souche V_{17} âgée de 25 jours et au bout d'un temps determiné on fait des prélèvements et on ensemence des flacons contenant le milieu de SAUTON sans métal. Les observations obtenues sont reproduites dans le tableau 6.

TABLEAU 6.

Cultures de B. tuberculeux ayant été en contact avec le zinc ou le cadmium. 40 mg de Zn ou de Cd par 100 cc de liquide.

Repiquage au bout de *n* jours	Durée de l'expérience en jours	Témoins mg	Zinc mg	Cadmium mg
6	25	485,0	515,5	211,0
		667,5	416,0	222,0
		602,0	601,5	198,0
		347,5	302,5	126,0
Total		2102,0	1835,5	757,0
10	26	615,0	706,3	253,3
		648,3	555,0	411,8
		719,3	697,7	366,0
		727,3	692,0	225,0
Total		2709,9	2651,0	1256,1
15	25	657,0	697,0	179,0
		691,5	663,5	95,0
		783,2 [1])	751,0	92,5
		663,6 [1])	738,0	102,0
Total		2795,3	2849,5	468,5
30	25	551,1 [1])	573,5	73,5
		608,7 [1])	568,0	76,5
		669,8 [1])	482,0	68,5
		545,6 [1])	409,1 [1])	78,0
Total		2375,2	2032,6	296,5
45	28	675,0	43,0	103,5
		629,0	94,0	136,5
		763,5	270,0	61,5
		627,0	82,0	46,5
Total		2694,5	489,0	348,0
Total (20 cultures)		12676,9	9857,6	3126,1
Moyennes		633,8	492,9	156,3

[1]) valeur calculée.

On peut résumer le résultat de l'analyse de la variance comme suit (tableau 7). L'influence de cadmium et de zinc est considérable comme l'a montré

TABLEAU 7.
Analyse de la variance.

Sources de variation	Somme des carrés	Somme des carrés	Degrés de liberté n	Variances moyennes
Cultures parallèles		67 386	3	
Traitements		3 511 563	14	
Métaux	2 407 989		2	1 203 995 **
Temps	508 676		4	127 169 **
Metaux × temps	594 898		8	74 362 **
Traitements × parallèles		192 281	35	
Métaux × parallèles	36 227		5	7 245
Temps × parallèles	54 059		10	5 406
Métaux × temps × parallèles	101 995		20	5 100
Total		3 771 230	52	

la très grande valeur de la variance moyenne (écart quadratique moyen). Le poids moyen des cultures à cadmium: $\bar{p} = 156{,}3$ mg est beaucoup plus petit que celui des cultures à zinc: $\bar{p} = 492{,}9$ mg et celui-est également inférieur à celui des cultures témoins: $\bar{p} = 633{,}8$ mg. Une différence entre ces moyennes peut être considérée significative si elle est supérieure à

$$\sqrt{\frac{7{,}245 \times 2}{20}} \times 2{,}571 \text{ mg} = 69{,}2 \text{ mg}.$$

Les trois souches de B. tuberculeux sont donc des souches différentes. L'action inhibitrice de cadmium est beaucoup plus élevée que celle de zinc.

Le temps joue également un rôle considérable. Voici les poids moyens obtenus dans des temps différents, l'indice désigne la durée en jours de contact de B. tuberculeux avec le métal avant le prélèvement.

$$\bar{p}_6 = 391{,}2 \text{ mg}$$

$$\bar{p}_{10} = 555{,}4 \text{ mg}$$

$$\bar{p}_{15} = 509{,}4 \text{ mg}$$

$$\bar{p}_{30} = 392{,}0 \text{ mg}$$

$$\bar{p}_{45} = 294{,}3 \text{ mg}.$$

Une différence entre ces moyennes supérieure à

$$\sqrt{\frac{5406 \times 2}{12}} \times 2{,}228 \text{ mg} = 66{,}9 \text{ mg}.$$

doit être considérée comme significative. On constate une diminution progressive de récolte à mesure que le temps de contact avec le métal est plus grand. Cette diminution est très marquée à partir de 15 jours.

Le temps joue également un rôle considérable sur les trois cultures différentes: témoin, zinc et cadmium. La figure 2 et le tableau 8 montrent

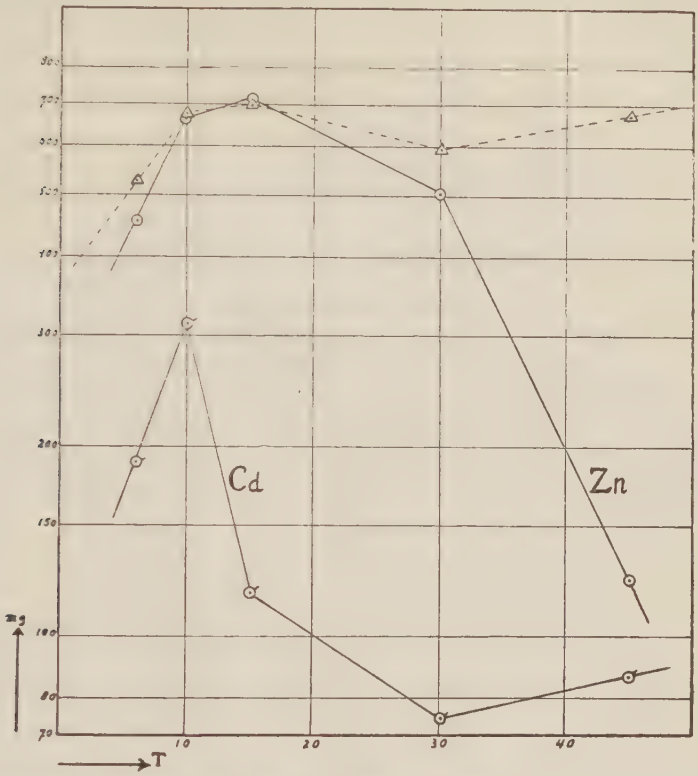


Fig. 2. Relation entre le poids moyen en mg et le temps de repiquage T en jours de B. tuberculeux ayant été en contact avec le zinc ou le cadmium.
— — — — — Courbe témoin.

TABLEAU 8.

Poids moyen en mg des cultures de B. tuberculeux ayant été en contact avec le zinc ou le cadmium.

Repiquage au bout de n jour	6	10	15	30	45
Témoins	525,5	677,5	698,9	593,8	673,6
Zinc	458,9	662,8	712,4	508,2	122,3
Cadmium	189,3	314,0	117,1	74,1	87,0

clairement cette influence. Une différence entre ces moyennes est significative si elle est supérieure à

$$\sqrt{\frac{5100 \times 2}{4}} \times 2{,}086 \text{ mg} = 105{,}3 \text{ mg}.$$

On constate dans la figure 2 que la courbe du Cd a une forme semblable à celle du Zn. Le poids moyen obtenu par Cd est dans tous les cas, sauf le dernier, inférieur à celui de Zn. L'action toxique de Cd se manifeste

donc à partir du sixième jour ou peut-être plus tôt. Par contre l'action toxique de Zn se manifeste seulement à partir de trente jours, elle se rapproche de l'action de Cd au bout de 45 jours. En ce qui concerne la courbe témoin on peut remarquer que le poids moyen obtenu au sixième jour est inférieur à celui obtenu au 10, 15 et 45ème jour. Ce résultat est peut-être en relation avec la phase latente de B. tuberculeux ou le microbe ne se multiplie pas.

b. Expérience réalisée avec 20 mg de Zn ou de Cd par 100 cc de milieu synthétique de SAUTON.

On répète l'expérience précédente avec 20 mg de métal et une souche

TABLEAU 9.

Cultures de B. tuberculeux ayant été en contact avec le zinc ou le cadmium. 20 mg de Zn ou de Cd par 100 cc de liquide.

Repiquage au bout de *n* jours	Durée de l'expérience en jours	Témoins mg	Zinc mg	Cadmium mg
15	95	709,0 692,5 744,0 706,5	677,0 771,6 730,0 746,5	677,5 745,5 707,0 707,1
Total		2852,0	2925,1	2837,1
35	34	670,8 395,0 601,4 [1]) 636,7 [1])	697,5 666,5 708,0 726,5	614,5 481,0 659,0 711,0
Total		2303,9	2798,5	2465,5
Total (8 cultures)		5155,9	5723,6	5302,6
Moyennes		644,5	715,5	662,8

TABLEAU 10.

Analyse de la variance.

Sources de variation	Somme des carrés	Somme des carrés	Degrés de liberté *n*	Variances moyennes
Cultures parallèles		22 087	3	
Traitements		78 526	5	
Métaux	**21 710**		**2**	10 855
Temps	**45 614**		**1**	45 614
Métaux × temps	**11 202**		**2**	5 601
Traitements × parallèles		63 562	13	
Métaux × parallèles	**17 198**		**6**	2 866
Temps × parallèles	**38 405**		**3**	12 802
Métaux × temps × parallèles	**7 959**		**4**	1 990
Total		164 175	21	

V_{12} âgée de 30 jours. Après prélèvement et ensemencement sur milieu de SAUTON au bout de 15 et 35 jours on laisse la culture maintenant se développer plus longuement, 95 et 34 jours au lieu de 25 à 28 jours. On constate dans les tableaux 9 et 10 que le résultat est maintenant négatif. Ce résultat négatif doit être attribué à trois facteurs: 1. la quantité de métal ajouté est dans ce cas 20 mg au lieu de 40 mg dans l'expérience précédente, 2. la longue durée de la culture et 3. les cultures mères sont largement ensemencées. Un résultat comparable a été déjà obtenu avec le thallium, (1946). On peut en conclure comme dans le cas de thallium, que le B. tuberculeux est modifié après le contact avec le zinc ou le cadmium qui se manifeste par une diminution de croissance après transplantation et que cette modification est réversible.

5. *Addition de 20 mg de zinc ou de cadmium au cours de la croissance du B. tuberculeux.*

L'expérience est réalisée avec une souche V_9 âgée de 45 jours. Au bout d'un temps déterminé on ajoute 20 mg de Zn ou de Cd stérilisé et on détermine en même temps le poids de culture témoin. On laisse ensuite

TABLEAU 11.

Addition de 20 mg de zinc ou de cadmium par 100 cc de liquide au cours de la croissance de B. tuberculeux.

Addition de métal au bout de *n* jours	Poids des cultures témoins au bout de *n* jours mg	Zinc mg	Cadmium mg
15	158,0	653,9	731,4
	293,0	622,8 [1])	801,7
	298,9	522,4	844,4
	236,4 [1])	561,5	813,4
Total	986,3	2360,6	3190,9
25	417,8	711,2	613,5
	582,3	750,0	689,6
	449,0	721,3	771,4
Total	1449,1	2182,5	2074,5
36	655,8		741,6
	583,5		562,5
	640,0		759,1
Total	1879,3		2063,2
50	742,7		
	770,0		
	716,7		
Total	2229,4		
Grand total	6544,1	4543,1	7328,6
Moyennes	503,4	649,0	732,9

TABLEAU 12.
Analyse de la variance.

Sources de variation	Somme des carrés	Somme des carrés	Degrés de liberté n	Variances moyennes
Cultures parallèles		25 035	2	
Traitements		852 221	11	
Métaux	308 901		2	154 451 **
Temps	122 203		3	40 734 **
Métaux × temps	421 117		6	70 186 **
Résiduelle		61 725	14	4 409
Total		938 981	27	

les cultures avec métal et trois cultures témoins dans l'étuve pendant 50 jours et on termine l'expérience.

Les observations obtenues sont reproduites dans le tableau 11. L'analyse de la variance (tableau 12) montre le résultat suivant. La différence entre les poids moyens du témoin et du zinc est égale à 145.6 mg, elle est significative puisqu'elle est supérieure à la différence significative: 93.3 mg. De même entre les moyennes du témoin et du cadmium. Ce résultat doit être attribué au poids total du témoin qui représente la somme de récoltes de différents âges. Par contre la différence entre les moyennes de Zn et de Cd n'est pas significative puisqu'elle est inférieure à la différence significative: 98,9 mg.

Comparons maintenant les poids moyens obtenus dans des temps différents.

Poids moyens	Différences	Différences significatives
$\bar{p}_{15} = 544{,}8$ mg		
	89,2 mg	88,6 mg
$\bar{p}_{25} = 634{,}0$ mg		
	23,1 mg	105,6 mg
$\bar{p}_{36} = 657{,}1$ mg		
	86,0 mg	140,4 mg
$\bar{p}_{50} = 743{,}1$ mg		

On constate que le poids augmente en même temps avec le temps, cette augmentation doit être attribué à des cultures témoins de différents âges.

TABLEAU 13.
Poids moyens des cultures de B. tuberculeux après l'addition de zinc ou de cadmium au cours de la croissance.

Addition de métal au bout de n jours	15	25	36	50
Poids moyens des cultures témoins au bout de n jours	246,6	483,0	626,4	743,1
Zinc	590,2	727,5		
Cadmium	797,7	691,5	687,7	

L'influence du temps sur l'addition de métal est résumé dans le tableau 13. Voici les différences significatives entre les différentes moyennes: *a*. entre les moyennes du 15éme jour: 100,7 mg, *b*. entre les moyennes du 15ème jour et les autres groupes: 153,4 mg, *c*. entre les moyennes du 25,36 et 50ème jour: 116,3 mg. On en conclut: la courbe de croissance de cultures témoins montre une forme normale ce qui permet de faire la comparaison des autres cultures avec cette courbe. L'addition de zinc au 15ème et au 25ème jour n'arrête pas la croissance du B. tuberculeux, elle la ralentit cependant si l'on ajoute le zinc au 15ème jour. Par contre, l'addition de cadmium au 15, 25 et 36ème jour n'a aucune influence sur la croissance de B. tuberculeux.

Nous remercions vivement M. E. VAN DER LAAN, Professeur agrégé de statistique au Landbouwhoogeschool à Wageningen de ses précieux conseils dans l'analyse de la variance.

Kamerlingh Onnes Laboratorium, Leiden.

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Medicine. — *Audiometry by means of a continu audiogram.* By J. VAN EBBENHORST TENGBERGEN, M.D. (Communicated by Prof. A. DE KLEYN.)

(Communicated at the meeting of September 27, 1947.)

The methods for the determination of the auditory threshold have obtained a high degree of refinement and accuracy by means of the audiometer. Previously being committed to tuning fork, flute and monochord, one was enabled by the audiometer to determine the intensity of the auditory threshold in decibel-measure with almost pure tones.

The first audiometers were constructed for a series of fixedly established gauge tones, generally mounting in octaves. The construction of the audiometer forming the audible tones by means of interferency of two high frequent vibration systems, which can be varied in frequency with regard to each other, made it possible to reproduce the whole audible tone-region with continu variable intensity. Both frequency and intensity can be closely read.

The practical use of the audiometer bears traces of the previous period as the principle of fixed gauge tones, whose auditory threshold is determined as accurately as possible, is still used.

Before discussing the audiogram we have to consider the determination of the auditory threshold for *one* pitch.

The determination of the auditory threshold can be made in two manners: the tone is intensified in loudness till it is heard or the tone is decreased from distinct audibility till the testperson states it cannot be heard any more. The values thus found are not the same, because the testperson only recognizes the auditory threshold when the latter is crossed. In intensifying from inaudible to audible, the moment when the sound is heard is mentioned, thus a point just above the auditory threshold. When the intensity is decreased the moment on which the sound is no longer heard, is mentioned, thus a point just below the auditory threshold. In the first procedure the value found is somewhat too large, in the second somewhat too small.

The auditory threshold sought for lies between these two values. The difference in both values is increased by the following facts:

a. Reaction time of test-person and examinator. Especially when the attention of the test-person flags, this time plays a role.

b. The identifiability of a sound increases when the pitch is known beforehand, by which the attention is already drawn to this tone. When the strength of the sound is intensified from inaudible to audible the sound must have crossed the auditory threshold already markedly before the tone is recognized.

The identifiability of a weak sound increases when a intermittent tone is used. It is well known that the determination of the auditory threshold is strongly dependent on tiredness of the test-person.

The usual audiogram is obtained in determining the auditory threshold for a certain amount of tones. The values thus found are put in diagrams, in which the obtained points are connected by a broken line. It is evident that the line thus constructed does not reproduce the actual auditory threshold, as nothing is known about the tones lying between the examined points. The beginning and end of a steep rise or fall in the audiogram are not exactly determined. The accuracy can be considerably increased by augmenting the amount of gauge points markedly. However, what is gained in exactness is partly lost by tiredness of the test-person.

Improvement of the method was sought for by starting from a constant intensity by which the whole audible tone region is passed through. This is a.o. done in the detail audiogram of VAN DISHOECK. The test-person mentions the point at which the increasing tone is first heard and the point at which the sound has completely disappeared. These points can be put into diagrams at once. By varying the intensity step by step the appearance and vanishing points can be taken down for each following intensity level. The obtained points are now connected by a broken line which represents the ultimate audiogram. The whole audible tone region is now systemeticaly searched, so that narrow dips find expression in the audiogram.

Applying the observations about the determination of the auditory threshold to this method, we notice the following facts. When the region of frequency is passed through from low to high, too high values are found for the appearance and vanishing points. When passing from high to low, the inverse is found. It is evident that here also reaction time of examinator and test-person plays a part. The appearance point is observed with more difficulty as the attention was not drawn beforehand to a certain pitch. An advantage is the ever-varying pitch during the examination by which the attention remains more intense. A disadvantage is the fact that the region, already heard in a lower intensity level, must be passed again for each following intensity level. This causes an unnecessary tiredness of the hearing.

Wheras the principle of the ordinary audiogram is founded on a step by step increasing frequency with continu variable intensity, the second method is based on continu variable frequency with step by step increasing intensity.

A combination of these two methods is: continu variable frequency with continu variable intensity. I tried to find a solution in the form of a registration apparatus, connected to the dialling knobs of the audiometer.

The apparatus (fig. 1 and 2) is formed by a registration cylindre, fastened to the frequency knob of the audiometer. By means of a sensitivity-regulation the cylindre, and consequently also the frequency knob of the audiometer, can be slowly and evenly moved. Across the cylindre a stylus

moves parallel with the axis of this cylindre. The stylus is, by means of a chord-transference, moved by the intensity knob of the audiometer. In order to move the intensity knob accurately along small excursions, it is connected to a large disk in front of the apparatus.

The procedure is as follows. The patient himself serves the intensity knob by means of the large disk of the apparatus. The patient is instructed to seek and hold the border between hearing and not-hearing with small about-movements. After short exercise it appears to be fairly easy to let the noise hover across the auditory threshold. The patient then hears a very soft intermittent tone. If the intensity level is too high, the sound doesn't disappear, if too low, no sound-blasts are observed.

The investigator now slowly moves the registration cylindre, by which the whole tone region is passed through, while the patient is keeping the auditory threshold by small about-movements of the intensity disk. These movements are followed by the stylus. The curve thus obtained is a zigzag line, moving around the actual auditory threshold. The deviation of the auditory threshold is about 2—4 db., to both sides, dependent on the ease with which the patient observes the intensity variations.

For the registration ordinary millimetre paper is used. The proportions of the apparatus are calculated in such a way that 1 mm in vertical direction corresponds with a difference in intensity of 1 db. The normal auditory threshold for air and bone conduction is delineated beforehand on the paper; at regular intervals the frequency is marked. The curve thus obtained is shown in fig. 3.

The zigzag line makes a correct determination of the auditory threshold difficult. The actual auditory threshold can be outwardly estimated and sketched. Better results, however, are obtained by a small extension of the apparatus causing the stylus to look itself for the average of the zigzag line.

In order to reach this the stylus is not directly connected to the chord (see fig. 4).

In the chord *a* a steel spring (*c*) is fastened on a level with the stylus (*b*). This spring is kept at a constant distance of the stylus by hanging it on a small gallows which is fastened perpendicularly on the stylus. The proper connection between chord *a* and stylus is made by a small chord *e* connecting the extremities of the spring with the stylus. By making the length of the chord *e* greater than the distance between the extremities of the spring it is possible to move the chord *a* along some distance without moving the stylus. If, for instance, this tolerance is 4 mm to both sides, the stylus is only moved when the chord *a* crosses this distance of 4 mm. The relations being thus chosen that 1 mm of spring movement corresponds with 1 db., the test-person can move the intensity forward and backward from 4 db. below to 4 db. above the auditory threshold; at the same time the stylus remains in the middle i.e. on the wanted auditory threshold.

If the test person appears to be able to notice the auditory threshold easily, i.e. smaller deviations of the zigzag movements, the cord *a* can be

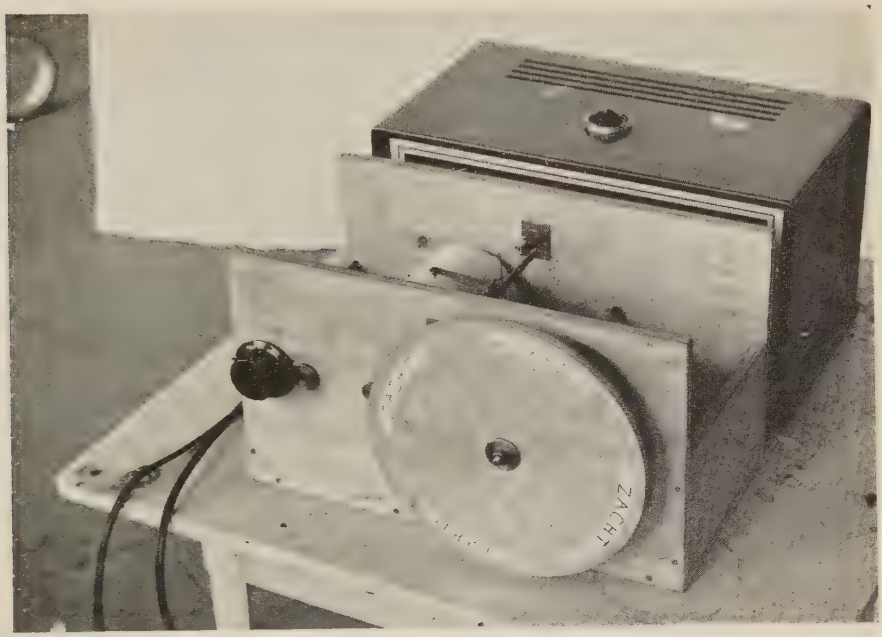


Fig. 1.

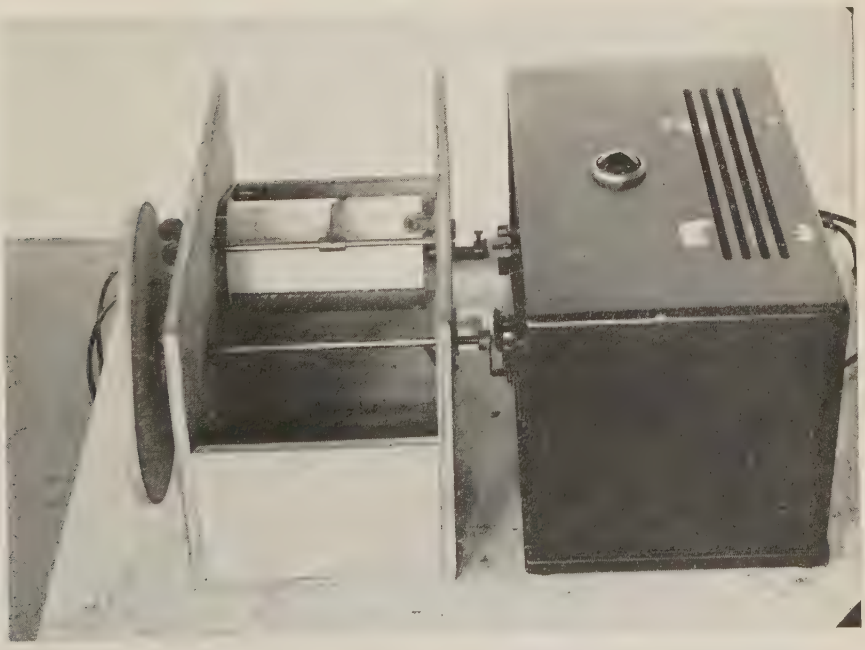


Fig. 2.

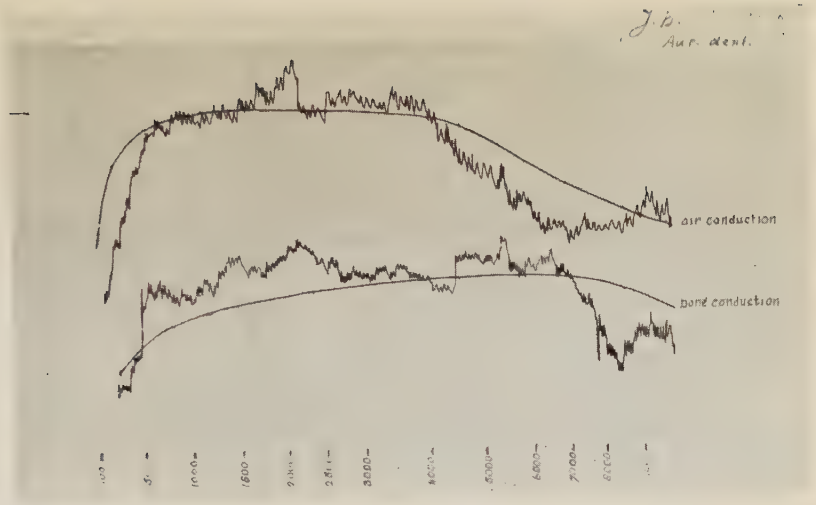


Fig. 3.

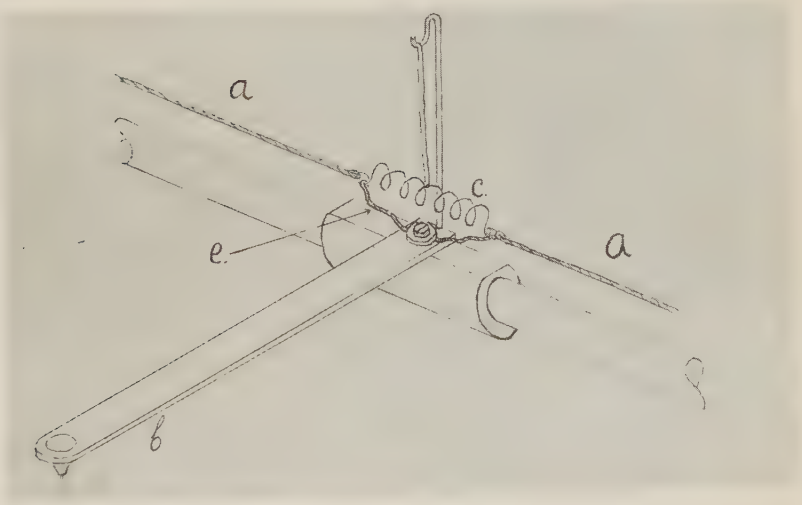


Fig. 4.

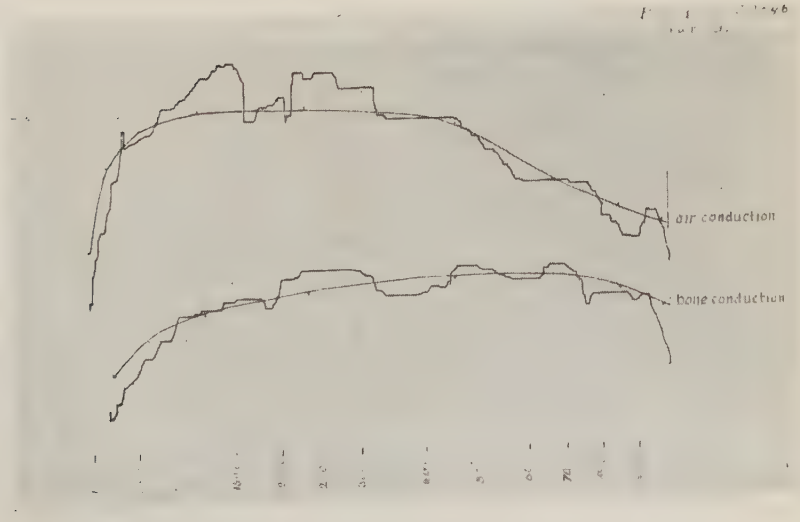


Fig. 5.

tightened by means of a chord-tightner, fixed elsewhere in the apparatus. The spring c is now extended and the tolerance in chord e has become smaller. In this way the apparatus can be adjusted to individual requirements.

The curve thus obtained shows small notches only slightly differing from the actual audiogram (fig. 5).

We considered to make the apparatus completely automatic by moving the registration cylindre mechanically with a fixed velocity. This, however, should be injurious to the preciseness of the audiogram. The investigator's task is to follow the movements of the stylus accurately. When in the audiogram a steep fall or rise begins, the patient appears not to be able to follow the auditory threshold easily owing to too large excursions. The frequency movement must then immediately be slackened or stopped till the patient has adapted himself to the new level. If no important rise or fall exists, the movement can be slightly accelerated in order to make the investigation as short as possible.

Intelligent patients can serve the frequency knob themselves, which will improve the preciseness.

Summarizing the attainments of this method we find the following.

1. The whole region is searched with a continu variable intensity. This is so far approximated as the quality of the audiometer permits. A quality of our audiometer is a non-completely continu intensity arrangement but an arrangement mounting with small jerks. This is probably caused by a too rough winding of the variable resistence. By this large notches are found in the curve.

2. It is a direct-graphic method. It is not necessary to make the curve with interpolation. The db.-measure can be read in mm from the vertical distance from the curve to the line for normal hearing.

3. Reaction time of patient and investigator are almost eliminated.

4. It makes no difference whether the frequency runs from low to high or inverse.

5. The attention of the patient remains intense, because he contributes actively, while the observation is facilitated by the intermittent tone.

6. The whole tone-region is passed through but once, the investigation thus taking the same time as a normal audiogram and not causing a greater tiredness of the hearing.

Preliminary experiences.

It appears possible to make a continu audiogram of each patient with a normal intellect. Children of 10 years and older gave no particular difficulties. When the intellect was very low, the curve showed many irregular movements. The question remains, however, whether the continu audiogram is in this respect inferior to the normal audiogram. In the normal audiogram

the eventual result is a series of numbers presumed to represent the actual auditory threshold. The extent of the inexactness cannot be concluded from the numbers themselves. By this a greater exactness is supposed than is really the case. The continu audiogram, however, by abnormally large deviations or abnormal straightness of the curve, immediately betrays that the necessary reserve must be exercised in reading it.

Not every patient appeared to be able to notice the auditory threshold with the same facility. In cases of undamaged hearing this caused generally more difficulties than in cases with loss of hearing. Patients, suffering from perception deafness and regression, can very easily recognize the auditory threshold by means of the rapid rise of the loudness-sensation in increasing energy.

Chemistry. — *The mechanism of isomerisations of unsaturated fatty acids by SO_2.* By J. H. DE BOER[1]), J. P. W. HOUTMAN[2]) and H. I. WATERMAN[3]).

(Communicated at the meeting of October 25, 1947.)

§ 1. *Introduction.*

Systematic investigations by WATERMAN and collaborators on the polymerisation of linseed oil[4]) and of linoleic and linolenic esters by the action of SO_2 have shown[5]) that prior to polymerisation there is a shift of one or more double bonds in the molecules. This isomerisation leads from a system of isolated double bonds to a system with conjugated double bonds. The reaction is restricted to those unsaturated fatty acids which have in their molecules a system of two double bonds separated by a central CH_2-group. As an example we mention the isomerisation of linoleic ester:

$$CH_3(CH_2)_4CH{:}CH{-}CH_2{-}CH{:}CH(CH_2)_7COOR \rightarrow CH_3(CH_2)_5CH{:}CH{-}CH{:}CH(CH_2)_7COOR$$

It is not easily understood why such an isomerisation should be catalysed by SO_2. It is the purpose of this note to discuss a possible mechanism which may offer an explanation. We will start our considerations with a short discussion of two other catalytic reactions by which this isomerisation can be performed.

§ 2. *Conjugation by hydrogenation catalysts.*

It is known that isomerisation of these molecules to form systems with conjugated double bonds also proceeds at the surface of hydrogenation catalysts, like paladium[6]), nickel[7]), platinum on active carbon or nickel on kieselguhr[8]).

A hydrogenation catalyst, such as nickel, may give rise to a shift of the

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[3]) Laboratory for Chemical Engineering, University, Delft (Holland).

[4]) H. I. WATERMAN and C. v. VLODROP, J. Soc. Chem. Ind., **55**, 333 T (1936).

[5]) H. I. WATERMAN, C. v. VLODROP and M. J. PFAUTH, Verfkroniek, **13**, 130 (1940). British Patent 544, 482, April 15th 1942 (H. I. WATERMAN and C. v. VLODROP to I.C. I).

[6]) C. W. MOORE, J. Soc. Chem. Ind., **38**, 320 T (1919).

[7]) H. I. WATERMAN, C. v. VLODROP and W. J. TAAT, Chimie et Industrie, **44**, 285 (1940).

[8]) H. I. WATERMAN and M. J. v. TUSSENBROEK, Chem. Weekblad, **26**, 410, 566 (1929); **27**, 146 (1930).
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double bond in mono-unsaturated molecules also provided gaseous hydrogen is present[9]). For the conjugation isomerisation of the system of two double bonds, separated by a central CH_2-group the presence of gaseous hydrogen, however, is not necessary[10]).

The isomerisation may — formally — be understood in these cases to be caused by the transfer of a hydrogen *atom* of the central CH_2-group *a* of the typical grouping:

$$\begin{array}{c} R-CH=CH-CH_2-CH=CH-R' \\ a-2\ a-1\ \ a\ \ a+1\ a+1 \end{array}$$

to the metal. It is known that metallic hydrogenation catalysts may easily take up hydrogen atoms as well as transfer them other molecules.

The resulting radical

$$R-CH=CH-\underset{|}{CH}-CH=CH-R'$$

may be considered as one of the forms of a resonance hybrid, other contributing forms of which are

$$R-\underset{|}{CH}-CH=CH-CH=CH-R'$$

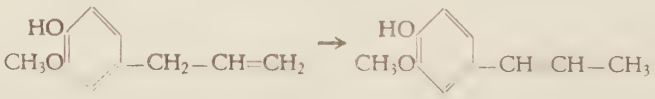
$$\text{and } R-CH=CH-CH=CH-\underset{|}{CH}-R'.$$

When this radical takes up a hydrogen atom from the metal again, we may, therefore, expect this hydrogen to be added either to carbon atom *a*–2, to *a* or to *a* + 2. An addition to *a*–2 or *a* + 2 is even more probable because of the resonance energy of the conjugated system in the resulting molecule[11]).

Even if every molecule collides and reacts with the metal surface only once, we may already expect a product with more than $^2/_3$ of the molecules in the conjugated form.

§ 3. *Conjugation by alkali.*

The isomerisation under consideration may also often be performed by the action of aqueous or alcoholic alkali, as is well known in the technically important isomerisation of eugenol to iso-eugenol by alkali[12]):



[9]) G. H. TWIGG, Proc. Roy. Soc., (London), A 178, 106 (1941).

[10]) See for one of the latest articles: S. B. RADLOVE, H. M. TEETER, W. H. BOND, J. C. COWAN and J. P. KASS, Ind. Eng. Chem., 38, 997 (1946).

[11]) The picture developed here runs completely parallel to the picture of the mechanism of oxidation of similar systems, as developed by FARMER and collaborators, see e.g. E. H. FARMER, H. P. KOCH and D. A. SUTTON, J. Chem. Soc., 1943, 541, and also § 4 of this article.

[12]) In a case like this, as in many others, the position of one of the double bonds of the system is fixed by being part of an aromatic ring, hence the shift can take place to one side only.

or in the transfer of α-carotene into β-carotene by heating in an alcoholic solution with sodium-ethylate [13]).

The isomerisation under the influence of alkali, which can also be obtained readily in the case of linoleic or linolenic acids [14]), is often considered to be performed by adding the elements of water: OH and H to e.g. the carbon atoms $a + 1$ and $a + 2$ respectively and by subsequently losing OH and H again by carbon atoms $a + 1$ and a respectively. In some, similar, cases the corresponding hydroxy compounds are isolated. Vinyl acetic acid, when boiled with caustic soda, is converted into crotonic acid and β-hydroxy-butyric acid [15]):

$$CH_2{=}CH{-}CH_2{-}C\overset{=O}{\underset{-OH}{}} \rightarrow \begin{cases} CH_2{-}CH{=}CH{-}C\overset{=O}{\underset{-OH}{}} \\ \text{and} \\ CH_3{-}CHOH{-}CH_2{-}C\overset{=O}{\underset{-OH}{}} \end{cases}$$

With a metallic hydrogenation catalyst, therefore, the reaction starts by taking away something of the molecule (a H-atom); with alkali-hydroxides as catalysts the first step seems to be to add something to one of the double bonds. Neither of the simple starting mechanisms seem to serve us, however, in the case of SO_2 as isomerisation catalyst.

§ 4. *Conjugation during oxidation.*

Apart from the ability to be isomerised easily, the systems under consideration also show a higher rate of oxidation than other systems with double bonds. GUNSTONE and HILDITCH [16]) found that the ratio of the rates of oxidation by oxygen of the methyl esters of oleic-, linoleic- and linolenic acids at 20° C is 1 : 12 : 25.

Systematic investigations by FARMER and collaborators [17]) have suggested that oxidation of systems with nonconjugated double bonds leads to the formation of hydroperoxides at methylene groups adhacent to the double bonds. The autoxidation reaction proceeds as a chain-reaction of the radical type; the radical is formed by the loss of a hydrogen atom from this α-methylene group:

$$R{-}CH{=}CH{-}\underset{|}{CH}{-}R'$$

The radical, however, is a resonance hybrid, the other contributing form being:

$$R{-}\underset{|}{CH}{-}CH{=}CH{-}R'$$

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[13]) P. KARRER and E. JUCKER, Helv. Chim. Act., **30**, 266 (1947).

[14]) J. P. KASS and G. O. BURR, J. Am. Chem. Soc., **61**, 3292 (1939); **62**, 1796 (1940). T. F. BRADLEY and D. RICHARDSON, Ind. Eng. Chem., **32**, 963 (940); **34**, 237 (1942).

[15]) It may be remarked that also in this example, where one of the double bonds is not a C = C- but a C = O double bond, the shift can only take place to one side.

[16]) F. D. GUNSTONE and T. P. HILDITCH, J. Chem. Soc., 836 (1945).

[17]) See e.g. E. H. FARMER, Trans. Far. Soc., **38**, 340 (1942); E. H. FARMER and A. SUNDRALINGAM, J. Chem. Soc., 121 (1942; 125 (1943); E. H. FARMER and D. A. SUTTON, J. Chem. Soc., 139 (1942); 119, 122 (1943).

When the radical is saturated by -OOH, it is easily seen that part of the resulting hydroperoxide-molecules have the double bond at the original position; in part of them, however, the double bond is shifted.

When applied to the system of two double bonds, separated by one CH_2-group, the result of the oxidation is that part of the formed hydroperoxides possess a system of conjugated double bonds [18]). It is interesting to note that also in the enzymitic oxidation of linoleic acid and of esters of linoleic- and linolenic acids by lipoxidase, a high proportion of oxidation products with conjugated double bonds is formed [19]).

As BOLLAND [20]) points out, the most likely formation of the starting radical is by losing a hydrogen atom from the central CH_2-group (carbon-atom *a* in our notation), because the resonance energy associated with the resulting radical (three contributing forms with the "free" valency at carbon atoms *a*–2, *a* and *a* + 2 respectively) is much higher than when the radical is formed by losing a hydrogen atom from either carbon atom *a*–3 or *a* + 3:

$$\underset{a-3}{R-CH_2-}\underset{a-2}{CH=}\underset{a-1}{CH-}\underset{a}{CH_2-}\underset{a+1}{CH=}\underset{a+2}{CH-}\underset{a+3}{CH_2-R'}$$

In these latter cases there are only two contributing forms to the resonance hybrids, namely either a free valency at one of the carbon atoms *a*–3 and *a*–1 or at one of the carbon atoms *a* + 3 and *a* + 1; the resonnance energy is estamined to be 18.7 Kcal/mol in this case, as against 30.5 Kcal/mol when the hydrogen is taken from carbon atom *a* [21]), [22]).

Before the autoxidation of α-methylene groups to hydroperoxides can proceed, radicals with free valencies on these α-methylene groups have therefore to be formed. It is not until such radicals are formed that the real chainreaction sets in. FARMER [23]) suggests that the initial reaction starts with the addition of an oxygen molecule to one of the carbon atoms of a double bond, thus leading to a bi-radical: —

$$-R-\dot{C}H_2-CH=CH-R' + O_2 \rightarrow -R-CH_2-\underset{\displaystyle O-O\cdot}{CH}-CH-R'$$

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[18]) E. H. FARMER, H. P. KOCH and P. A. SUTTON, loc. cit. [11]); J L. BOLLAND and H. P. KOCH, J. Chem. Soc., 445 (1945).

[19]) S. BERGSTRÖM, Arkiv Kemi, Mineral. Geol., **21** A No. 14 and No. 15 (1935), see R. T. HOLMAN, Archives of Biochemistry, 519 (1946).

[20]) J. L. BOLLAND, Proc. Roy. Soc., A **186** 218 (1946).

[21]) J. L. BOLLAND and W. J. C. ORR, I.R.I. Trans., **21**, 133 (1945).

[22]) This result is completely in harmony with the well-known fact that such a central CH_2-group is an "active" one, the hydrogen of which is "mobile". This is especially the case when the CH^2-group is situated between C = O double bonds, as in acetyl-acetone, malonic ester, etc., or between one C = C and one C = O double bond.

[23]) E. H. FARMER, Trans. Far. Soc., **42**, 228 (1946).

Subsequent collisions of such a bi-radical with other molecules may then lead to the transfer of a hydrogen atom of the α-methylene group of these other molecules to the free valencies of the bi-radical, thus leaving a free valency at the α-methylene group behind.

§ 5. *Differences between the action of oxygen and of SO_2.*

When we compare the reaction with oxygen and the action of SO_2 we must bear in mind a few significant differences. Firstly there is no SO_2-containing ultimate product isolated in these reactions, secondly SO_2 produces a shift of double bonds only in those systems where two double bonds are separated by a central CH_2-group, thirdly there are no signs that the reaction with SO_2 has an autocatalytic character. Let us look into each of these three points separately:

1. The ultimate products do not contain SO_2. This, of course, does not mean that SO_2 could not be temporarily added and split off again during the reaction. Addition of SO_2 is known to occur at systems with conjugated double bonds, cyclic sulphones are isolated and well studied; they all lose SO_2 at higher temperatures, thus leaving the original hydrocarbons behind [24]). Although a careful examination of the crystal structure of one of these compounds (β-isoprene-sulphone) has revealed that the structure is probably a resonance hybrid of several forms, it is nevertheless certain that the compound is formed by sulphur-carbon bonds and not by oxygen-carbon bonds [25]).

2. SO_2 produces only a shift of double bonds in a system with two double bonds, separated by a central CH_2-group. As we have discussed above, oxygen produces hydroperoxides with shifted double bonds already from compounds with one double bond only.

SO_2, however, though leaving the double bond in its place, may produce a cis-trans-isomerisation, in a system with one double bond. Whilst the shift of double bonds to produce a conjugated system in linoleic acid proceeds at temperatures of about 180° C—200° C in liquid SO_2, oleic acid is transformed into elaeidic acid when heated with liquid SO_2 at 100° C—140° C for a longer time. This latter reaction is a strong indication that a temporary addition compound of SO_2 to one of the carbon atoms of the double bond may be formed.

3. There are no signs that the reaction has an autocatalytic character. From the study of the reaction one gets the impression that every con-

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[24]) J. BÖESEKEN and E. DE R. VAN ZUYDEWIJN, Rec. trav. chim., **56**, 1047 (1937); **57**, 445 (1938). H. J. BACKER and J. A. BOTTEMA, Rec. trav. chim., **51**, 294 (1932); H. J. BACKER and J. STRATING, Rec. trav. chim., **53**, 525 (1934); **54**, 618 (1935); H. J. BACKER, J. STRATING and A. J. ZUITHOFF, Rec. trav. chim., **55**, 761 (1936); H. J. BACKER and C. C. BOLT., Rec. trav. chim., **54**, 538 (1935); H. J. BACKER, J. STRATING and C. M. H. KOOL, Rec. trav. chim., **58**, 778 (1939).

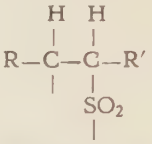
[25]) E. G. COX and G. A. JEFFREY, Trans. Far. Soc., **38**, 241 (1942).

version goes via a direct action of SO_2 rather than via a chain-reaction which is initiated by SO_2.

§ 6. *Suggested mechanism for the cis-trans-isomerisation.*

The above considerations lead us to the following suggestions for the explanation of the cis-trans-isomerisation of oleic acid by SO_2:

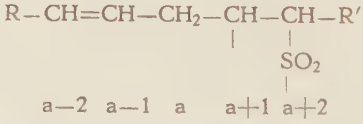
SO_2 may be added to one of the carbon atoms of the double bond, forming a bi-radical with one free valency on the sulphur and a second one on the other carbon atom of the original double bond:



In this bi-radical there is not only a single bond between the two carbon atoms, but one of these carbon atoms carries only three substituents. If, therefore, the period of existence of the SO_2-addition product is long enough, rearrangements of the mutual position of the two carbon atoms and of the substituents on one of them may occur[26]). When subsequently SO_2 splits off again the double bond may be restored in such a way that either a cis- or a trans-position of the two large substituents R and R' will result. Consequently a partial cis-trans-isomerisation may have taken place as determined by the energy relations of the resulting products. It is not necessary that the large substituents R and R' actually rotate round the temporarily single bond.

§ 7. *Suggested mechanism for the SO_2 catalysed conjugation.*

The catalytic action of SO_2 in the conjugation isomerisation starts probably in the same way. In the resulting bi-radical in this case, however, there is a CH_2-group (*a*) left, still "activated" by the other remaining double bond:



The transfer of a hydrogen atom from CH_2-group *a* to the sulphur atom, attached to carbon-atom $a + 2$ seems to be quite feasible, also from sterical considerations. The resultant free valency on carbon atom *a* and the free valency on carbon atom $a + 1$ would saturate each other and form a

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[26]) The cis-trans-isomerisation by bromine atoms in the case of maleic- and fumaric acids is also caused by the formation of radicals, in which one of the carbon atoms carries only three contituents, see e.g. W. HÜCKEL, Theoretische Grundlagen der Organische Chemie, Leipzich 1931, Band I, page 269.

double bond which is conjugated with the one between carbon atoms $a-2$ and $a-1$:

$$R-CH=CH-CH=CH-\underset{\displaystyle HSO_2}{\underset{|}{CH}}-R'$$

This sulphone, in splitting off SO_2, gives:

$$R-CH=CH-CH=CH-CH_2-R'$$

§ 8. *Influence of the orientation by polar forces.*

It may be asked whether the polar character of SO_2 might play a role in this suggested mechanism. SO_2 carries a relatively large dipole, the moment of which is 1.61 Debye-units. The positive pole is on the sulphur side of the triangular molecule (valency angle O-S-O about 125°, distance O-S 1.37 A). The combination of a double carbon-carbon-bond and an adjacent CH_3- or CH_2-group also gives rise to local dipole moments which, depending on the symmetry conditions, may or may not result in a total dipole moment of the molecule as a whole. The CH_3- of CH_2-groups form the positive poles of such partial dipoles, the negative charges are more or less located on the carbon stoms of the double bond.

In the mutual attraction of the molecules, prior to the addition of SO_2 to an isolated double bond, the sulphur of the SO_2 will therefore be directed towards one of the C-atoms of the double bond. This is in accord with the experimental fact that in such an addition (and also in the addition of bisulphite) C—S links are always formed and not C—O links.

In systems of two double bonds, separated by one $-CH_2-$ group, this central CH_2-group is the site of a positive charge and the carbon atoms $a-2$ or $a-2$ are to be considered as the sites of negative charges. When SO_2 is added, the sulphur atom is directed either to carbon-atom $a+2$ or to $a-2$ and will be added to one of these C-atoms. One of the oxygen atoms of SO_2 is orientated into the direction of the central CH_2-group and the possibility might be considered for one of the hydrogen atoms to be added to such an oxygen atom, rather than to the sulphur-atom as in our consideration above. The resultant sulphinic acid would, however, easily transfer into the corresponding sulphone [27]):

$$R-CH=CH-CH=CH-\underset{\displaystyle HOSO}{\underset{|}{CH}}-R' \rightarrow R-CH=CH-CH=CH-\underset{\displaystyle HSO_2}{\underset{|}{CH}}-R'$$

or it would, when losing SO_2, transfer its hydrogen to the carbon atom to which it is attached.

The polar properties of SO_2, therefore, assist the addition mechanism to follow the suggested course. The typical feature of the catalytic action of SO_2 in the cis-trans-isomerisation and in the conjugation isomerisation is,

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27) T. P. Hilditch, J. Chem. Soc., **93**, 162 (1908).

in our opinion, however, a radical reaction. This radical reaction does not start a chain reaction, as in the case of oxygen.

Summary.

SO_2 catalyses the cis-trans-isomerisation of unsaturated fatty acids, such as oleic acid. At somewhat higher temperatures it also catalyses the conjugation of linoleic and linolenic acids.

Both catalytic reactions may be explained by assuming that SO_2 forms an addition product, which has the character of a bi-radical. Intramolecular rearrangements and subsequent splitting off of SO_2 result in the isomerisations.

When the typical features of this and other isomerisation catalysts are compared, the following differences may be pointed out:

a) In the isomerisation by a heterogeneous catalyst a hydrogen atom is taken away from the molecule and when another hydrogen atom is added to the resultant radical this addition may take place at another carbon atom.

b) In the isomerisation by means of alkali, a OH-group and a H-atom may be added to a double bond, whereupon the OH-group and a hydrogen atom from another carbon atom are removed.

c) In the isomerisation by SO_2, a molecule of SO_2 is added to the double bond and performs the transfer of a hydrogen atom of one carbon atom of the molecule to another carbon atom.

Chemistry. — *Contribution to the explanation of motory and desintegration phenomena in complex coacervate drops in the electric field.* By L. DE RUITER and H. G. BUNGENBERG DE JONG.

(Communicated at the meeting of September 27, 1947.)

A. *Introduction.*

In a previous article [1]) it was suggested that local changes in the interfacial tension (σ) coacervate-medium might play an essential part in the occurrence of the motory and desintegration phenomena, which have been observed in complex coacervate drops in the electric field and in diffusion fields of substances, which have a swelling or densifying influence on the coacervate.

It was supposed that the assumed σ-changes in the electric field would be due to variations of pH caused by polarisation of the surface of the drops. No immediate data however were available at that time concerning the influence of diverse variable factors on σ.

No more had the postulated pH changes been directly demonstrated. Some data concerning the value of σ and the influence thereon of mixing proportion and salt concentration in gum arabic (A)-gelatine (G) systems have since been obtained [2]).

In the present article some experiments will be mentioned, which were carried out on gum arabic-gelatine systems in order to demonstrate the postulated pH changes. Although we did not succeed therein (which may be due to the fact that we worked with buffered systems), we found indications for the existence of another factor, which may be responsible for the assumed changes of σ, viz. a local shift of the composition of the coacervate in the electric field.

Finally it will be undertaken to give, in the light of these recent data, a more detailed explanation of the phenomena observed in coacervate drops in electric and diffusion fields.

B. *Description of phenomena in the electric field.*

In our experiments we used systems prepared by mixing 2 % A- and 2 % G sols at 40° C. These stock sols had been buffered with a mixture of sodium acetate and acetic acid at pH 3.7 [3]) and were of the following composition:

20 gr. G, resp. A, (airdry)
50 cc 0,1 n. sodium acetate
50 cc 1 n. acetic acid
920 cc dest. water.

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[1]) H. G. BUNGENBERG DE JONG and E. G. HOSKAM, Proc. Ned. Akad. v. Wetensch., Amsterdam, **44**, 1099 (1941).

[2]) L. DE RUITER and H. G. BUNGENBERG DE JONG, Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, **50**, 836 (1947).

[3]) H. G. BUNGENBERG DE JONG and E. G. HOSKAM, Proc. Ned. Akad. v. Wetensch., Amsterdam, **45**, 585 (1942).

Whenever in the following pages a e.g. "70 % *A* system" is mentioned, a system is meant consisting for 70 % of stock sol 2 % *A* and for 30 % of stock sol 2 % *G*.

For every single observation a small quantity of equilibrium liquid in which coacervate drops were suspended was brought on a starched [3a]) heated (about 40° C.) glass surface under the microscope. The electric current was applied to the preparation by means of the electrodes mounted on the objective of the microscope, which have been described before [3a]). We used fields ranging from ± 0,5—5 V/cm: the intensity of the current has hardly any effect on the phenomena. In very weak fields only they are less clearly seen, whereas in too strong fields they follow one another so rapidly that accurate observations are impossible.

The phenomena occurring in electrophoretically charged (− or —) drops in the electric field may be shortly described as follows (in the normal sequence):

1. In alternating fields the originally circular drops become elliptical in form (short axis parallel to direction of field). This is called the BÜCHNER-effect. No other effects of alternating fields have been observed. The BÜCHNER effect occurs in direct current fields as well (see fig. 1—4), but the elliptical form may become more or less irregular, when the effect of movements within the drops is superposed on it (fig. 3).

2. In direct current fields a cloud of small vacuoles originates on one side of the drops, at some distance under their surface. This we called the "coarse vacuolisation". On which side it will occur is determined by the sign of the electrophoretic charge of the drop and the direction of the field (see fig. 1).

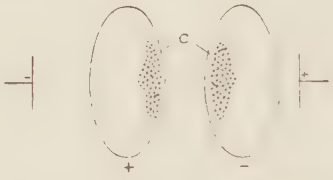


Fig. 1. *c* = coarse vacuolisation.

3. On the other side of the drop at the extreme periphery a single layer of very small and very closely packed vacuoles is formed, the socalled "peripheral vacuolisation". These vacuoles move as soon as they are formed in the direction of the poles of the drops (i.e. the points where the long axis cuts the surface). At the aequator however the formation of the small vacuoles is continued as long as the layer can be clearly seen (fig. 2, *p*).

4. In the mean time, on the side where the coarse vacuolisation is found, in the equilibrium liquid outside the drop and at a certain distance from its surface, a number of very small coacervate droplets is formed (fig. 2, *a*).

[3a]) See note 1.

5. During these events a movement of the coarse vacuoles to the inner parts of the drop begins. This movement is fastest at the aequator. The



Fig. 2. a = coacervation in the equilibrium liquid;
c = coarse vacuolisation;
p = peripheral vacuolisation.

movement of the peripheral vacuoles is continued; they pass the poles and begin to move along the opposite surface of the drop. Then however it seems as though they are arrested by the coarse vacuolisation: they turn off and near both poles a small vortex is formed. Often the drops slightly bulge out there (fig. 3). In the meantime the formation of coarse vacuoles has gone on in their original areal, and soon the drop is quite filled up with vacuoles, which makes further observation difficult. The movements slow

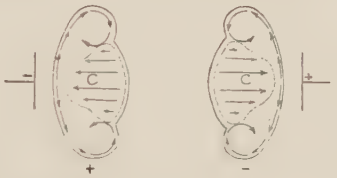


Fig. 3. Movements of vacuoles in a coacervate drop in the electric field (schematically).
c = coarse vacuolisation.

down and become very irregular. The vacuoles increase in size and number and continually break through to the outside. The regular form of the drop is totally destroyed.

Somewhat different are the phenomena in direct current fields observed in electrophoretically neutral drops (about 48 % A). It is comparatively difficult to realize this case and it was observed only a few times. The most striking phenomenon is that the movements of the coacervate within the drop form a "four-quadrant-system" (fig. 4). Less clear are the vacuolisation processes in this case. In one preparation we initially observed coarse vacuolisation on one side, then, when after a short interruption the



Fig. 4. Schematical drawing of morphological phenomena in an electrophoretically neutral drop in which vacuolisation of both types was observed on both sides.

current was closed again, on both sides of the drop a peripheral and a coarse vacuolisation (fig. 4). In one case also we observed the formation of small coacervatedrops in the equilibrium liquid on both sides of the drop.

A transport of coarse vacuoles across the drop, parallel to the direction of the field was never observed in electrophoretically neutral drops. In some cases however they showed a slow movement in the direction of the poles (fig. 4).

C. *Observations in the electric field on coacervates coloured with pH indicators.*

These experiments were carried out in order to ascertain whether indeed pH changes occur in the coacervate surface in the electric field. We used the following indicators:

Indicator	pH interval	acid-alkali colour change
Bromophenol blue	3,0—4,6	yellow-blue
Congo red	3,0—5,2	blueish-red
Methyl orange	3,1—4,4	red-yellow

It was necessary to use the indicators in relatively high concentrations, (e.g. 5 cc of a 0.4 % neutralized solution of bromophenol blue in 10 cc of the stock sols) otherwise the difference in intensity of colour between coacervate and equilibrium liquid was insufficient. An inevitable consequence was that the coacervate was slightly vacuolized already at the beginning of the experiment.

The colour differences described hereafter are, in as far as observed through the microscope, although indeniable, never very striking. Favorable light therefore is an essential factor in these experiments. For observation on bromophenol blue and congo red: daylight, for Janus green: light from an ordinary electrical bulb, proved preferable. Of equal importance was the moment chosen for the observation: as soon as vacuolization becomes too intensive, it is very difficult to see the colour differences.

In the experiments with methyl orange no colour change was ever observed in the coacervate drops on closing the current.

Experiments with other indicators and with the dye Janus green (2 cc of 0.1 % solution in 10 cc) gave the following results (see next page):

In several experiments with bromophenol blue it was observed that on reversal of the direction of the field the originally still blue side turns greenish and the other one blue again.

The experiments (see table next page) do prove indeed that in the electric field colour changes occur in the drops, which are determined by the direction of the field and are independent of the electrophoretical charge of the drop. The possibility that this is due to pH changes however is excluded by the fact that bromophenol blue changes to its more acid colour on the

Indicator	Colour of coacervate	Mixing proportion	Elec. troph. charge	Change of colour in coacervate drop at the side of the	
				cathode	anode
Br. ph. Blue	blue	± 1 A : 1 G	—	stays deep blue	turns greenish
		id.	—	" " "	" "
		3 A : 7 G	+	" " "	" "
		id.	+	" " "	" "
		id.	+	" " "	" "
		1 A : 2 G	+	" " "	" "
		id.	+	" " "	" "
		id.	+	" " "	" "
		id.	+	" " "	" "
		id.	+	" " "	" "
		id.	+	" " "	" "
		2 A : 1 G	—	" " "	" "
		id.	—	" " "	" "
		id.	—	" " "	" "
		id.	—	" " "	" "
Congo red	violet	± 1 A : 1 G	—	turns red.	
		G rich	+	too weakly coloured, no changes.	
		id.	+	no evident change	
		A rich	—	no evident change	
		id.	—	turns slightly reddish	
		id.	—	purple	dirty blue green *)
Janus green	blue	2 G : 1 A	+	violet	greenish-blue
		2 A : 1 G	—		greenish

*) Very large drop of coacervate.

anodal side of the drop, whereas congo red does the same on the kathodal side of the drop! Moreover methyl orange shows no colour changes at all.

In this connection the following observations are important (see next page):

It is true that the pH of the 2 stock sols is very slightly different, as the buffer can only be used in low concentration [4]). This however cannot be the cause of the observed colour differences: the colour of stock sol G with bromophenol blue does not occur at all in the colour-interval of this indicator.

Besides it was found that addition of great quantities of buffer (i.e. neutralisation of the supposed pH shift) did not in the least reduce the colour difference between the 2 stock sols when coloured with Janus green.

The experiments mentioned above indicate that systems containing gelatine are coloured abnormally by bromophenol blue, congo red and Janus green (which in the case of indicators is generally known as "protein errors" of the indicators).

[4]) H. G. BUNGENBERG DE JONG and E. G. HOSKAM, Proc. Ned. Akad. v. Wetensch., Amsterdam, loc. cit. (1942).

Mixing proportion	Indicator	Colour
100 % Stock sol A (pH 3.7)	Br. ph. blue	dirty blue green
100 % Stock sol G (pH 3.7)	id.	clear deep blue
Acetate buffer (pH 3.7)	„	dirty blue green
1 A : 2 G eq. liquid	„	clear deep blue
coacervate *)	„	blue
2 A : 1 G eq. liquid	„	greenish
coacervate	„	bleu green
100 % Stock sol A (pH 3.7)	Congo red	blue
100 % Stock sol G (pH 3.7)	id.	clear red
Acetate buffer (pH 3.7)	„	blue
G rich coacervate	„	reddish violet
A rich coacervate	„	violet
100 % Stock sol A (pH 3.7)	Janus green	green blue (electr. light!)
100 % Stock sol G (pH 3.7)	id.	violet
Acetate buffer (pH 3.7)	„	green blue
G rich coacervate	„	greenish blue
A rich coacervate	„	colours **)

*) In these experiments the colour of the coacervates was observed in about 0,2 mm thick layers.

**) These colours are difficult to distinguish.

Now the explanation of the colour changes observed in coacervate drops in the electric field is obvious. It is to be expected that in the electric field a transport of the coacervate components will occur, gelatine molecules moving in the direction of the kathode and gum arabic molecules moving in the direction of the anode. Consequently the kathodal side of the drop will get relatively rich in *G*, the anodal side relatively rich in *A*. The observed colour changes are in perfect agreement with this hypothesis: the colour changes on the cathodal side always indicate an increase of the relative quantity of gelatine.

Indicator	Colour at pH 3.7	Colour in Stock sol G.	Colour change in coac. drop on	
			kathodal side	anodal side
Br. ph. blue	blue green	clear deep blue	blue	greenish
Congo red	blue	violet	red purple	greenish blue
Janus green	greenblue	violet	violet	green blue

In consequence of this transport of *A* and *G* within the drop in opposite directions, the 2 sides of each drop might become oppositely charged. We did however not yet succeed in demonstrating this directly.

D. *Explanation of the morphological phenomena in coacervate drops in the electric field.*

The experiments which were described above indicate that in a direct current field A—G coacervate drops will become relatively rich in A on the anodal side and relatively rich in G on the kathodal side. Now measurements [5]) of the interfacial tension (σ) coacervate-equilibrium liquid have proved that such changes in the composition of a coacervate will always be accompanied by a change in σ, the direction of which is determined by the fact that σ reaches a maximum value at the optimal mixing proportion (i.e. the mixing proportion at which the coacervate volume is maximal, which nearly coincides with the point at which reversal of charge occurs).

So in the first place it can be stated that the σ changes, postulated before for the explanation of the motory phenomena, will indeed occur, though in consequence of another mechanism than previously assumed. Moreover the observed shifts in colloid composition are responsible for a number of "desintegration" phenomena as will be apparent from the following paragraphs in which the explanation of these phenomena is given as far as possible at the present time.

1. The BÜCHNER effect will probably be due to the accumulation of opposite electrical charges on the surfaces of the drop facing the electrodes, in consequence of differences in conductivity or dielectrical constant between drops and equilibrium liquid [6]). This will cause the deformation. Every deformation of a spherical body increases its area and is therefore opposed by the surface tension. As in our case the deforming energy has a finite value, equilibrium will be reached as soon as these two energies are equal to each other. This agrees with our observations.

2. The origin of the coarse vacuoles may be explained as follows:

In fig. 5 a schematical phase diagram (isotherm) for $A + G - H_2O$ systems is given. The two loops, which are parts of one closed curve, represent the compositions of the coacervates (the big loop) and of the equilibrium liquids (the small loop near the origin) respectively. When we start from a positive coacervate drop (situated at $-$), the change in composition on the kathodal side of the drop will be represented by k (simultaneous increase of G and decrease of A), the change that tends to occur at the anodal side by a (simultaneous decrease of G and increase of A). This vector a is however directed towards the interior of the heterogeneous region, representing two-phase systems.

At the anodal side therefore the system will separate into a smaller quantity of equilibrium liquid and a larger quantity of coacervate, i.e. the coacervate will vacuolise (see fig. 1). No unmixing however will occur

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[5]) L. DE RUITER and H. G. BUNGENBERG DE JONG, Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, **50**, 836 (1947), see fig. 4 and 5.

[6]) E. H. BÜCHNER and A. H. H. VAN ROYEN, Kolloid Z. **49**, 249 (1929).

on the kathodal side, vector k being directed towards the interior of the homogeneous region.

Similar argumentation for a negatively charged drop (situated at — in fig. 5) leads to the conclusion, that vacuolisation will occur on the kathodal side. This is in full agreement with the experiments (fig. 1).

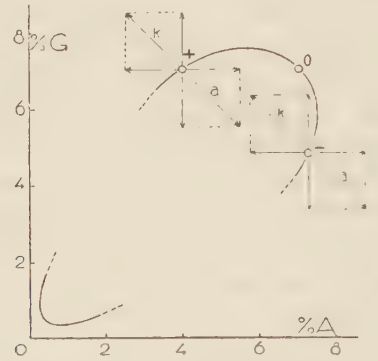


Fig. 5. Schematical representation of a phasediagram for 2 % A—2 % G mixtures. The shift which tends to occur in the composition of positive (+) and negative (—) coacervate drops on the sides facing the kathode (k) and anode (a) respectively has been indicated. The lengths, absolute as well as relative, of the vectors have been arbitrarily chosen and probably exaggerated. It will be clear however that, whatever their real length be, in positive drops vacuolisation will occur at the anodal side, in negative drops at the kathodal side, the corresponding vectors being directed towards the interior of the heterogeneous region.

Moreover fig. 6 gives some indication that in favorable circumstances it will be possible to obtain coarse vacuolisation in electrophoretically neutral drops either at the cathodal or at the anodal side. This will depend on the intensity of the transport within the drop, about which no quantitative data are available. It will be clear however that the observed case in which vacuolisation was seen on both sides at the same time must have

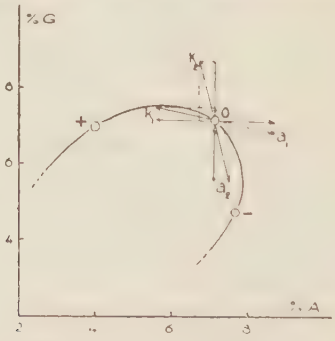


Fig. 6. Schematical representation of the coacervate part of a phasediagram for 2 % A—2 % G mixtures. It is indicated that when the relative and absolute rates of colloid transport within the drop remain within certain limits vacuolisation may occur either at the kathodal or at the anodal side (k_1 and a_2 resp.). In each of these cases however simultaneous vacuolisation on the other side of the drop is impossible, as is indicated by the vectors a_1 and k_2.

been due to the previous treatment of that preparation: on the kathodal side vacuolisation can normally occur only when the transport of A is faster than that of G (fig. 6, k). This however precludes vacuolisation on the anodal side, as therefore it is necessary that the transport of G prevails on that of A (fig. 6, a).

3. At the present time the explanation of the peripheral vacuolisation still offers difficulties, though it may be stressed that it occurs at that side of the drop where in consequence of the disturbance of the equilibrium the coacervate tends to dissolve.

4. The movements of the peripheral vacuoles however can be explained. In fig. 7—9 the relation between the interfacial tension coacervate — equilibrium liquid and the mixing proportion of the colloids (at constant pH) has been represented graphically. It should be noted that the σ values in the present experiments will have been slightly higher than those men-

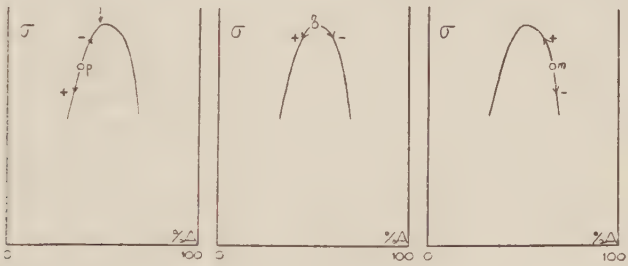


Fig. 7, 8 and 9. Illustrations of the changes in σ on anodal (—) and kathodal (+) side of electrophoretically positive (fig. 7, *p.*), neutral (fig. 8, *n.*) and negative (fig. 9, *m.*) coacervate drops in consequence of changes of composition. The arrow in fig. 7 indicates the point where reversal of charge occurs. (48 % A)

tioned in a previous article [7]), as we worked now with systems in which the $Na^{\cdot}$ concentration of the buffer amounted to 5 maeq. p.l. only instead of 10 maeq. Qualitatively however this has no influence.

In fig. 7, p may represent the interfacial tension of a positive coacervate drop. It is evident that in the electric field σ of this drop will decrease at the cathodal side (fig. 7, +) but increase at the anodal side (fig. 7 ,—) as a consequence of the transport of A and G within the drop.

Now it is a wellknown fact that substances tend to move from places with lower to places with higher interfacial tension. It is therefore to be expected that the coacervate in the surface of a positive drop in the electric field will move from the cathodal side (i.e. the side where the peripheral vacuoles originate) to the anodal side, which is confirmed by the experiments: the peripheral vacuoles are taken along by this current, and also subperipheral layers partake in the same movement.

The movement of the surface of the drop is demonstrated too by a

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7) L. DE RUITER and H. G. BUNGENBERG DE JONG, loc. cit.

creeping movement of the drop over its substrate in the direction of the cathode. [8])

In the same way it can be explained that the surface of a negative coacervate drop moves from anodal to cathodal side (see fig. 9 in which *m* represents a negative drop and + and — the changes on kathodal and anodal side respectively).

5. The formation of a "4-quadrant-system" in an electrophoretically neutral drop can be explained in the same way. Its interfacial tension may be represented by *n* in fig. 8, as maximal σ and reversal of charge nearly coincide. In this case it is evident that on both cathodal and anodal side σ will decrease, so that on both sides the coacervate will move to the poles of the drop (where σ does not change).

6. Another factor however will influence the movements of the vacuoles. Apart from being moved passively by the currents in the coacervate drop they also move actively in the electric field (as was known already [9])), behaving as though they had an electrical charge of the same sign as the drop in which they are contained. This factor will contribute to the movement of the coarse vacuoles across the drop and to the formation of a vortex near both poles of the drops. In electrophoretically neutral drops the vacuoles will of course also be neutral, so that in this case the movements are exclusively due to changes of σ. It is perfectly clear now that this is the only case in which a movement of coarse vacuoles in the direction of the poles of the drop was to be observed: In electrophoretically charged drops σ is exactly maximal in the region where the coarse vacuoles are formed, so no movement from there to the poles could occur, nor was it ever observed although we paid special attention to it.

7. It also seems to be possible to give an explanation of the formation of small coacervate drops in the equilibrium liquid near bigger coacervate drops in the electric field.

In figure 10, the equilibrium liquids of a positive, a negative and a neutral coacervate have been represented in a phasediagram. The transport of colloids in the electric field through the equilibrium liquid itself will cause in a point near the anodal side of a positive drop a very slight decrease of A and an equally slight increase of G, as the equilibrium liquid contains only very small quantities of colloid in comparison to the coacervate. In the drop itself however a great quantity of A is accumulated on the anodal side, part of which will be solved in the equilibrium liquid. The net result will be a considerable rise of A content in the equilibrium liquid, whereas the percentage G does not or nearly not change. This result is represented

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[8]) See note 1.

[9]) H. G. BUNGENBERG DE JONG and W. A. L. DEKKER, Biochem. Z. **221**, 403 (1930).

by *a* (in fig. 10, +) from which can be seen that coacervation will occur in the equilibrium liquid.

For the same reasons on the cathodal side of a positive drop an increase

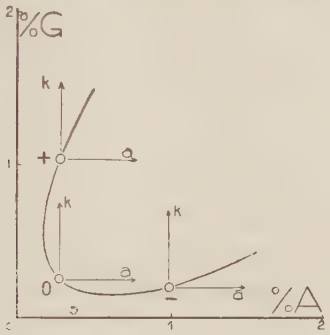


Fig. 10. Schematical representation of the equilibrium liquid part of a phasediagram for 2 % **A**—2 % G mixtures. The vectors indicate the changes of composition of the equilibrium liquid that will occur near the kathodal (*k*) and anodal (*a*) side of electrophoretically positive, neutral and negative coacervatedrops. The major part of these changes is due to the solution of colloid from the drops into the liquid. The relatively slight changes due to tranport of colloids of the equilibrium liquid have been omitted. The length of the vectors is arbitrarily chosen. It is evident however that in strongly positive and strongly negative systems coacervation can occur only near the anodal resp. kathodal side of the drop, whereas in electrophoretically neutral systems and in those that are near the equivalent mixing proportion, coacervate may be formed in the equilibrium liquid near both surfaces of the coacervate drops.

of *G* will occur, whereas *A* is practically not changed. This change may be indicated by *k* (in figure 10, +). Here no coacervation will occur.

Further it is to be expected, that in systems where *A* is in excess i.e. negative coacervates, coacervation in the equilibrium liquid will occur near the kathodal sides of the coacervate drops (fig. 10, —) and finally in neutral coacervates, coacervation in the equilibrium liquid may occur on both sides (fig. 10, 0).

All this is in perfect agreement with the results of our experiment (compare fig. 2).

E. *Phenomena in swelling or densifying media.*

The data we obtained concerning the influence of the salt concentration on the interfacial tension coacervate-medium, are sufficient to confirm in some points the previously given hypothesis, which sought the explanation of motory phenomena in coacervate drops in diffusion fields of swelling or densifying substances in local changes of σ [10]).

In normal gum arabic-gelatine complex coacervates, σ decreases with increasing salt concentration. This agrees with the picture given previously of the movements of vacuoles in coacervate drops (on which NaCl evi-

10) H. G. BUNGENBERG DE JONG and E. G. HOSKAM, loc. cit. (1941).

dently had a swelling influence) in a NaCl diffusion field. The assumption is made here that the addition of urea and resorcinol in these experiments did not influence the qualitative behaviour. This appears to be justified by the fact that the presence of these non electrolytes did not qualitatively alter [11]) the general characteristics of the complex coacervation, the latter depending solely on the interaction of the electric charges of both macromolecular colloids.

Summary.

1. Drops of complex coacervates, suspended in their equilibrium liquids show a number of morphological phenomena in the electric field. These phenomena are described in greater detail than hitherto published.

2. In a previous article it was supposed that in the explanation of the motory phenomena local changes of the interfacial tension (σ) coacervate medium might play an essential part. It was assumed that these changes of σ would be due to pH changes by polarisation of the surface of the drops. Experiments were therefore carried out in order to demonstrate these pH changes in gum arabic (A)-gelatine (G) coacervates.

3. No changes in pH could be proved, which may be due to the use of buffered stock sols. These experiments however indicated that in the electric field gum arabic is accumulated at the anodal and gelatine at the cathodal side of the drop in consequence of electrophoresis of these components within the drop. As most of the indicators we used in these experiments were found to have "protein errors", on account of which systems rich in G had another colour than those rich in A, these shifts of the mixing proportion on both sides of a drop manifested themselves in slight colour differences of the parts of the drops facing the electrodes.

4. It appears from now available data concerning σ that these changes, and not, as far as is now known, changes in pH, can be held responsible for the σ-variations postulated for the explanation of the motory phenomena.

5. Moreover it is demonstrated that the changes in mixing proportion themselves are also important factors in the explanation of a number of the socalled desintegration phenomena.

6. Starting from the results of these experiments and the data on σ explanations have been given for the greater part of the morphological phenomena that have been described.

Laboratory for Medical Chemistry of the University, Leiden.

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11) H. G. BUNGENBERG DE JONG and E. G. HOSKAM, loc. cit. (1942).

Mathematics. — *The Growth-Curve.* By J. W. N. Le Heux. (Communicated by Prof. A. Pannekoek.)

(Communicated at the meeting of October 25, 1947.)

1. In T. Brailsford Robertson's "Chemical basis of Growth and Senescence" (No. 1) the growth $\frac{dN}{dt}$ of an organism N in a time t is given by an equation of the form

$$\frac{dN}{dt} = N(b - aN)$$

where a and b are constants. Introducing the maximum value of $N = \frac{b}{a} = A$, we get

$$\frac{dN}{dt} = aN(A - N) = bN\left(1 - \frac{N}{A}\right)$$

with the solution

$$^{e}\log \frac{N}{A-N} = b(t - t_1)$$

or

$$N = A\,\frac{e^{b(t-t_1)}}{1 + e^{b(t-t_1)}}$$

t_1 is the time, corresponding to $N = 0.5$ A.

Now, sets of values for N and t being given by experiment and admitting, that the given relation holds true, the most probable values of the constants A, b and t_1, may be calculated by the method of least squares.

In this way and using ordinary logarithms, Robertson finds f.i. the equation

$$^{10}\log \frac{N}{320 - N} = 0.127\,(t - 1.57)$$

from the following experimental data:

weight of male British infants in ounces

148	169	194	219	234	252	269	276	283	300	303	314

time in months

1	2	3	4	5	6	7	8	9	10	11	12

If — as often occurs — only an approximate comparison of theory and observation is required, the constants may be found in a much shorter way by using a graphical method, that will be described in the following lines.

This method procures a quick and easy way to test a given graph (if a relation of the foregoing form is suggested) by the construction of the growth-curve or sigmoïd

$$N = A \frac{e^{b(t-t_1)}}{1+e^{b(t-t_1)}}$$

in form well-known from different publications on biological and chemical subjects, but whose application seems largely to have been overlooked.

In a recent work on empirical equations (No. 2) the author says: "Some data take the form of an elongated S when plotted, the curve being characterized by a very small initial slope followed by a period of rapidly increasing slope which gives way to an interval of nearly constant slope succeeded by a period when the rapidly decreasing slope approaches zero. The growth of population and production statistics for iron and steel and for rayon are typical of this class. In the case of the manufacture and sale of some new product, consumer acceptance is slow at first, as shown by the small slope of the sales-time curve. Advertising increases the slope greatly. Later a period of stabilization is encountered and the sales-time curve levels off. The Gompertz equation

$$y = ab^{c^x} \quad (\text{or } y = a + ab^{c^x})$$

has been found to be satisfactory in representing such data ".

Especially in questions of growth of living organisms, the sigmoïd, that is not at all mentionned here, should be preferred to the Gompertz curve.

2. Construction of the curve $N = A \frac{e^{b(t-t_1)}}{1+e^{b(t-t_1)}}$.

The plotting of the scale $\frac{e^n}{1+e^n}$ (growth-scale) with modulus A, against a regular t-scale (n-scale with modulus $\frac{1}{b}$) on ordinary coordinate paper results in a S-shaped curve, well-known as "sigmoïd" or "logistic" or "growth-curve".

Raising the horizontal t-axis over 0,5 A, the equation becomes

$$N' = 0.5\,A \frac{e^{bt}-1}{e^{bt}+1} = 0.5\,A\,th\frac{bt}{2}$$

proving, that growth, as defined by the equation

$$\frac{dN}{dt} = N(b - aN)$$

may be expressed by a tangens hyperbolicus.

From a nomographical point of view, the scale $A \frac{e^n}{1+e^n}$ is a projective scale. It may be obtained by projecting the graduations of the scale e^n (n from $-\infty$ to $+\infty$) on a horizontal axis OX from the centrum $(-1, A)$

on a vertical axis OY (fig. 1). The result is a symmetric scale, for the graduations $\frac{e^p}{1+e^p}$ and $\frac{e^{-p}}{1+e^{-p}}$ are equidistant from the centre of the scale, because

$$A\,\frac{e^p}{1+e^p} = A\,\frac{1}{e^{-p}+1} = A - A\,\frac{e^{-p}}{1+e^{-p}}.$$

Consequently, to construct this scale only the part $0 - 1$ (n from $-\infty$ to 0) needs to be projected.

A plot of this symmetric scale against the regular t-scale shows a curve, that being turned about its centre through an angle of 180°, is indistinguishable from the original curve.

The sigmoïd is of use in the two following problems:

1e. The testing of an experimental smooth curve through plotted points and the determining of its equation.

2e. The construction of the curve, given by an equation of ROBERTSON

$$^{10}\log \frac{N}{A-N} = b\,(t-t_1).$$

3. In the first place, we suppose, that a rather great number of data is known, as in the case already mentionned:

weight of male British infants in ounces

148	169	194	219	234	252	269	276	283	300	303	314

time in months

1	2	3	4	5	6	7	8	9	10	11	12

These data refer to the first or infantile cycle — the second or juvenile and the third or adolescent cycles of human growth are partially fused with one another and do not permit of such precise formulation as the infantile cycle. The analysis of such a complex curve into its constituent cycles, each defined by three mutually independant parameters A_1, b and t_1 is — as ROBERTSON remarks — a matter of considerable difficulty and tedium.

The conclusion of the first year of post-natal growth in man coïncides approximately with the conclusion of a cycle of growth and therefore, the existence of the relation $\frac{dN}{dt} = N\,(b - a\,N)$ may be suggested.

The average weight of British male infants at 12 months of age is 314.3 ounces, hence we may take A as being 320 ounces.

0.5 A = 160 ounces is reached in a time t_1

160 lies between 149 (1 month) and 169 (2 months), so

$$t_1 = 1 + \frac{160-148}{169-148} = 1.57 \text{ months.}$$

On rectangular axes OX and OY (fig. 2) a line PQ parallel to OY is

located at a distance $t_1 = 1{,}57$ cm from 0. The length of $PQ = A = 320$ (16 cm). Now a growth-scale is laid off along PQ. This growth-scale is taken from a "modulus-chart", which must be prepared beforehand in the same manner as a logarithmic modulus-chart in nomography. The construction is as follows. Because the growth scale is symmetric, mark off half such a scale with a modulus of 40 cm along the line AB (fig. 3).

The graduations are found by construction or by calculation from $N = 40 \frac{e^{-n}}{1+e^{-n}}$ and for values of n from 0—5 (because $40 \frac{e^{-6}}{1+e^{-6}} \sim 1$ mm) with subdivisions of 0.1.

Project a pencil of lines from the graduations on AB to the point C, 20 cm from AB on a line at right angles to AB in B, bearing a half-centimeter scale. The slant lines will cut the vertical modulus lines in scales of moduli between 0 and 40 cm.

For practical use, the chart is cut into two parts, which are rejoined as in fig. 3.

When no modulus-chart is disposable, the points numbered $-5, -4 \ldots\ldots 0 \ldots\ldots + 4, + 5$ are sufficiently located by the distances 1, 2, 5, 12, 27, 50, 73, 88, 95, 98, 99 from the point $(-\infty)$ for a modulus of 100 mm.

Through the dividing-points of the growth scale on the line PQ lines are drawn parallel to the X axis.

The regular t scale on the X axis has a modulus $\frac{1}{b}$.

Suppose that $B(t-t_1) = 1$ for $t = t_2$.

Then $N = A \frac{e}{1+e} = 0.731\ A = 0.731.\ 320$ ounces $= 234.24$ ounces (11.7 cm).

t_2 is calculated from the data in the same manner as t_1.

234.24 lies between 234 (5 months) and 252 (6 months).

So $t_2 = 5 + \frac{234.24-234}{252-234} = 5.013$ months and $t_2 - t_1 = 3.44$ months $= \frac{1}{b}$.

Now, the X axis must be divided in equal parts of 3.44 cm with the zero-point in Q.

Drawing lines through the dividing points parallel to OY and joining up the intersecting points of corresponding lines by a smooth curve, the sigmoïd, corresponding to the data, appears. Fig. 2 gives a comparison between the observed and the constructed values of N.

Extrapolation is not permitted: about the epoch of birth, there is a slight arrest in growth and for the pre-natal months, this curve has no significance.

From $\frac{1}{b} = 3.44$ it follows, that $b = 0.29$ and as we have found $A = 320$ and $t_1 = 1.57$, the equation becomes

$$^e\log \frac{N}{320-N} = 0.29\,(t-1.57)$$

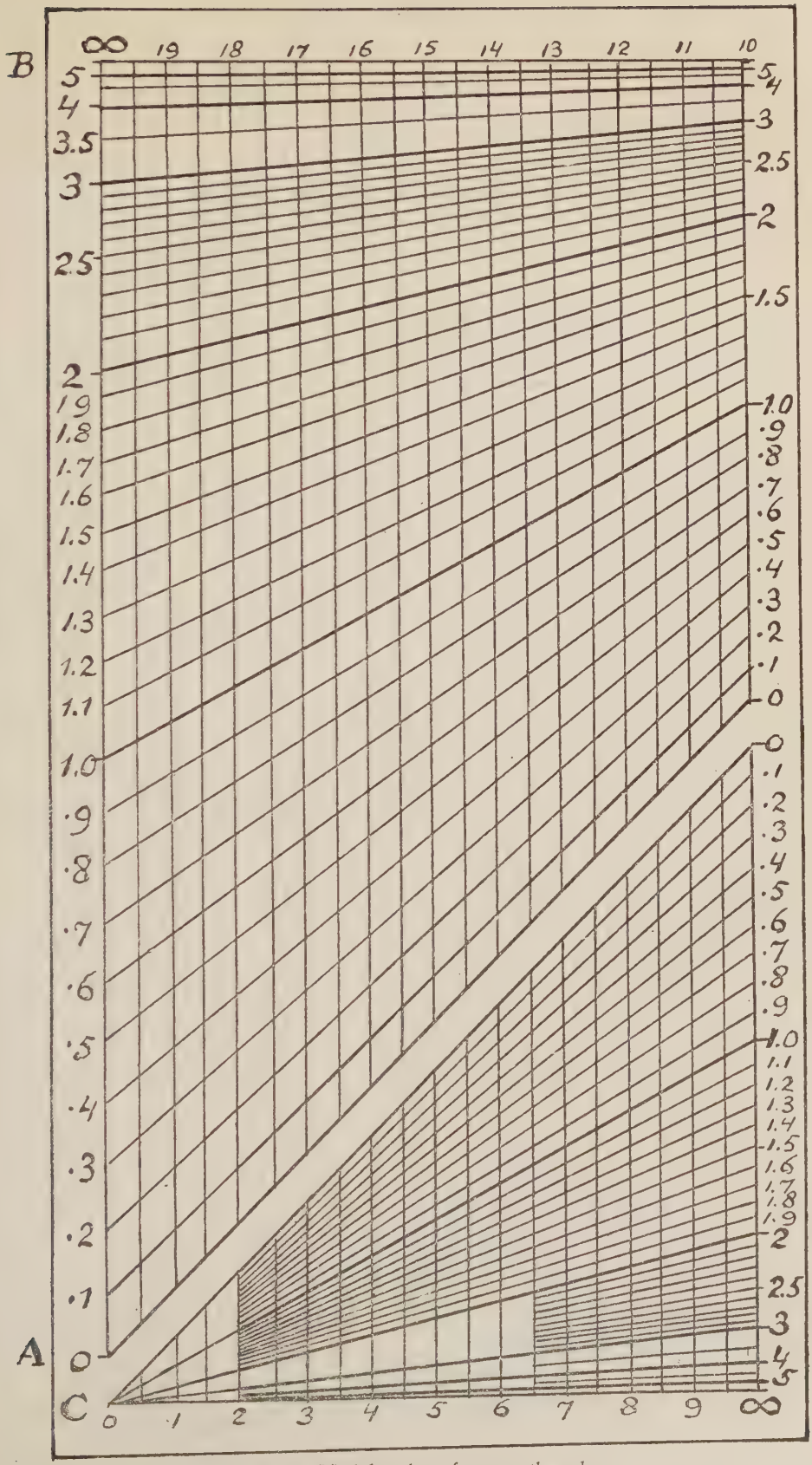


Fig. 3. Modulus-chart for growth-scale.

or

$$^{10}\log \frac{N}{320-N} = 0.434 \times 0.29\,(t-1.57) = 0.128\,(t-1.57).$$

ROBERTSON finds with the method of least squares for $b = 0.127$. Instead of calculating the values of N from the equation, the lengths of the ordinates may be measured in the carefully drawn curve, using the growth-scale with graduations of 0.1.

4. In the second place we suppose, that a given graph gives rise to the question, whether this curve may be a sigmoïd?

The S shaped curve has two parallel (horizontal) parts with a distance A. If the (shortened) parts are nearly parallel, we must take in this "trial and error" method a possible value of A.

The X axis is laid along the lower part and a vertical axis through the intersecting point of the curve with the line $Y = 0.5$ A.

A growth-scale with the modulus A is laid off along the vertical axis and through the dividing points 1, 2, 3, 4 and 5, horizontal lines are drawn till they meet the curve. If the lengths of these lines are also in ratio 1, 2, 3, 4 and 5, the curve will be a sigmoïd. In fig. 4, this method is applied to the curve of pre-natal growth (length) of child in THOMPSON's "on Growth and Form", according to HIS's dates (the same curve in Encyclopaedia Britannica, 14th ed. vol. 10) (No. 3 and 4).

THOMPSON writes: "a beautifully regular one, nearly symmetrical on either side of the point of inflexion. A curve, for which we might well hope to find a simple mathematical expression".

This hope can be realized. Fig. 4 shows, that the lower part fits exactly a sigmoïd with a time-unity measured in the figure $\frac{1}{b} = 1.2$. It is impossible, to find a combination of the same growth-scale with a regular t-scale for the upper part of the given curve (points 5 — 10), but we see that, omitting point 5, the points 6 and 4 belong to the same curve as points 1 — 4, while the points 7, 8 and 9 fit another (dotted) sigmoïd with a time unity $\frac{1}{b} = 1.4$. So this graphical method suggests the idea of a variable b: by one reason or another, its value may change into another during a certain interval and return to the initial value.

In fig. 4, $A = 500$ mm, $t_1 = 5$ months, $\frac{1}{b} = 1.2$ months, $b = 0.83$ (for $\frac{1}{b} = 1.4$, $b = 0.71$). The general equation is

$$N = 500\,\frac{e^{0.83(t-5)}}{1+e^{0.83(t-5)}}$$

with the restriction, that 0.71 must be taken instead of 0.83 in the interval $t = 6$ to $t = 10$. Fit for calculation:

$$^{10}\log \frac{N}{500-N} = 0.36\,(0.31)\,(t-5).$$

Another instance is the graph for pre-natal growth of child in weight (after Vignes, fig. 5).

The modulus of the growth-scale $A = 3500$ gram, $t_1 = 7$ months.

The lower part is exactly a sigmoïd with $\frac{1}{b} = 1$ month, $b = 1$.

The upper part is nearly a sigmoïd with $\frac{1}{b} = 0.7$, $b = 1.43$.

The equation is:

$$N = 3500 \frac{e^{(t-7)}}{1 + e^{(t-7)}}$$

or

$$^{10}\log \frac{N}{3500 - N} = 0.434\,(t-7)$$

with 1.43 t instead of t and 0.621 instead of 0.434 for the last three months.

An inquiry into the growth of Phycomyces (fig. 6a after ERRERA, Nr. 6) shows a complex curve with a first cycle of slow growth ending when the production of a sporangium begins and a second cycle of very rapid growth [1]).

We will try to find a sigmoïd, corresponding to each cycle.

According to the given numbers, the first curve may have a maximum value $A = 8.5$ mm. Now the second sigmoïd is plotted by diminishing the ordinates of the original curve with 8.5 beginning with $t = 24$. We find $A_2 = 125 - 8.5 = 116.5$ mm.

On a vertical axis through the intersection-point of the dotted curve with the line $y = 0.5\ A_2 = 57.3$, a growth-scale is laid off with a modulus 116.5 and horizontal lines are drawn through the dividing points till they meet the dotted curve. Evidently, it is impossible to get a regular t-scale for the whole curve, but here also the graphical method enables us to see, what is not revealed by calculation. The dividing point 1 of the growth-scale suggests a modulus $\frac{1}{b} = 7.2$ for the t-scale (above the point of inflexion) and this modulus fits the inner part of the dotted curve, while the outer part corresponds to a modulus $\frac{1}{b} = 6$ (below the point of inflexion).

For the first sigmoïd $A_1 = 8.5$, $t_1 = 12$, $\frac{1}{b_1} = 3$ (as read from the graph), $b_1 = 0.33$.

For the second sigmoïd $A_2 = 116.5$, $t_1 = 53.4$, $\frac{1}{b_2} = 6$ (7.2), $b_2 = 0.167$ (0.139).

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[1]) I am indebted to Prof. Dr. V. J. KONINGSBERGER for informations about Phycomyces.

Fig. 6*b* shows the two curves

$$N_p = 8.5\,\frac{e^{0.33(t-12)}}{1+e^{0.33(t-12)}} + 116.5\,\frac{e^{0.167(t-53.4)}}{1+e^{0.167(t-53.4)}}$$

and

$$N_q = 8.5\,\frac{e^{0.33(t-12)}}{1+e^{0.33(t-12)}} + 116.5\,\frac{e^{0.139(t-53.4)}}{1+e^{0.139(t-53.4)}}$$

with the experimental data, partly belonging to each curve. For calculation, we have:

$$\begin{cases} {}^{10}\log\dfrac{N_1}{8.5-N_1} = 0.143\,(t-12) \\ {}^{10}\log\dfrac{N_2}{116.5-N_2} = 0.07\,(t-53.4) \\ N_p = N_1 + N_2 \end{cases} \qquad \begin{cases} {}^{10}\log\dfrac{N_1}{8.5-N_1} = 0.143\,(t-12) \\ {}^{10}\log\dfrac{N_2}{116.5-N_2} = 0.06\,(t-53.4) \\ N_q = N_1 + N_2. \end{cases}$$

$$0.434 = {}^{10}\log e.$$

$$0.143 = 0.434 \times 0.33 \qquad 0.07 = 0.434 \times 0.167 \qquad 0.06 = 0.434 \times 0.139.$$

A comparison between calculated and observed lengths of Phycomyces gives:

		first day				second day			
	hours	6	12	18	24	6	12	18	24
length in mm.	$b = 0.07$	1.1	4.5	7.8	9.5	11.1	15.2	21.1	42.9
	observed	1.25	4	7	10	10.5	14	28	48
	$b = 0.06$	1.2	4.8	8.3	10.5	12.9	18.1	28.4	46

		third day				
	hours	6	12	18	24	6
length in mm.	$b = 0.07$	69.6	95.1	111.5	119.4	122.8
	observed	69	90	107	120	124
	$b = 0.06$	69.2	91.6	107.6	116.7	121.1

5. The construction of the complex curve

$${}^{10}\log\frac{N_1}{185-N_1} = 0.0252\,(t-69)$$

$${}^{10}\log\frac{N_2}{90.5-N_2} = 0.01125\,(t-170)$$

$$N = N_1 + N_2$$

found by ROBERTSON for the growth in weight of male, unmated white rats, measured by DONALDSON, runs as follows:

$${}^{e}\log\frac{N_1}{185-N_1} = 2.3 \times 0.0252\,(t-69) = 0.058\,(t-69).$$

$$b_1 = 0.058 \qquad \frac{1}{b_1} = 1725.$$

$$^e\log \frac{N_2}{90.5 - N_2} = 2.3 \times 0.01125\,(t-170) = 0.026\,(t-170).$$

$$b_2 = 0.026 \qquad \frac{1}{b_2} = 38.6.$$

Axes OX and OY at right angles.

Construct the growth-scales with moduli $A_1 = 185$ and $A_2 = 90.5$ on lines P_1Q_1 and P_2Q_2 parallel to OY on distances of 69 and 170 from 0. Construct regular scales on OX with moduli $\frac{1}{b_1} = 17.25$ and $\frac{1}{b_2} = 38.6$ with zero-point resp. in Q_1 and Q_2. Draw the curves by joining up points of intersection of corresponding lines through the graduations on OX and OY and add the ordinates with the same abscissae.

Of course, one cm of the X axis may represent another number than one cm of the Y axis.

6. *Conclusion.*

In many cases, growth may be defined by the equation

$$\frac{dN}{dt} = BN\left(1 - \frac{N}{A}\right).$$

A is the constant maximum value of N, dependent upon nutritional and other conditions, constituting the environment of the growing organism.

B expresses the specific velocity of the growth-process itself. Commonly, it has the constant value b but sometimes a slight alteration is possible during a certain period of the growth-process.

The relation between N and t (b constant) is

$$N = A\,\frac{e^{b(t-t_1)}}{1 + e^{b(t-t_1)}}$$

t_1 is the time for reaching half the maximum.

To construct this "growth-curve", plot a growth-scale with a modulus A from a modulus-chart against a regular scale with a modulus $\frac{1}{b}$.

The coordinates of the point of inflexion are (t_1, 0.5 A).

To calculate the constants from the experimental data for N and t, the following scheme may be used

$$0.5\,A = \ldots \quad t_1 = \ldots$$

$$0.731\,A = \ldots \quad t_2 = \ldots$$

$$t_2 - t_1 = \frac{1}{b} \qquad b = \ldots$$

To calculate N for a given t:

$$^{10}\log \frac{N}{A-N} = b\,(t-t_1)\,.$$

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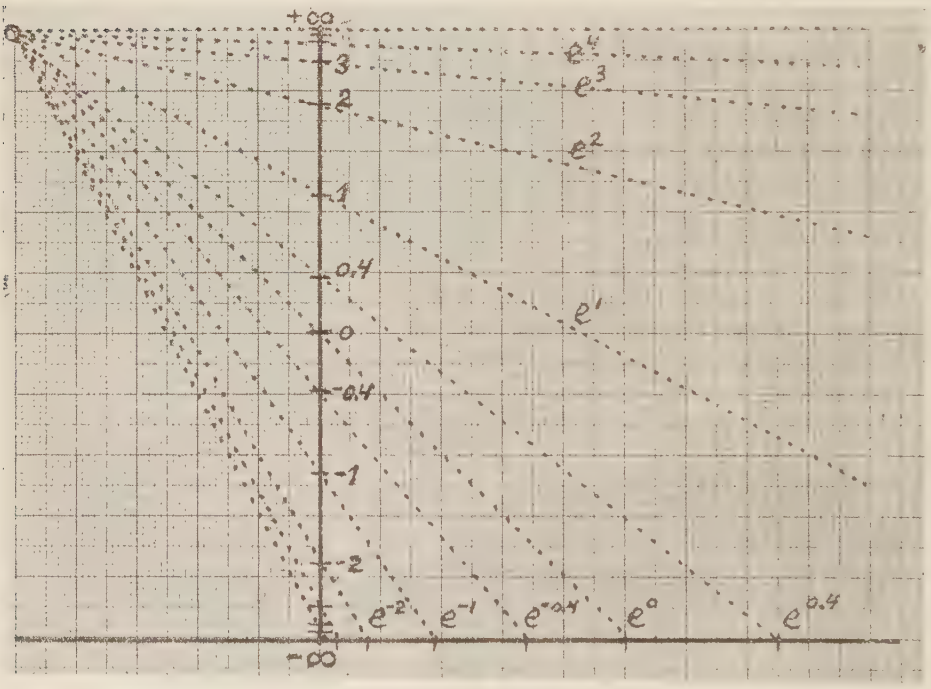


Fig. 1. The symmetric growth-scale.

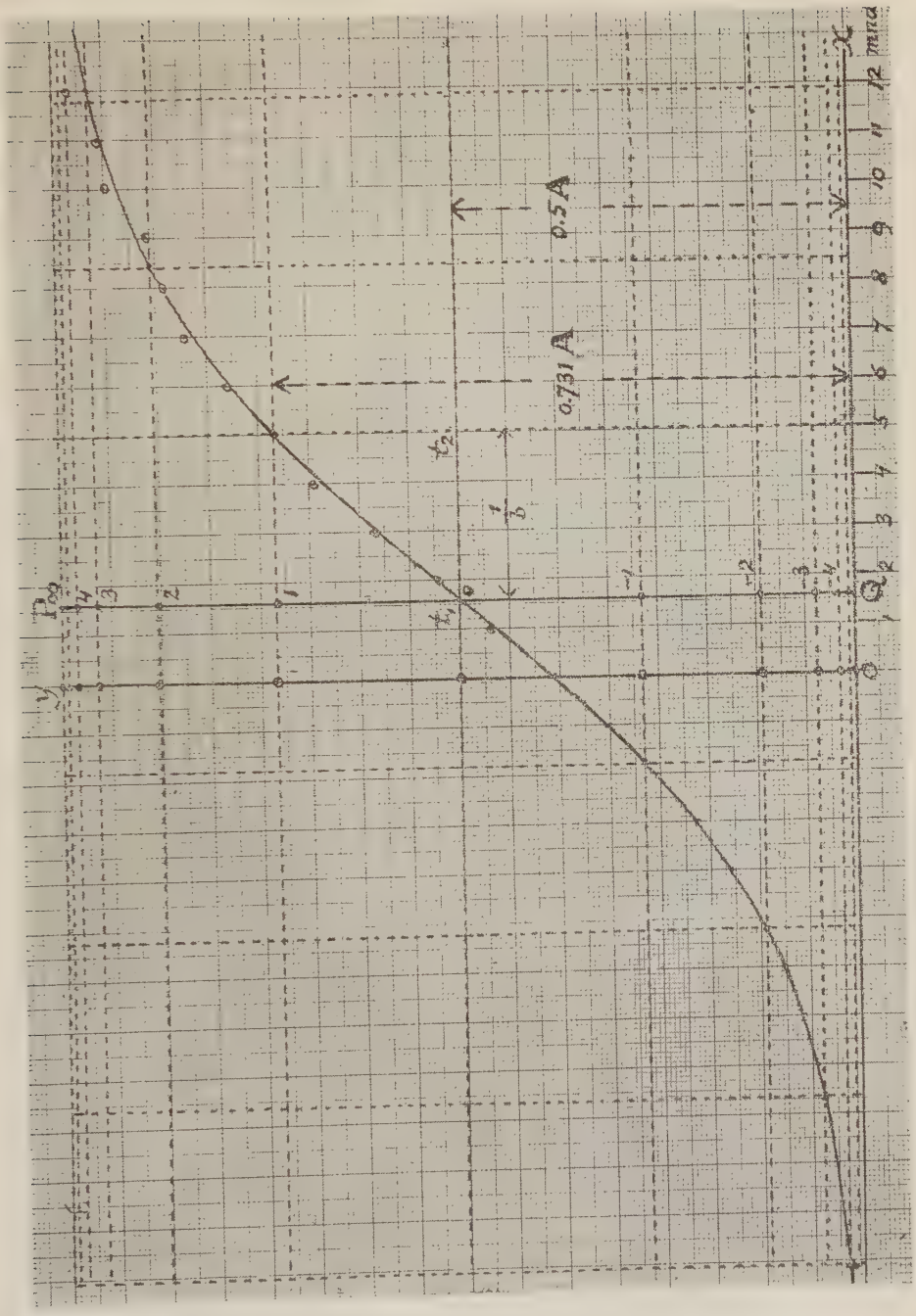


Fig. 2. Post-natal growth in weight.

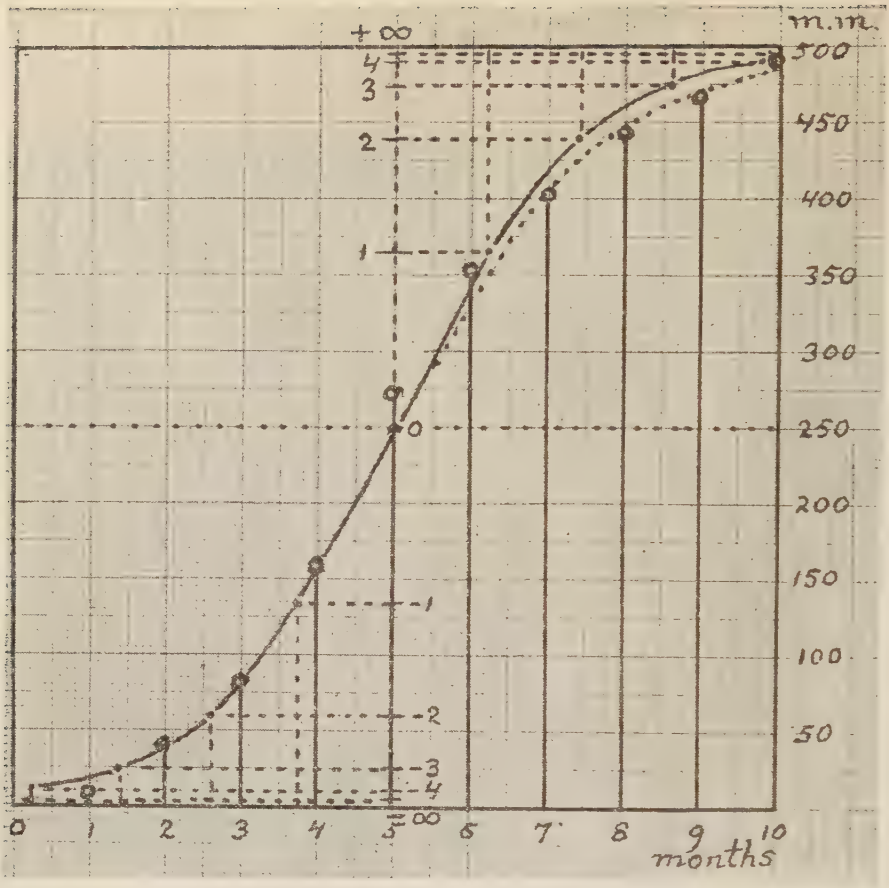


Fig. 4. Pre-natal growth in length.

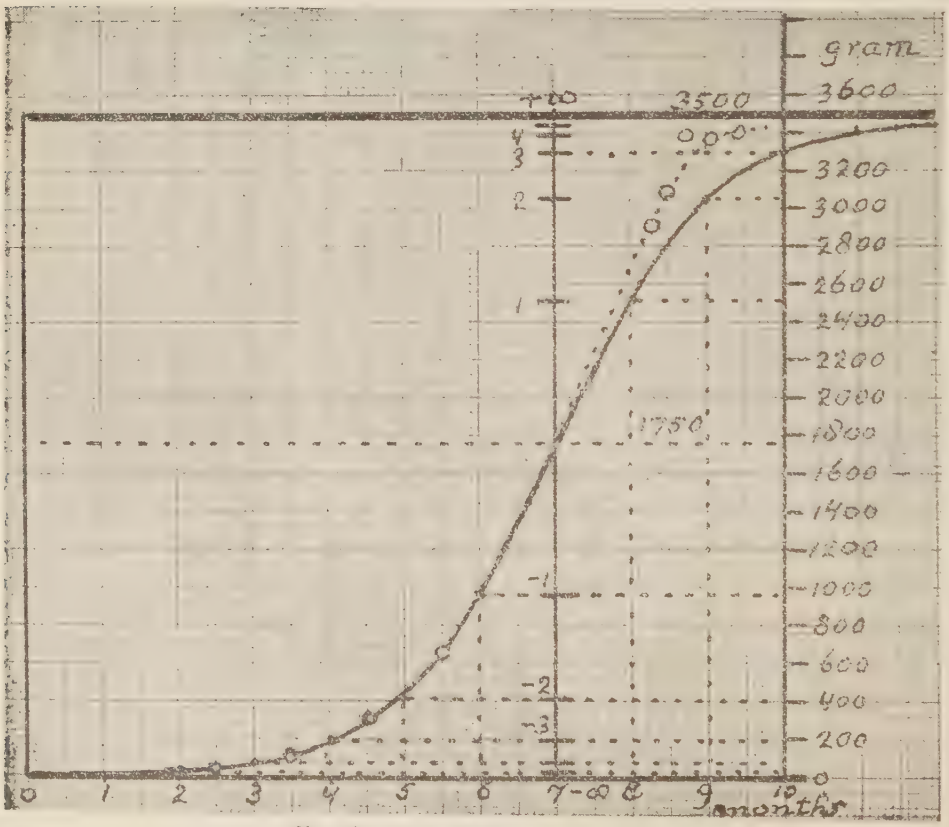


Fig. 5. Pre-natal growth in weight.

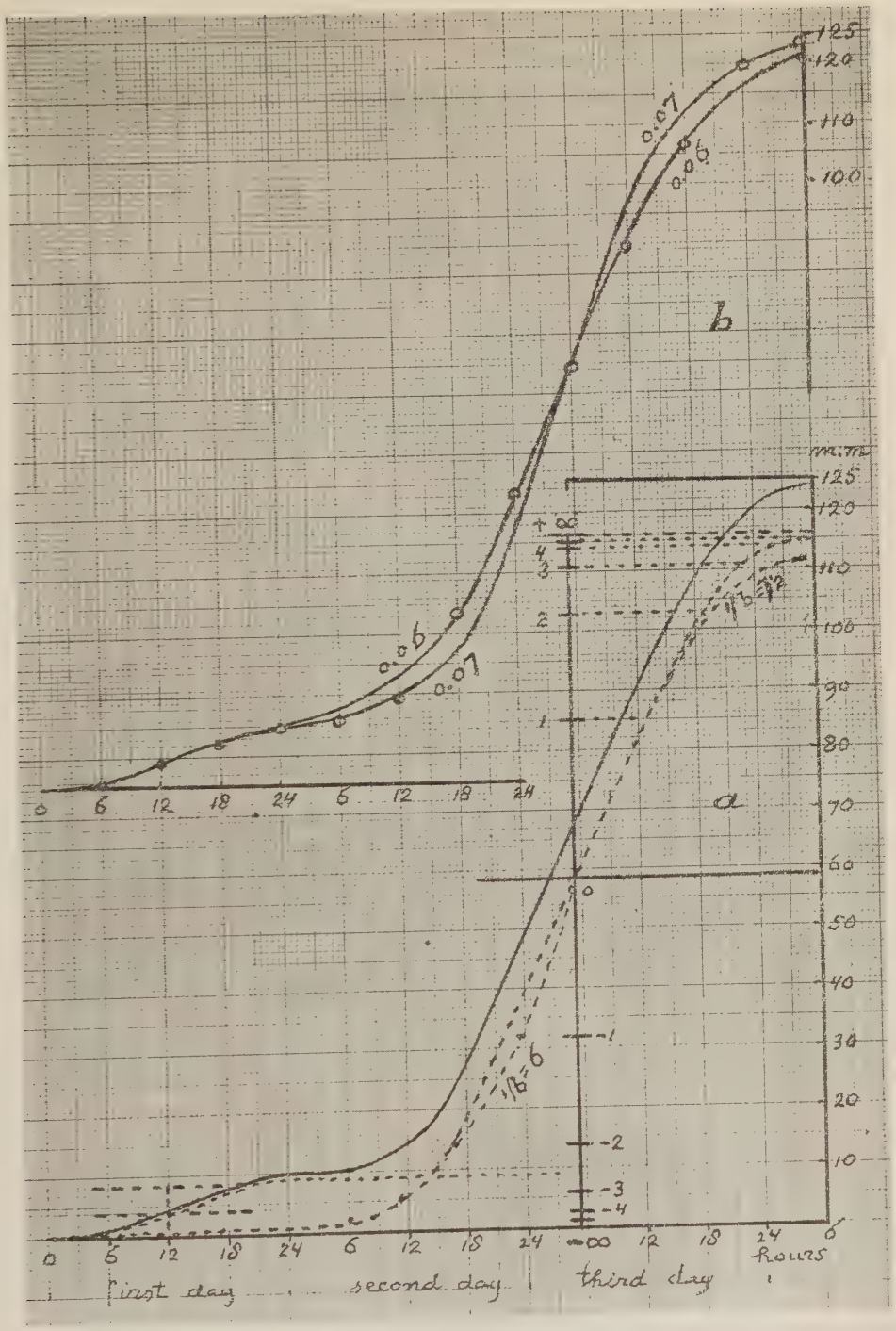


Fig. 6. Growth of Phycomyces.

Mathematics. — *Functional Logic without Axioms or Primitive Rules of Inference.* By K. R. POPPER. (Communicated by Prof. L. E. J. BROUWER.)

(Communicated at the meeting of October 25, 1947.)

I

The customary systems of modern lower functional logic, such as *Principia Mathematica*, or the systems of HILBERT–ACKERMANN, HILBERT–BERNAYS, or HEYTING, etc., proceed in the following way.

(a) Undefined primitive *formative signs* of the system under construction are laid down — at least *one* (either alternative denial or conjoint denial) for propositional logic and *one* (the universal or existential quantifier) for quantification theory.

(b) Unproved primitive propositions or axioms are laid down — at least one for propositional logic, and one for quantification theory.

(c) Primitive rules of inference (such as the *modus ponens*) for the formative signs are laid down, at the very least one for propositional logic and one for quantification theory.

(d) Some further very general primitive rules of inference (such as some principles stating that the inference relation is transitive and reflexive) which do not refer to formative signs are assumed, either explicitly or, more often, tacitly.

The procedure here described is not always strictly adhered to, to be sure; the axioms (b) for example, may be expressed in the form of rules of inference (c), etc. — But to my knowledge, the scheme sketched is the one most frequently adopted.

In an earlier publication [1], I have outlined a method by which the procedure can be considerably simplified. This method consists, in the main, in this.

It is observed that the traditional procedure must make use of a language of communication — *the metalanguage* — in which it discusses and

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[1]) Cp. *New Foundations for Logic* (MIND, vol. LVI, 1947, No. 223, pp. 193—235). In this paper there are a number of *errata*, of which some that may cause trouble to readers of the present paper may be listed here: In D5.2 and D5.3, replace the last occurrences of "*a*" and "*b*" by "*b*" and "*c*" respectively. In D5.4, replace by "*c*" the two last occurrences of "*a*" without subscript. Rules 6.1 and 6.02 must be introduced by the condition "If for every z, $a \| a(\frac{y}{z})$ and $b \| b(\frac{y}{z})$, then". The right hand side of D6.2 should be replaced by: "(for every b and z: $(((b \| b_x^{\backprime}) \mathbin{\&} b \| b_y^{\backprime}) \to a,\ b(\frac{z}{x})/b(\frac{z}{y}))$ & ((for every c and u: $((c \| c_x^{\backprime}) \mathbin{\&} c \| c_y^{\backprime}) \to b,\ c(\frac{u}{x})/c(\frac{u}{y}))) \to b/a)))$". In D6.1, "*b*" must be twice replaced by "*a*". In D7.1 and in D7.2, the first two occurrences of "*a*" must be replaced by "$a_y^{\backprime}$" and the third by "$b_y^{\backprime}$". Ultimately, in the footnote, line (C) on p. 228, replace "$=$" twice by "$\neq$".

describes the linguistic system under consideration — the *object language*. It is further observed that all systems are bound to use metalinguistic concepts such as "The conclusion *a* is derivable from the premises $a_1, a_2, \ldots, a_n$", and "If *a* is demonstrable then *b* is demonstrable", etc.

Now it can be shown that, if we have at our disposal the first of these two metalinguistic ideas, i.e., the concept of deducibility (deducibility from *n* premises is an $n + 1$-termed metalinguistic predicate), characterized by a few trivial rules of inference of the kind (d), such as the transitivity and reflexivity principles which are assumed in all the customary systems, then the following holds:

(a′) No sign such as a sign for alternative denial or for a quantifier need be laid down as primitive, since names for the alternative denial and the quantifiers and all the other formative signs of an object language can be defined in terms of deducibility.

(b′) No unproved axioms or primitive propositions need to be assumed, since the axioms or primitive propositions of the usual systems can be shown to be all demonstrable (on the basis of an explicit definition of demonstrability in terms of deducibility).

(c′) No primitive rules of inference of kind (c) need to be laid down, since all the rules of inference pertaining to the formative signs of the system follow from the definitions of these signs.

In other words, given the concept of deducibility and a few primitive rules of kind (d) to characterize it (I have called an independent system of such rules a "basis"), everything else can be obtained by way of explicit definitions in terms of deducibility ("inferential definitions").

In this way, a considerable simplification, as compared with the customary systems, is achieved. But, as will be shown here, we can even go further.

The inferential definitions of the conjunction (or the alternative denial) and of the result of one of the quantifications, for example, of universal quantification, can be reformulated in such a way as to incorporate *all* the rules of inference mentioned. In this way, we can get rid of even the few trivial primitive rules (d) which were left in the previous approach; in other words, we obtain the whole formal structure of logic from metalinguistic inferential definitions alone.

II

Our theory is completely metalinguistic.

Our universe of discourse consists of expressions of certain object languages; we consider especially the *statements* (open or closed) and the *name-variables* of these languages.

We shall use, in the metalanguage, "*a*"; "*b*"; "*c*"; etc.; "a_1"; "a_2"; "a_n"; "b_1"; "b_2"; "b_m"; etc., as variables whose values are the statements (open or closed) of an object language under consideration, and "*u*"; "*v*"; "*w*"; "*x*"; "*y*"; "*z*"; as variables whose values are the name-variables of that object language.

We shall use two undefined concepts: (1) deducibility and (2) substitution.

(1) The notation

$$\text{“}a_1, \ldots, a_n/b\text{”}$$

will be used in order to express the assertion "From the statements $a_1, a_2, \ldots, a_n$ of the object language, the statement b of that language can be deduced." The sign "/" is an $n+1$-termed predicate of the metalanguage.

(2) The notation

$$\text{“}a\left(\tfrac{x}{y}\right)\text{”}$$

will be used as a (variable) metalinguistic name of the statement which is the result of substituting, in the statement a (open or closed), the variable y for the variable x, wherever it occurs. $a\left(\tfrac{x}{y}\right)$ is identical with a if x does not occur in a.

These two are the only specific metalinguistic primitive signs used in our metalanguage.

Apart from the specific metalinguistic or logical signs "/" and "$a\left(\tfrac{x}{y}\right)$", we shall use, in the metalanguage, the normal means of expression, including "and", "if and only if", etc.

The following *abbreviations* will be used in the metalanguage:

"$\ldots \rightarrow \ldots$" for "if ... then ..."
"$\ldots \leftarrow\!\rightarrow \ldots$" for "... if, and only if ..."
"$\ldots \,\&\, \ldots$" for "... and ...".

Also, the expression "for every ..." (followed by variables) will be used, and "$=$" and "$\neq$" (flanked by "x", "y", etc.). Brackets will be used in the customary way.

We do not undertake here to formalize our metalanguage. But it may be mentioned that the only logical rules needed in the metalanguage (except where we treat modalities) are those of the positive part of the HILBERT–BERNAYS calculus of propositions as far as they pertain to "if-then", "if, and only if", and to "and" (i.e. groups I, II, IV), and the rules for identity. The rules for negation need not be assumed (only some of their applications to "$x \neq y$"); but we need rules for universal quantification, especially the rule of specification: "If (for every $a_1, \ldots, a_n$: $(F(a_1, \ldots, a_n))$) then $F(b_1, \ldots, b_n)$."

III

Apart from the primitive signs, *defined* signs are introduced by definitions. These are, except the sign "//" for mutual deducibility, variable metalinguistic names of certain statements (open or closed), viz., either of so-called compounds or of the results of quantification.

We shall use the following signs, to be defined later:

1. "$a//b$" will be used to express that a/b and b/a.
2. "$a \wedge b$" will be used as name of the conjunction of a and b.
3. "$a > b$" will be used as name of the conditional statement with the antecedent a and the consequent b.
4. "$a \vee b$" will be used as name of the disjunction of a and b.
5. "a^c" will be used as name of the classical negation of a.
6. "a^i" will be used as name of the intuitionist negation of a.
7. "$(Aza)\ (^x_y)$" will be used as name of the statement which results from first applying to a universal quantification with regard to z and then substituting, in the result, y for x wherever it occurs.
8. "$Az(a(^x_y))$" will be the name of the statement which results from the universal quantification of $a(^x_y)$ with regard to z.
9. "$(Eza)\ (^x_y)$" is like 7, but with existential quantification.
10. "$Ez(a(^x_y))$" is like 8, but with existential quantification.

We shall not lay down primitive rules or axioms etc. for our undefined terms, but we shall, instead, frame three of our explicit definitions in such a way that the few rules needed to characterize our undefined terms emerge from these definitions. These three definitions will be fundamental to the others, and will be called "Basic Definitions" (DB).

The Definition (DB1), of "//", is basic for all others; (DB2), of "$\wedge$", is basic for propositional (and therefore also for functional) logic; and (DB7), of "$(Axb)(^x_y)$", is basic for quantification theory. Each of the other definitions of propositional logic presupposes DB1 and DB2, and D8 to D10 presuppose, in addition, DB7. None of them except D10 presupposes other than Basic Definitions. The only definitions in which defined terms (other than "//") are used are D8 and D10, which use terms defined in D7 and D9 respectively. (The relation between D8 and D7, or D10 and D9, is somehow akin to recursiveness; but these are the only definitions needed in which anything resembling recursiveness occurs.)

IV

We now proceed to the definitions of the concepts mentioned. The first definition, of "//", uses only "$\longleftrightarrow$" apart from "/". All the later ones make use of "//" also. They have all the form

"$a//$ the definiendum $\longleftrightarrow$ (for every ..: (...))"

That is to say, we do not define, e.g., conjunction, but rather the logical force of conjunction. (I have discussed this problem at length elsewhere [1,2].)

Here is the list of the definitions.

General.

(DB1) $a//b \longleftrightarrow$ (for every c: $(a/c \longleftrightarrow b/c)$)

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2) Cp. *Logic Without Assumptions* (Aristotelian Soc. Proc., 1947), esp. section (7).

Propositional Logic.

(DB2) $a//b \wedge c \longleftrightarrow$ (for every $a_1, \ldots, a_n$: $((a_1, \ldots, a_n/a \longleftrightarrow (a_1, \ldots, a_n/b$ & & $a_1, \ldots, a_n/c))$ & $(b/c \rightarrow (a_n, \ldots, a_1/b \rightarrow a_1, \ldots, a_n/c))$ & $(a_1, \ldots$ $\ldots, a_n/c \rightarrow a_1, \ldots, a_n, b/c)$ & $a_1/a_1))$

(D3) $a//b > c \longleftrightarrow$ (for every d: $(d/a \longleftrightarrow b, d/c))$

(D4) $a//b \vee c \longleftrightarrow$ (for every d: $(a/d \longleftrightarrow (b/d$ & $c/d)))$

(D5) $a//b^c \longleftrightarrow$ (for every d: $(a, b/d$ & $(a, d/b \rightarrow d/b)))$

(D6) $a//b^i \longleftrightarrow$ (for every d: $(a, b/d$ & $(b, d/a \rightarrow d/a)))$

Note [3]: In the presence of D5, D6 is equivalent to D5; in the absence of D5, D6 is weaker than D5.

Quantification Theory.

(DB7) Provided $x \neq y$,

$a\binom{x}{y}//(Axb)\binom{x}{y} \longleftrightarrow$ (for every c, and every u, v, and w:

$((c\binom{x}{y}/a\binom{x}{y} \longleftrightarrow c\binom{x}{y}/b\binom{y}{x})$ & $((v \neq x$ & $y \neq u) \rightarrow (c\binom{x}{u})\binom{y}{v}//(c\binom{y}{v})\binom{x}{u})$ &

& $(c\binom{y}{x})\binom{y}{w}//c\binom{y}{x}$ & $(c\binom{u}{v})\binom{v}{w}//(c\binom{u}{w})\binom{v}{w}$ & $(c\binom{u}{v})\binom{w}{v}//(c\binom{w}{v})\binom{u}{v}$ &

& ((for every z: $(b//b\binom{u}{z}$ & $c//c\binom{u}{z}))) \rightarrow (b//c \rightarrow b\binom{v}{u}//c\binom{v}{u}))$ & $c\binom{u}{u}//c))$

(D8) $(Aua)\binom{x}{y}//Az(a\binom{x}{y}) \longleftrightarrow ((u = x \rightarrow y = z)$ & $(u \neq x \rightarrow u = z))$

(D9) $a\binom{x}{y}//(Exb)\binom{x}{y} \longleftrightarrow$ (for every c: $(a\binom{x}{y}/c\binom{x}{y} \longleftrightarrow b\binom{y}{x}/c\binom{x}{y}))$

(D10) $(Eua)\binom{x}{y}//Ez(a\binom{x}{y}) \longleftrightarrow ((u = x \rightarrow y = z)$ & $(u \neq x \rightarrow u = z))$.

V

I shall now briefly outline the method by which we obtain, from this system of inferential definitions, the systems of rules of inference which were sketched in my earlier papers referred to.

Since

(5.01) $a/c \longleftrightarrow a/c$

is true whatever the meaning of the predicate "/" may be, we obtain, substituting a for b, from DB1

(5.02) $a//a$

and therefore

(5.03) $b \wedge c//b \wedge c$

This allows us to obtain from DB2, by substituting "$b \wedge c$" for "a", the set of five rules

(5.20) $a_1, \ldots, a_n/b \wedge c \longleftrightarrow (a_1, \ldots, a_n/b$ & $a_1, \ldots, a_n/c)$

(5.201) $b/c \rightarrow (a_n, \ldots, a_1/b \rightarrow a_1, \ldots, a_n/c)$

---

[3]) This remark which follows from the comments made in *New Foundations* upon rule 4.2e, qualifies an assertion made there on p. 220.

(5.202) $a_1, \ldots, a_n/c \rightarrow a_1, \ldots, a_n, b/c$

(5.203) see below

(5.204) a_1/a_1

Rule 5.203, viz.:

(5.203) $a_n, \ldots, a_1/c \rightarrow a_1, \ldots, a_n/c$

can be easily obtained from 5.201 and 5.204. Now it is quite obvious that the system of the four rules 5.201 to 5.204 is equivalent to the following four rules 5.21 to 5.24:

(5.21) $a_1, \ldots, a_n/b \rightarrow (b/c \rightarrow a_1, \ldots, a_n/c)$

(5.22) $a_1, \ldots, a_n/b \rightarrow a_1, \ldots, a_{n+1}/b$

(5.23) $a_1, \ldots, a_n/b \rightarrow a_n, \ldots, a_1/b$

(5.24) a/a

But this set, together with 5.20, is equivalent to one which I have elsewhere [1] described as "Basis II", and which can be shown to provide a sufficient basis for propositional logic, and especially for the definitions D3 to D6. In the presence of 5.20, the rules 5.21 to 5.24 characterize our first undefined concept, "/", sufficiently for operating with it successfully.

By the same method, i.e., by substituting "Axb" for "a" (and "a" for "c") in DB7, we obtain the rule

(5.70) $a\binom{x}{y}/(Axb)\binom{x}{y} \longleftrightarrow a\binom{x}{y}/b\binom{y}{x}$

and together with it the following six rules which characterize our second undefined concept, "$a\binom{x}{y}$", sufficiently for operating with it successfully. These rather trivial rules are

(5.71) If $v \neq x$; $x \neq y$; and $y \neq u$, then: $(a\binom{x}{u})\binom{y}{v}//(a\binom{y}{v})\binom{x}{u}$

(5.72) If $x \neq y$, then: $(a\binom{x}{y})\binom{x}{z}//a\binom{x}{y}$

(5.73) $(a\binom{x}{y})\binom{y}{z}//(a\binom{x}{z})\binom{y}{z}$

(5.74) $(a\binom{x}{y})\binom{z}{y}//(a\binom{z}{y})\binom{x}{y}$

(5.75) If, for every z, $a//a\binom{y}{z}$ and $b//b\binom{y}{z}$, then: $(a//b \rightarrow a\binom{x}{y}//b\binom{x}{y})$

(5.76) $a\binom{x}{x}//a$

If we now consider also definitions other than basic ones then we find that, in the presence of 5.20 to 5.24, each of the definitions DB1 to D6 is equivalent to the corresponding rule of inference in the following list:

(5.1) $a//b \longleftrightarrow (a/b \;\&\; b/a)$

(5.2) $a \wedge b/c \longleftrightarrow a, b/c$

(5.3) $a/b > c \longleftrightarrow a, b/c$

(5.4) $a \vee b/c \longleftrightarrow a/c \;\&\; b/c$

(5.5) $a^c, a/b \;\&\; (a^c, b/a \rightarrow b/a)$

(5.6) $a^i, a/b \;\&\; (a, b/a^i \rightarrow b/a^i)$

Turning to the non-basic definitions of Quantification Theory, we find that from D8, we obtain the two rules

(5.81) If $x \neq z$, then: $(Azb)(^{x}_{y})//Az(b(^{x}_{y}))$

(5.82) If $x = z$, then: $(Azb)(^{x}_{y})//Ay(b(^{x}_{y}))$

which I have described elsewhere [4] as "rules of substitution", and which relate the two concepts defined in DB7 and D8 to one another. Corresponding remarks hold for D9 and D10.

It will be seen that the two Basic Definitions DB2 and DB7 are somewhat complicated. Yet I think that the procedure of incorporating not only 5.20 into a definition of "$\wedge$" but also the rules which make it possible to operate with 5.20, is perfectly natural. Take DB1. The meaning of "$\wedge$" is determined not by 5.20 alone, but only by 5.20 in connection with 5.21 to 5.24; it seems therefore appropriate to incorporate 5.21 to 5.24 into our definition of conjunction. Analogous remarks hold for the six rules incorporated, together with 5.70, in DB7. (See also the next section, VI.)

It may be asked why conjunction is singled out, of all the propositional compounds, to serve as basic. The answer is given in an earlier publication [1]: conjunction (more precisely, rule 5.20) allows us to simplify the transitivity principle, and such a simplification is necessary if we wish to incorporate the transitivity principle into a definition which can be formulated with the means of expression described in section II. Nevertheless, conjunction is not the only compound which can be used in this way. Alternative Denial can be used as well.

In the presence of DB1 and DB2, alternative denial can be defined in various ways, for example:

(D↓) $a//b \downarrow c \longleftrightarrow$ (for every d_1: ((for every d_2: $(d_2/d_1 \longleftrightarrow (d_2/b\ \&\ d_2/c))) \longleftrightarrow$ (for every d_3: $(a, d_1/d_3\ \&\ (a, d_3/d_1 \rightarrow d_3,\ d_3/d_1)))))$

All that is necessary for strengthening this definition in such a way that it can serve as basic instead of DB2 is (1) to replace "d_2" by "$a_1, \ldots, a_n$", and (2) to incorporate the rules 5.201 to 5.204 in a manner precisely analogous to the way in which they are incorporated in DB2.

VI

Against our procedure of incorporating the rules 5.201 to 5.204 into DB2, and 5.71 to 5.76 into DB7, the objection may be raised that it does not lead to homogeneous definitions. I do not think this objection sufficiently important to discuss it here at length. But I wish to mention that at least one attempt to define the intuitive idea of homogeneity which underlies this objection leads to the following result.

Rules 5.20 and 5.204 alone are (in the presence of DB1) sufficient for proving that the left hand side of DB2 follows from the right hand side;

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[4]) Cp. *New Foundations for Logic*, notes to pp. 227 and 228.

the other rules, 5.201 and 5.202, which are together with 5.20 and 5.204 incorporated in DB2, are not needed for this purpose. This shows, it may be said, that they should be taken out of DB2, and stated seperately, as primitive rules. Corresponding remarks may be made about DB7. Here rule 5.70 alone is sufficient (in the presence of DB1 and DB2) for establishing that the left hand side of DB7 follows from the right hand side. 5.71 to 5.76 are superflous for this purpose.

In order to meet this objection, we can easily transform DB2 and DB7 in such a way that the objection no longer holds. All that is necessary for this purpose is to replace, in DB2, the expression

$$\text{“}(a_1, \ldots, a_n/b\text{”}$$

by the expression

$$\text{“}((b/c \to (a_1, a_2/b \to a_3, \ldots, a_1/c)) \to a_n, \ldots, a_1/b)\text{”}.$$

The resulting definition is then “homogeneous” in the sense that all rules (components) tied up in the definiens are now needed for showing that the left hand side follows from the right hand side. Similarly, in DB7, we need only replace the expression

$$\text{“}(c(\tfrac{x}{y})/a(\tfrac{x}{y}) \longleftrightarrow c(\tfrac{x}{y})/b(\tfrac{y}{x}))\text{”}$$

by the expression

$$\text{“}(x \neq u \neq y \to ((c(\tfrac{u}{u}))(\tfrac{y}{x})/(a(\tfrac{u}{u}))(\tfrac{y}{x}) \longleftrightarrow (c(\tfrac{y}{y}))(\tfrac{y}{x})/(b(\tfrac{y}{y}))(\tfrac{x}{y})))\text{”}$$

in order to achieve the corresponding result. In other words, homogeneity of the kind here indicated (and the same seems to hold for some other kinds) can always be attained in a trivial way, at the price of making the formulae more complicated. It is therefore hardly worth worrying about.

It may be mentioned, in this connection, that there are quite a number of equivalent ways of writing DB2 and DB7. The shortest formula to take the place of DB2 known to me is the following. (We assume that $n + r \geq n \geq 1$.)

(DB2′) $a//b \wedge c \longleftrightarrow$ (for every $a_1, \ldots, a_{n+r}$: $((a_1, \ldots, a_n/a \longleftrightarrow (a_1, \ldots, a_n/b \,\&\, a_1, \ldots, a_n/c)) \,\&\, (a_n, \ldots, a_1/b \to (b/c \to a_1, \ldots, a_{n+r}/c)) \,\&\, b/b)$).

VII

Our system is sufficient for lower functional logic in the sense that its rules of inference permit all logical transformations which are permissible by force of the usual lower functional calculi, and that they allow us to establish the demonstrability of all formulae which are demonstrable in these calculi.

In order to establish sufficiency, it is advisable to introduce the one-termed metalinguistic predicate

$$\text{“}\nvdash a\text{”}$$

which we read as "a is demonstrable". It can be defined by

(D11) $\quad \vdash a \longleftrightarrow$ (for every b: (b/a))

With the help of the means at our disposal, it is easy to show that this definition is equivalent to the formula

(7.11) $\quad \vdash a \longleftrightarrow a > a/a$

This formula could be used as a definition. Provided we do not object to the use of the defined term "$a > a$" in the definiens, it may even be said to be preferable because it uses *one* variable only, and avoids metalinguistic quantification.

The sufficiency of our system can now be established in the following sense.

Whenever a "proposition" such as

$$"p \supset p"$$

can be "asserted" in the system, say, of *Principia Mathematica*, then the corresponding formula, in our case

$$"\vdash a > a"$$

can be obtained in our metalinguistic system, by using only our definitions.

Whenever the formula in question is intuitionistically valid, it is only necessary to use, besides Basic Definitions, the definitions of the symbols actually occurring in the formula. If it is intuitionistically invalid but classically valid, then, even if no symbol of negation occurs in the formula, D5, the definition of classical negation, is needed for establishing its demonstrability.

VIII

If we wish to apply our system to some object language L, then certain existential postulates must be added which correspond to the definitions, such as

(P2) If b and c are both statements of L, then there exists a statement a of L such that $a//b \wedge c$, and that DB2 is satisfied.

OR:

(P5) If b is statement of L, then there exists a statement a of L such that $a//b^c$, and that D5 is satisfied.

(And so on.)

Of course, P2 and P5 suffice for propositional logic, if we wish to use classical negation, or even *one* postulate demanding the existence of $a//b \barwedge c$ for every b and c. (This is the real advantage of using alternative denial.)

IX

Our method can be easily extended so as to comprise modal functions. We can introduce the six symbols [5]:

(N) "*Na*" for "*a* is necessary".
(I) "*Ia*" for "*a* is impossible".
(L) "*La*" for "*a* is logical" (i.e. either necessary or impossible).
(C) "*Ca*" for "*a* is contingent" (or factual).
(P) "*Pa*" for "*a* is possible".
(U) "*Ua*" for "*a* is uncertain" (or not necessary).

These explanations are inexact, since the six symbols are actually intended to serve as *names* of statements of the object language. "*Na*", for example, should be explained, more lengthily but more precisely, as the name of a statement which asserts the necessity of the state of affairs described by *a* (or the necessity of the proposition expressed by *a*).

For the definitions, we must use "or" in the metalanguage (abbreviated by "*V*"), based on the appropriate rules. We shall also use "$\nvdash a$" and "$\nearrow a$" (i.e. "*a* is refutable"), defined by

(D12) $\qquad \nearrow a \longleftrightarrow$ (for every b: (a/b)).

But "$\nvdash$" and "$\nearrow$" are used here as abbrevations only, and should be eliminated with the help of (D11) and (D12).

(For the development of the theory of modality, metalinguistic negation is also needed, together with at least its intuitionistically valid rules.)

Our six definitions are:

(DN) $a//Nb \longleftrightarrow ((\nvdash a \; V \nearrow a) \;\&\; (\nvdash a \longleftrightarrow \nvdash b))$
(DI) $a//Ib \longleftrightarrow ((\nvdash a \; V \nearrow a) \;\&\; (\nvdash a \longleftrightarrow \nearrow b))$
(DL) $a//Lb \longleftrightarrow ((\nvdash a \; V \nearrow a) \;\&\; (\nvdash a \longleftrightarrow (\nvdash b \; V \nearrow b)))$
(DC) $a//Cb \longleftrightarrow ((\nvdash a \; V \nearrow a) \;\&\; (\not\nearrow a \longleftrightarrow (\nvdash b \; V \nearrow b)))$
(DP) $a//Pb \longleftrightarrow ((\nvdash a \; V \nearrow a) \;\&\; (\not\nearrow a \longleftrightarrow \nearrow b))$
(DU) $a//Ub \longleftrightarrow ((\nvdash a \; V \nearrow a) \;\&\; (\not\nearrow a \longleftrightarrow \nvdash b))$

The main interest of these definitions lies in the fact that our method allows us here again to avoid the use of defined concepts, such as negation, in the definiens. Accordingly, any sub-set of these definitions can be combined, for example, with positive logic, avoiding the use of classical or of any negation. It is also possible, for example, to introduce only "*Ia*"

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[5]) This list of six modal functions corresponds to the table given by R. CARNAP, *Meaning and Necessity* (1947), p. 175. Cp. also my *Logic Without Assumptions* where, in note to 7.2, a definition of impossibility is given which is equivalent to the one given here, and simpler. The more complicated definition given here has however the advantage of belonging to a complete set of six analogous definitions for all the modal functions.

(impossibility), and to compare the result with the intuitionist use of negation, etc. Our formulae yield, for example,

(9.1) $$I(Ia)/\!/Pa$$

thus justifying an intuitively obvious theorem (much used by intuitionism).

It appears that from these and our previous definitions taken together, all the valid rules of modal logic (including quantification theory) can be obtained).

X

The question may be raised whether our method is not so powerful merely because we employ fairly strong logical means in our metalanguage. It is therefore worth mentioning that, if we replace our system of definitions (e.g. DB2) by the system of rules which can be obtained from them (e.g. 5.20 to 5.24; note also that DB1 and D11 can be replaced by 5.1 and 7.11 respectively which do not need any quantifiers, and that corresponding adjustments can easily be made in the definitions of the modalities), then there is no need whatever to operate with quantifiers in the metalanguage. (This result is, perhaps, of some interest in connection with the decision problem). The logical rules of operating in the metalanguage may be reduced, in this way, to the rules pertaining to "$\rightarrow$" and to "$\&$". (If we split up our rules 5.1 to 5.4 etc. sufficiently, even "$\&$" may be avoided.) But no system of logic can be constructed without using at least "$\rightarrow$" in the metalanguage (just as we have to use "/"); for the transitivity principle — in its simplest form, "$a/b \rightarrow (b/c \rightarrow a/c)$" — must be capable of being stated, in some way or other, in the metalanguage of every system: for example, in those systems which (like NICOD's and QUINE's) use *one* undefined primitive sign of the object language for propositional logic.

Our results show that *no* such undefined sign of the object language (nor anything representing it in the metalanguage) need be assumed in addition to these necessary metalinguistic means of expression and to our two undefined specific logical concepts, and that these two concepts themselves, although undefined, can be, indirectly, characterized by two explicit definitions, viz. those of conjunction (or alternative denial) and universal quantification, respectively.

We can therefore say:

Every metalanguage which possesses the means of expression necessary to formulate the most general rules of inference (such as the principle of transitivity of inference in the simple form 5.21) possesses all the means necessary for constructing the whole propositional logic, including the logic of modalities.

Every metalanguage which, in addition, possesses a variable name of the statement which is the result of the substitution of x for y in a statement a, also possesses all the means necessary for constructing quantification theory.

Biochemistry. — *Oleate systems containing potassium chloride in which the KCl concentration is still too low for coacervation.* III. *Influence of KCl and alcohols on the elastic behaviour.* By H. G. BUNGENBERG DE JONG and G. W. H. M. VAN ALPHEN.

(Communicated at the meeting of March 29, 1947.)

1. *Introduction.*

The high viscous sols originated by KCl addition in a definite concentration trajectory preceding coacervation, show elastic phenomena. Shaking such a high viscous system in a test tube, fixes shorter or longer the air bubbles in the gelatinous mass. By rotating the testtube with a pull round the axis, — say nearly 45° — one may observe a pendulation of the air bubbles several times to and fro.

Leaving now the testtube alone, all the air bubbles, even the very small ones, ascend and after a shorter or longer time (e.g. some hours) they have left the high viscous system.

We therefore get the impression that the system acquires elastic properties in a mechanical way (the pull), becoming thus in this way a gel, but losing these elastic properties when left alone. This peculiar behaviour thus may be called in a way the contrast of thixotropic behaviour (where a system is fluid when shaked, but becomes solid when left alone).

Though we aim to study this elastic behaviour more in detail by more accurate methods later on, we here give the results obtained by a very primitive technic. These results concern the same variables, which have been investigated in the two previous communications [1]) with regard to the viscosity of oleate systems containing KCl, viz. KCl concentration and influence of alcohols.

2. *Measurement of the elastic properties.*

In the time of oscillation will be implied in the next the time between two turningpoints of an airbubble. The oscillation time appears to depend on the nature of the oleate system and on the dimensions of the cylindrical tube containing the system.

This was investigated at a system obtained by mixing in a stoppered bottle: 25 cc oleate sol + 28 cc KCl 3.8 N + 47 cc dest. water. (The oleate sol was prepared as in communication II from 40 gr. Na oleate + 2000 cc dest. water + 200 cc KOH 2N).

The mixture is shaken vigorously until a homogenous system is obtained.

It is of a high viscosity and represents nearly the viscosity maximum in fig. 2 of communication I.

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1) H. G. BUNGENBERG DE JONG and G. W. H. M. VAN ALPHEN, Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, **50**, 849 (1947) (I) and **50**, 1011 (1947) (II).

This system is now put to different heights in a measuring glass of 24 mm internal diameter, waited for the ascension of the very large airbubbles and measured (roomtemp.). The technic of measuring is as follows. The cylindrical tube is turned with a pull over a limited — not too large — angle.

When an airbubble, situated near the wall of the tube, takes its first turning point, a stopwatch is set on. The number of the next turning points, just visible with the naked eye, is counted and when the oscillations are practically finished, the stopwatch is stopped.

In this way a number n for the number of observed turning points and a time t (sec) for the total duration of the oscillation phenomenon is obtained. This measurement is repeated ten times and from the average values for n and t the oscillation time t/n may be computed. After some experience one gets satisfactory reproducible data.

At the first place the oscillation time of a bubble appeared to be dependent on the column height of the oleate system: if the column height amounts to less than 2 × the diameter of the cylindrical tube, too small values for t/n will be obtained.

But if the column height exceeds 2 × the diameter of the tube, t/n appears to be practically independent from the height. We therefore take column heights of the oleate system, being at least 2.5 × the diameter of the tube and perform the measurements on air bubbles situated in the upper zone.

Still further was investigated the influence of the width of the bottle, always taking care that the column height amounted to at least $2\frac{1}{2}$ × the width of the cylindrical tube.

Next table presents the complete data, in order to demonstrate one instance of the measurements.

TABLE I.

Diameter of the cylindrical tube in mm													
60		47		43		35.5		28		18		8.7	
t	n	t	n	t	n	t	n	t	n	t	n	t	n
9.0	5	9.4	7	8.4	7	7.6	7	5.4	7	3.6	7	1.0	5
8.2	5	9.6	7	8.2	7	8.2	8	5.0	7	3.4	7	1.2	5
8.8	5	9.0	7	7.0	6	8.0	8	6.8	8	3.4	7	1.2	6
10.0	6	9.2	7	7.8	7	8.4	8	4.8	7	3.8	7	1.2	5
10.0	6	9.6	7	7.8	7	8.0	8	5.6	8	3.2	7	1.2	5
10.0	6	9.2	7	8.0	7	8.4	8	5.0	7	3.4	7	1.2	5
10.6	6	10.6	8	8.6	7	8.4	8	5.4	7	3.6	7	0.8	4
10.8	6	10.8	8	8.4	7	8.2	8	5.2	7	3.6	7	0.8	4
12.4	7	10.0	8	8.6	7	8.0	8	4.8	7	3.8	7	1.0	4
12.4	7	12.4	9	9.2	7	8.2	8	6.0	8	3.4	7	1.0	4
mean 10.2	5.9	10.0	7.5	8.2	6.9	8.14	7.9	5.4	7.3	3.44	7	1.06	4.7
t/n 1.73		1.33		1.19		1.03		0.74		0.49		0.23	

If the oscillation time t/n is plotted against the diameter of the cylindrical tube, the result is a straight line through the zero of the diagram i.c. the oscillation time is proportional to the diameter of the tube.

We might be inclined to derive from these measurements another characteristic for the elastic behaviour of oleate systems viz. the damping, being the reciprocal value of the total number of observed turning points. It may be seen from the table, that this number n has a value of approximately 7 at tubes of 47—18 mm diameter, while it is lower at smaller or larger tubes.

These lower values are probably erroneous and may be caused by difficulties in observing the oscillations. The difficulty with too narrow tubes consists in the small amplitude and the small oscillation time. In very wide tubes on the other hand, too much airbubbles are visible on different depths at the same time. As all these bubbles differ in amplitude during the oscillation, a too large uncertainty of observation exists.

Taking the above into consideration, we believe to conclude that the damping, for which we take the reciprocal value of *n*, is independent from the diameter of the tube.

But it will be clear that the reproducibility of n and also of t will be much less satisfactory than of t/n.

The observer gets experienced in the just visible observation of a turning point.

(Compare the increase of n in the columns of the table for 60 and 47 mm).

Using a magnifying glass, increases the observation of the number of turning points.

Hence, n has to be observed in the same way, which was done always with the naked eye.

By means of a small microscope however, it was investigated if during the whole oscillation phenomenon, the oscillation time remains really constant, i.c. if we have to do with a damped harmonic oscillation.

Therefore were measured — at an oleate system showing by microscopic observation 10 turning points —, the total time for 10, 9, 8, etc. until 2 turning points, and these times plotted against the number of turning points. A straight line through the zero of the diagram results, but only if the amplitude at the beginning is sufficient small.

When the amplitude is larger, the oscillation times of the first oscillations are relatively too large, reaching a normal value again at the next oscillations.

As this detail was not yet known to us at the performance of the now following measurements, no account could be yet kept with it.

We established however, that the systematic error, which is caused by the latter in the determination of t/n was of small importance compared by the very large alterations of the oscillation time when varying the KCl concentration or adding alcohols. When however by a variable, t/n is only but little decreasing while n is very strongly decreasing at the same time, this systematic error may accomplish, that t/n remains apparently constant.

3. *Changes of the elastic behaviour of the oleate systems by increasing the KCl concentration.*

For these and the next measurements always was used a cylindrical tube of 28 mm diameter. Mixtures were being prepared of constant oleate concentration and increasing KCl conc. as amply described in communication I. (See fig. 2.)

In the table, the added quantities KCl 3,8 N, concern in the same way 20 cc mixtures (though for the filling of the measuring tube 40 cc of this mixture had to be prepared).

Thus the results (table II) may be compared at once. Moreover the roomtemp. (25°) was the same as in the measurements of the viscosity in communication I.

TABLE II.

cc KCl 3.8 N in 20 cc mixture	t sec. (mean)	n sec. (mean)	t/n	Remarks
4	—	0	~ (?)	pulls no threads.
4.5	3.36	2	1.68	threads very thin.
4.75	3.54	3	1.18	threads very thin.
5	4.10	3.8	1.08	threads thin.
5.25	4.72	4.8	0.98	
5.5	4.80	5.5	0.87	
5.75	5.30	7	0.76	gradual transition to short
6	5.09	6.9	0.74	and thick threads.
6.5	5.46	8.1	0.67	
7	5.72	9.17	0.62	
7.5	5.86	9.9	0.59	
8	5.35	9.55	0.56	gradual transition to
8.5	3.84	7.4	0.52	long and thin threads.
8.75	2.92	5.5	0.53	
9	2.10	4	0.53	not yet coacervated.
9.25	—	—	—	coacervated.

The values t, n and t/n are plotted in the upper part of fig. 1 against the KCl concentration. At KCl values until 4 cc no elastic phenomena appear.

Where they occur for the first time (4.5 cc KCl) the oscillation time t n has its greatest value and decreases continuously when increasing the KCl concentration, to attain a horizontal level just before the coacervation limit (between 9 and 9.25 cc KCl).

The time, in which one may observe oscillations, and the total number of turning points, increases to a maximum and decreases afterwards again.

From choosing either t or n as a measure of the reciprocal value of the damping of the oscillation follows that the damping with increasing KCl concentration passes a minimum.

If we compare, the curves of t/n and t or n resp. with the viscosity curves (fig. 1 lower part), we notice only *one* correlation between elastic and viscous behaviour.

The elastic phenomena do appear for the first time at the inflexion point in the left ascending branch of the viscosity curve.

The added arrow indicates the KCl concentration (4.5 cc), at which the liquid flowing out of the viscosimeter starts to pull threads. There is no

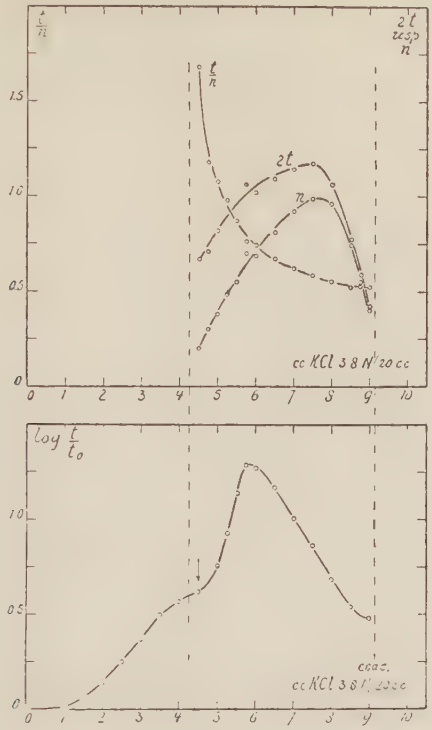


Fig. 1.

threadpulling at KCl = 4 cc. The left dotted vertical thus indicates nearly the "Thread limit" (KCl = 4.25 cc).

The right dotted vertical indicates in the same way the coacervation limit (at 9.13 cc KCl).

There is no reflection of the viscosity maximum in the t/n, t or n curves (no inflection points or extreme values).

In the same way there are no peculiarities in the viscosity curves at KCl concentration (7.5 cc KCl) at which t- or n-curves resp. show a maximum.

These facts will be important from the viewpoint of alterations in the structure of the "soap micells", occurring with increasing KCl concentration and finally resulting in coacervation. We will still withhold however from further speculations.

4. *Influence of normal alcohols on the elastic behaviour of oleate systems at the maximum of the viscosity KCl curve.*

Mixtures were being prepared corresponding to the maximum of the viscosity KCl curve all in the same way as in communication II, (5.75 cc

KCl 3.8 N per 20 cc final volume) to which had been added increasing concentrations of alcohols. Again measurements occurred in a measuring tube of 28 mm diameter. Roomtemp. amounted to 24—25°, fig. 2 shows the results as function of the logarithm of final alcohol concentrations (in mols per L). Arrows indicate the alcohol concentrations at which the thread-pulling propertie is lost (compare communication II).

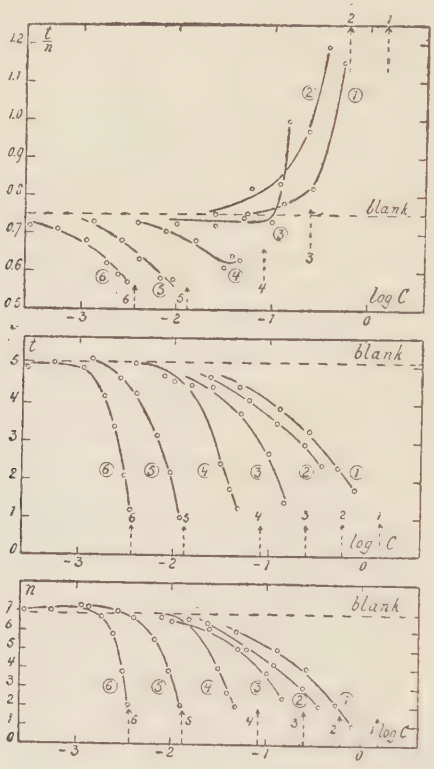


Fig. 2.

Summarizing the influence of alcohols on the elastic behaviour of oleate systems, we conclude:

1. A normal primary alcohol acts on the oscillation time t/n as well as on the damping (reciprocal value of t or n resp.) at the lower concentrations, the longer its carbon chain.

Compare e.g. in fig. 2 b the logarithmic distance of the curves 6 and 1 when t decreases on its half-value. This distance amounts to 2.18, i.e. hexylalcohol acts already in a 150 × smaller concentration as methylalcohol.

2. In the homologous series a change of action occurs concerning the influence on the oscillation time: methyl- and aethylalcohol increase t/n, n. butyl-, n. amyl- and n. hexylalcohol decrease t/n.

n. Propylalcohol acts as typical transition term, has small influence at first, but in higher concentrations the t/n curve ascends rapidly and is even situated then at the left side of the ethylalcohol curve.

Perhaps the n. butylalcohol curve tends also to ascend after a strong descending in the beginning.

3. No change of action occurs in the homologous series concerning the influence of alcohols on the damping. All t and n curves descend i.e. all alcohols increase the damping.

It may be useful to mention some points of correspondence and of difference, to get in the future a deeper insight into the nature of the elastic oleate systems and the structure of oleate coacervates.

ad. 1 : The same influence of the length of the carbon chain holds for the influence of alcohols *a*) on the volume of the oleate coacervate [2]) *b*) on the viscosity of oleate systems at KCl concentrations still to low for coacervation [3]).

ad. 2 : A change of action occurs in the same way with the influence of alcohols on the volume of oleate coacervates [2]).

Methyl- and aethylalcohol increase the volume, they act "opening". n. Butyl- and n. amylalcohol decrease the volume, they act "condensing". n. Propylalcohol behaves as transition term.

We also met with a change of action in high viscous oleate systems at the maximum of the viscosity KCl curve: methyl-, aethyl-, n. propyl-, and n. butylalcohol do *not* accomplish coacervation in contrast to n. amyl- and n. hexylalcohol [3]).

The transition term is shifted one term higher up in the homologous series, probably caused by the here smaller final KCl concentration.

ad. 3 : All six alcohols considerably decrease the high viscosity of the oleate system mentioned under ad 2) (followed by coacervation or not) [3]).

Summary.

1. A provisional method to characterise the elastic behaviour of high viscous oleate systems, containing KCl was indicated.

An orientating investigation was accomplished about the influence of KCl and of the first six terms of the homologous series of normal primary alcohols.

2. The elastic behaviour occurs only, when a definite KCl concentration is exceeded, nearly corresponding with the appearance of the phenomenon of thread-pulling and with the inflexion point on the ascending branch of the viscosity-KCl curve.

3. The oscillation time decreases with increasing KCl concentration.

The damping firstly decreases, passes a minimum and increases afterwards.

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[2]) H. G. BUNGENBERG DE JONG, G. G. P SAUBERT and H. L. BOOY. Protoplasma **30**, 1 (1938).

[3]) H. G. BUNGENBERG DE JONG and G. W. H. M. VAN ALPHEN, Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, **50**, 1011 (1947).

4. The elastic behaviour of maximum viscous KCl oleate systems is altered by alcohols. An alcohol acts in lower concentrations, the longer its carbon chain.

5. In the homologous series of the alcohol a change of action occurs with respect to the oscillation time. The first two terms increase, the 4th, 5th and 6th decrease the oscillation time.

n. Propylalcohol acts as transition term.

6. All six terms increase the damping.

7. Analogies to the influence of KCl and of alcohols on the viscosity of high viscous oleate systems and on the volume of oleate coacervates is briefly mentioned.

Botany. — *Uptake and transport of chlorine by parenchymatic tissue of leaves of Vallisneria spiralis.* II. *Analysis of the transport of chlorine.* By W. H. ARISZ.

(Communicated at the meeting of October 25, 1947.)

§ 1. *Introduction.*

After it had appeared that leaves of Vallisneria when exposed to the light are apt to take up chlorine from highly diluted concentrations of balanced salt solutions, the transport of the absorbed chlorine ions in the leaf was examined. With the simple phenomenon that a leaf brought into a KCl solution absorbs chlorine, it is an obvious assumption that this is done by the epidermal cells, which subsequently transport the absorbed chlorine to the deeper lying cells. Though it is imaginable that the ions diffuse inwardly along the cell walls and consequently the more inward cells can also obtain salt from the external solution, such a diffusion in the cell walls will, if it is possible, proceed extremely slowly, because the concentration of ions in the medium is low, and with these submersed leaves no suction of water from the outside to the inside takes place, in consequence of which salts might be transported together with the absorbed water. It is quite certain that the subepidermal cells absorb chlorine as well, because they increase their osmotic concentration to the same degree as epidermal cells, as appears from plasmolysis experiments. How this transport takes place is still unknown and it is no simple thing to investigate this by means of a suitable experiment. This problem is much easier to be solved, when we bring a part of the leaf in touch with a salt solution and next try to find out whether transport is possible to other parts of the leaf which are not in direct touch with the salt solution. Such leaflengths must have obtained their increase in salt through transport in the leaftissue from the part that is in direct touch with the salt solution. Such a research has been made in my laboratory by Miss KOK for Li-salts and caffeine while ARISZ and OUDMAN examined the transport of asparagine and caffeine by bringing a small part of the leaf in touch with an agargel to which the substances had been added. In this way we get a purely local absorption of the substances by a certain part of the leaf and the transport to the rest of the leaf can be examined. As the agar used contained some chlorine the rest of the leaf was not put between agar, but between moist filter paper. This prevents transpiration in the free part of the leaf so that no suction of liquid from the part lying in agar can occur. This method of administering the salt solution locally in an agargel is (for experimental reasons) much to be preferred to the method of bringing a part of the leaf into a separate box filled with a liquid, because the proper closing

off of this box, is always attended with difficulties owing to leakage or injury of the leaf.

Two series of experiments were made. The first series took place in 1944. The second series in 1946 and 1947. The results of these two series are somewhat different. With the experiments of 1944 the influence of wounding caused by dividing the Vallisneria leaves into smaller parts, had not yet been investigated. As already discussed in the first part of this publication, the wounding has some influence on the uptake, but it is only of a quantitative nature. Leaves which are used for absorption experiments some hours after the wounding, show a fair absorption which under favourable conditions does not depart much from that of material subjected to a longer preliminary treatment. With the transport, however, deeper lying differences appear, so that it is necessary to treat the two experimental series separately. This is a remarkable example of the fact how experiments which quantitatively gave most reliable results, still suggest an incorrect interpretation of a physiological process, owing to the fact that a physiologic factor was still unknown.

§ 2. *Method.*

In the experiments made in 1944 leaf lengths of 45 mms, of a width of 4 mms were put with one end for a distance of 8 mms between two 8 mm wide strips of agar (2 %) to which 1/100 mol KCl had been added. In the later experiments of this series a mixture of KCl and $CaSO_4$ was used in which the toxicity of the ions had proved to be slight, as a rule in an 1/100 mol Cl concentration. For one series 24 leaflengths of 45 mms were used, which were divided into 5 pieces of 8 mms when the experiment was finished. Besides at the ends about 2 mms were cut off which projected from the agar, in order to keep the airpassages of the leaves open for the sake of a good aeration. 24 identical 8 mm pieces, i.e. 19.2 cms leaf length were analysed together. The free parts of the leaf lengths were either placed between moist filterpaper on both sides or laid on moist filterpaper with the lower side only, a glass plate being placed over the

TABLE 1.

Absorption of chlorine in the light by leaves of Vallisneria lying between sheets of moist filterpaper. 25° C.

Directly analysed γ Cl in 20 cms leaf	Analysed after 24 hours between filterpaper γ Cl in 20 cms leaf.
384	377
390	377
377	380
380	383
377	377
av. 381,6 ± 5,5	av. 378,8 ± 2.8

upper side. Controlexperiments showed that from filterpaper nothing is taken up (Cf. Table 1).

In the second series of experiments in 1946 and 1947 leaf lengths of 7.5 cms were used which after the uptake were cut into pieces of 2.5 cms. Series of 8 pieces of 2.5 cms were analyzed together.

The variability of the various leaflengths was eliminated by combining in one series pieces of various leaves taken at various distances from the top. To examine the effect of light on separate zones tinfoil boxes were used, in which the ends of the leaf-lengths lay side by side in one box, so that only the absorption zone was darkened for a distance either of 8 mms or in the later experiments of 25 mms. The remainder of the leaf-lengths was kept outside the box and could be placed into the light without the pieces in the box being exposed to the light. Through clefts in the side wall the leaf parts that had to be darkened, were put in the tinfoil boxes, the somewhat projecting cover above the cleft preventing a direct entering of the light through the clefts into the box. In another model of tinfoil boxes 4 zones of 8 mms or 2 zones of 25 mms of the leaves could be darkened, while only one zone was illuminated. The whole set up was put in closed glass boxes, the bottoms of which were covered with water, so that the air was saturated with water vapour. The experiments were made in a room for constant temperature at 25° C.

§ 3. *First series of experiments.*

In these experiments the leaflengths were put in water for only a few hours after cutting. For an experiment 12 leaves were used of 27 cms length, each divided into 6 pieces of 4.5 cms. 24 lengths of 4.5 cms are further divided into 5 parts and form 5 series, each of 24 pieces of 8 mms. These 5 series are used as a blank to determine the chlorine present, when the experiment is started. In addition there are two more series each of 24 pieces of 4.5 cms length, used for transport experiments. From these leaf lengths of 4.5 cm 8 mms are placed between agar strips, $KCl + CaSO_4$ being present in the agar. When the uptake is finished, the lengths are cut into 5 pieces of 8 mms and each time 24 identical pieces are analyzed together.

In the first series of experiments the difference in uptake and transport in a medium with and without oxygen was examined. In a medium free from oxygen there is neither uptake nor transport of chlorine.

A great number of experiments were made on the influence of an exposure to light on the uptake and transport of chlorine. In the experiments of table 2 the leaf pieces were exposed in their entire lengths, while the controlpieces remained in the dark. In this experiment, which just as those following was made with 1/100 mol $KCl + CaSO_4$ the uptake in the light was greater than in the dark. For a further analysis of this influence of light the experiments have been made in the above mentioned tinfoil boxes. From the result of these experiments it may be concluded

TABLE 2.

Influence of light on the transport of chlorine. One series of leaves in the light, the other one in the dark. Uptake in γ Cl from agar with 1/100 mol KCl with addition of $CaSO_4$ during 24 hours. 25° C.

	Increase of Chlorine in γ.	
	in the light	in the dark
First part of 8 mms in contact with agar 2% containing KCl + $CaSO_4$	119	91
Second part of 8 mms	77	52
Third part of 8 mms	63	39
Fourth part of 8 mms	39	32
Fifth part of 8 mms	24	22

that the darkening in the tinfoil boxes has been sufficient. This appears from the following observation. In some experiments the first zone i.e. the *contactzone* was in the dark, the remainder of the leaf, which for convenience's sake we shall call the *free part* being put either in the light or in the dark. It now appears that the contactzone, which lies between the agarplates with the salt solution and which is in the dark, absorbs an equal amount no matter whether the free part is in the light or in the dark. Moreover it appears from tables 3 and 4 that exposure to light of the contactzone greatly affects the uptake. But as not any difference in uptake was found for the darkened contactzone, whether the free part is exposed to light or not, this proves that the contactzone receives no light on exposure of the free part; therefore the isolation in the tinfoil boxes is satisfactory.

TABLE 3.

Influence of an illumination of the contactzone. The free part of the leaves in the light. Uptake in γ Cl from agar with 1/100 mol KCl + $CaSO_4$ during 24 hours. 25° C.

		Increase of Chlorine in γ.			
		A		B	
First part of 8 mms in contact with agar 2% containing KCl + $CaSO_4$	contactzone	dark	light	dark	light
		31	169	55	208
Second part of 8 mms	free part in the light	19	69	28	98
Third part of 8 mms		9	15	1	58
Fourth part of 8 mms		2	9	11	24
Fifth part of 8 mms		9	2	14	18
		39	95	54	198

Tables 3 and 4 refer to the influence of an exposure of the contactzone of the leaf which is in touch with the agar and the salt. In table 3 the

free part is in the light, in table 4 in the dark. The influence of the illumination is especially clear in the experiments of table 3. Owing to exposure the uptake in the contactzone increases in two experiments from 31 to 169 γ Cl and from 55 to 208 γ Cl i.e. 461 % as an average; the uptake of the free part increases from 39 to 95 γ and from 54 to 198 γ Cl, averagely 305 %. Seeing this free part has to get the salt by transport from the contactzone this zone must have taken in this quantity too, so the uptake by the contactzone has increased in the aggregate from 70 to 264 γ Cl and from 109 to 406 γ averagely 375 %.

TABLE 4.

Influence of an illumination of the contactzone. The free part in the dark. Uptake in γ Cl from agar with 1/100 mol KCl + $CaSO_4$ during 24 hours, 25° C.

		Increase of chlorine in γ.	
First part of 8 mms in contact with agar 2 % containing KCl + $CaSO_4$	contactzone	dark	light
		92	120
Second part	free part in the dark	40	136
Third part		26	26
Fourth part		16	0
Fifth part		12	33
		94	195

The influence of the exposure of the contactzone on the transport to the free part (see table 4) when the free part has been darkened, is 207 %. The effect on the uptake in this case is 141 % and that on the entire uptake 175 %. This experiment corresponds with that of table 3; the effect, however, is less great in this case. We may therefore conclude that exposure of the contactzone increases the uptake of chlorine considerably, both in the contactzone and in the free part of the leaf.

TABLE 5.

Influence of an illumination of the free part of the leaf. The contactzone is in the dark. Uptake in γ Cl from agar with 1/100 mol KCl + $CaSO_4$ during 24 hours, 25° C.

		Increase of chlorine in γ.	
First part of 8 mms in contact with agar 2 % containing 1/100 mol KCl + $CaSO_4$	contactzone in the dark	117	120
		dark	light
Second part of 8 mms	free part	64	61
Third part of 8 mms		17	34
Fourth part of 8 mms		0	24
Fifth part of 8 mms		10	30
		91	149

Table 5 refers to the influence of the exposure of the free part on the uptake by the contactzone, which remains in the dark. When exposed the zones of the free part have a quantity of chlorine of 164 % against 100 % in the dark. This increase probably lies without the limits of error, though the results of these experiments were rather variable. From these experiments it appears:

1. An uptake of chlorine by the contactzone is constantly taking place, it is stronger in the light than in the dark.

2. A transport of chlorine takes place from the contactzone to the free part of the leaf.

3. Both uptake and transport are dependent on the presence of oxygen in the atmosphere.

4. Transport to the free part is greater, when the contactzone is exposed to light.

5. Transport from the unexposed contactzone to the free part of the leaf is probably greater, when this free part of the leaf is exposed to light, than when it is in the dark.

§ 4. *Second series of experiments.*

In these experiments we could avail ourselves of the experience that owing to the cutting of the Vallisneria leaves into pieces of an equal length and width the uptake is checked. From the experiments made on this subject it has appeared that after 24 hours this wound influence has vanished for the greater part, if the leaf lengths are not too short. Therefore the material was always subjected to a 24 hours'preliminary treatment in destilled water in this experimental series.

For an experiment 8 leaves were used of a length of 30 cms, each divided into 4 pieces of 7.5 cms. From these 3 series were formed, each consisting of 8 leaf lengths of 7.5 cms and 3 series consisting of 8 leaf lengths of 2.5 cms, which were used for the determination of the chlorine present, when the experiment was started. Usually these leaflengths were analyzed after a 24 hours'preliminary treatment in destilled water, as during the preliminary treatment the tissue may lose chlorine. Of the 7.5 cm leaflengths the first 2.5 cms, i.e. the contactzone, were placed between agar, while the free part, which had a length of 5 cms, was either placed between moist filter paper or was lying on moist filter paper and covered with a glass plate. As a precaution a little vaseline was put between the contactzone and the free part of the leaf in order to render diffusion along the surface of the leaf impossible. For aeration during the preliminary treatment air free from CO_2 was used. In some experiments air containing CO_2 was used for this purpose. Probably the uptake and the transport are less strong in this case, but this also depends on other circumstances.

On the whole there are 4 different ways of exposure.

1. The leaf lengths entirely exposed.
2. Entirely in the dark.
3. The contact zone exposed and the free part in the dark.
4. The contact zone in the dark and the free part exposed.

Three of these could be compared in one experiment.
Table 6 gives as an example the result of two experiments. For Fig. 1

TABLE 6.

Influence of exposure of the contactzone and of the free part of the leaf on uptake and transport. In each experiment three leaf-series A, B, C with different exposure. Only the first zone is in contact with agar 2 % containing 1/100 mol KCl + $CaSO_4$. The second and third zone are in moist air upon filterpaper. Duration of uptake 24 hours, 25° C. After the uptake the leaflengths were cut in 3 pieces and these were analyzed on Cl content. Pretreatment 24 hours in aerated destilled water in the light.

	A uptake in γ Cl	B uptake in γ Cl	C uptake in γ Cl
First zone of 2.5 cms	light 165	light 199	dark 67
Second zone of 2.5 cms	dark 12	light 64	dark 16
Third zone of 2.5 cms	dark 1	light 49	dark 5

	A uptake in γ Cl	B uptake in γ Cl	C uptake in γ Cl
First zone of 2.5 cms	dark 45	light 116	dark 26
Second zone of 2.5 cms	light 94	light 26	dark 5
Third zone of 2.5 cms	light 53	light —7	dark —14

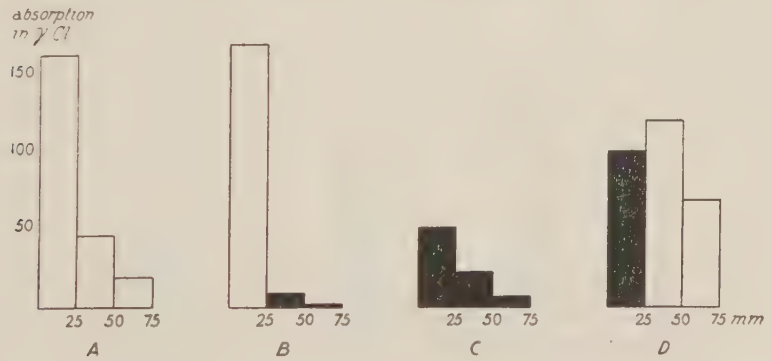


Fig. 1. Influence of exposure to light on uptake and transport of chlorine in leaves of Vallisneria. In *A* the whole leaf is in the light, in *B* only the first zone. In *C* the whole leaf is in the dark, in *D* the first zone in the dark and the free part in the light. The first zone of 25 mms is 24 hours in contact with a solution of 0.01 mol KCl + $CaSO_4$ in 2 % agar. The second and third zone are free in moist air on wet filterpaper. On the ordinate the increase in Cl-content. temp. 25° C. Pretreatment during 24 hours in aerated destilled water in the light.

the averages are used from a greater number of experiments. As discussed in the preceding part, the strength of the uptake is dependent on a great number of factors. As we saw, this obtains for uptake from solution as

well as from agar. Though in these experiments the conditions of exposure, temperature, etc. were kept as constant as possible, yet the differences between the experiments remain great, because the condition of the leaves, when gathered, greatly influences their behaviour and this is a factor which varies from day to day. A 24 hours'preliminary treatment cannot remove these differences.

In the dark, Fig. 1 C, the uptake is slight and so is the transport in every case. The introduction of the preliminary treatment does give much more regular results in this respect than had been obtained in the previous experimental series. The uptake in the light Fig. 1 A is considerably greater than in the dark, especially when during the preliminary treatment there was an exposure to strong light. In one case the transport was also slight in the light. As a rule the 2.5 cm zone bordering on the contact zone gets about 1/3 to 1/2 of the amount of chlorine absorbed in the contact zone. The third 2.5 cm zone gets but little chlorine, sometimes the amount of chlorine does not increase at all here.

If the contact zone is exposed and the free part in the dark, Fig. 1 B, the uptake by the contact zone is fairly normal, but the transport in the darkened free part is very slight.

If the contact zone is in the dark and the free part exposed, Fig. 1 D, a most remarkable phenomenon occurs, which was not found in this way in the first series of experiments (cf. I table 5). The contactzone absorbs more than is the case, when the free part is in the dark and the free part absorbs a great quantity of chlorine, not only the second but also the third zone.

It is noteworthy that in spite of the great variability in the strength of uptake and transport, which again strikes us in these experiments, qualitatively the same phenomena occur in all experiments. To state this it is desirable to study the results of table 7, in which 6 experiments have been

TABLE 7.

Influence of an illumination of the free part of the leaf. The contactzone is in the dark. Uptake from agar with 1/100 mol $KCl + CaSO_4$ during 24 hours. 25° C. Pretreatment of the leaves in destilled water during 24 hours in the light.

First zone of 2.5 cms	contactzone in the dark	147	129	72	88	48	45
Second zone of 2.5 cms	free part in the light	167	140	98	73	91	94
Third zone of 2.5 cms		63	63	46	51	40	53

comprised. Here we regularly find the phenomenon that exposure of the free part causes a strong transport, both to the second and to the third zone. This result deviates a little from what was found in the first experimental series of 1944. In the first experimental series it was only found that on darkening the contactzone the free part takes up more when

exposed than in the dark, but the differences were not very pronounced.

This has given rise to an investigation to find out whether this less pronounced effect in the first experimental series might be connected with the differences in preliminary treatment in the two experimental series. That is why in table 8 some results have been given of a treatment of leaf material subjected to a preliminary treatment in destilled water for periods of various length. In A there was no preliminary treatment, in B it lasted 16 hours. and in C 24 hours. The result of such an experiment, shown in table 8 and fig. 2, is clear. As already discussed before, the absorption is not only checked, but also the transport is strongly affected under the influence of the wounding. Only after a 24 hours'preliminary treatment the

TABLE 8.

Influence of the pretreatment during 0, 16 and 24 hours on the uptake of leaves, treated in the same way as in table 6.

		Pretreatment in dest. water		
		no A	16 hours B	24 hours C
First zone of 2.5 cms	contactzone in the dark	73	122	146
Second zone of 2.5 cms	free part in the light	16	62	157
Third zone of 2.5 cms		— 5	45	104

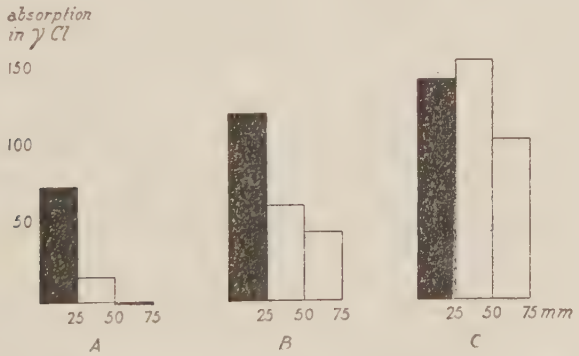


Fig. 2. Influence of pretreatment on the transport of chlorine. Exposure as in fig. 1 D. *A* no pretreatment, *B* 16 hours and *C* 24 hours pretreatment in destilled water. In *C* a strong transport to the free part of the leaves.

result is shown that the free part has increased its amount of chlorine considerably more than the contact zone, which is in direct touch with the medium. It is apparent that after 24 hours pretreatment in water there is an intensive transportation of chlorine from the contactzone to the free part of the leaf.

The results of this second experimental series are therefore a corroboration

of the conclusions drawn from the first experimental series and are a most valuable addition to them.

It appears that the wounding is not only the cause of an inhibition of the absorption, but has also a pronounced influence on the transport. This influence may be considered to contribute to the production of the great differences between the results found in these experiments. *There must be a protoplasmatic factor here which is influenced by the wounding and which is of a dominating influence on the transport.*

§ 5. *Influence of the length of the free part on the absorption by the contactzone.*

Already in the asparagine experiments in 1937 it was observed that in transport experiments made with leaves of a different length, of which a part of a same size acts as a contactzone (i.e. can absorb from the medium), more asparagine is absorbed according as the free part of the leaf is larger. As we saw in the first part of this publication this phenomenon may partly be due to the wounding, which has a stronger influence according as the pieces are shorter and the wound surfaces nearer to each other. For studying this problem it was therefore better not to compare the absorption by a length of 7.5 cms, of which only 2.5 cms are in touch with agar + KCl, with a short length of 2.5 cms which is entirely in touch with the agar, as in this case the wound stimulus might affect the two differently. It is better to compare with an equally large piece of 7.5 cms which is entirely in touch with the agar + KCl and to take for comparison 1/3 part of the quantity absorbed. Table 9 gives the result of such an experiment.

TABLE 9.

Comparison of leaves taking up chlorine only by the contactzone (B and C) with leaves taking up over the whole surface (A). Pretreatment 24 hours in dest. water. Absorption during 24 hours, 25° C, in the light.

	A		B	C
First zone 2.5 cms in agar + KCl	303	in agar + KCl	296	286
Second zone 2.5 cms in agar + KCl	300	between filterpaper	146	128
Third zone 2.5 cms in agar + KCl	300	between filterpaper	17	12
Uptake per zone 1/3 of the leaf	301 γ	total uptake	459γ	426γ

It appears that the two leaflengths B and C, which take up the chlorine in the first zone and transport it to the second and third, have accumulated in their first zone nearly as much chlorine as each third part of leaf A, which is over the whole surface in contact with the medium. Moreover they have transported 163 γ and 140 γ Cl to the second and third zones.

So the total uptake is 459 and 426 γ Cl, while one third part of leaf A gets only 301 γ Cl. This result proves that the transport to the free part of the leaf has not reduced the amount available for the contactzone. This experiment shows that the uptake in the free part of the leaf may be without influence on the accumulation in the contactzone itself.

In various other experiments it was found, however, that part of the transport to the second and third zones may occur at the expense of the accumulation in the contactzone, so that it makes the impression that there is a rivalry among the adjoining zones for the chlorine. The case mentioned first, however, is of essential importance.

In addition to the results already mentioned on page 1240 the following conclusions can be drawn.

1. The transport is strongly affected by the wounds caused by the cutting of the leaves. Though some transport in the first hours is possible a strong transport to the free part of the leaf is only found, when the leaves have been pretreated during 24 hours in destilled water.

2. When during the uptake the contactzone remains in the dark and the free part of the leaf is exposed, there results a strong accumulation in the free part of the leaf. The amount of chlorine present in the second zone is under these circumstances even greater than in the contactzone.

3. Contrary to the rules of a diffusionprocess, a zone lying at some distance from the medium containing the chlorine ions, can take up more chlorine than the zone which is in direct touch with the medium.

4. The amount of chlorine absorbed by the free part of the leaf must have passed the contactzone. In some experiments this resulted in a slighter accumulation in the contactzone, but it also occurred that there was no influence at all. This demonstrates that the uptake in the free part of the leaf is more or less independent of the accumulation in the contactzone.

Mathematics. — *Semantical considerations on intuitionistic mathematics* [1]. By E. W. BETH. (Communicated by Prof. L. E. J. BROUWER.)

(Communicated at the meeting of November 29, 1947.)

A. TARSKI, in a series of papers [2]), has introduced a new method, which has proved of great value to axiomatics. This paper is intended to point out the great importance of this so-called semantical method for the elucidation of the foundations of intuitionistic mathematics.

The semantical method may be considered as a synthesis of the mathematical and the metamathematical (or proof theoretical) methods, which so far dominated axiomatics.

The mathematical method is characteristic of that study of mathematics, which does not aim at an elucidation of its foundations, but only at the enlargement of our mathematical knowledge, as well as of older axiomatics, of which HILBERT's "Grundlagen der Geometrie" constitute a classical example. It is directed towards the mathematical entities — numbers, progressions, functions, sets — themselves and towards the properties and mutual relations of these entities; consequently it coincides with BROUWER's 'first order mathematics' [3]). In order to render its results communicable, one has to use names for the mathematical entities and for their properties and relations; the total of these names will be called mathematical terminology.

The metamathematical method — BROUWER's 'second order mathemathics', HILBERT's 'Metamathematik' or proof theory — is directed, not towards the mathematical entities themselves or towards their properties and relations, but towards the elements of mathematical terminology and towards the statements, definitions and proofs, obtained by combining these elements. In order to formulate its results, one has to make use, not of the names of mathematical entities and of their properties and relations, but of names for the elements of mathematical terminology; the total of these names will be called metamathematical terminology. It was CARNAP [4]), who stressed the necessity of a sharp distinction between mathematical

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[1]) This paper was written in July 1945, immediately after the liberation of our country; for external reasons, it could not be published earlier. I have preferred to publish it now in its original form, notwithstanding the publication, in the meantime, of S. C. KLEENE, "On the interpretation of intuitionistic number theory", Journal of symbolic logic **10**, 1945, and of L. E. J. BROUWER, "Richtlijnen der intuitionistische wiskunde", Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, Vol. **50** (1947).

[2]) A. TARSKI, "Der Wahrheitsbegriff in den formalisierten Sprachen", Studia philosophica **1**, 1935; "Ueber den Begriff der logischen Folgerung", Actes du Congrès Int. de Philos. scient., fasc. VII, Act. Scient. et industr. **394**, Paris 1936.

[3]) L. E. J. BROUWER, "Over de grondslagen der wiskunde", Amsterdam 1907, pp. 173—175.

[4]) R. CARNAP, "Logische Syntax der Sprache", Wien 1934.

terminology ("Objektsprache") and metamathematical terminology ("Syntaxsprache"). A great number of the problems of axiomatics — including the problem of consistency — belong to the field of metamathematics.

The semantical method deals with mathematical entities and their properties and relations as well as with the entities belonging to mathemathical terminology, that is to say, with the names of mathematical entities and of their properties and relations, and with mathematical statements, definitions and proofs. In order to deal with its results, we must have the disposal of a terminology, which embraces mathematical as well as metamathematical terminology (semantical terminology).

Even in the "ordinary" study of mathematics, one is not always able to evade the application of the semantical method. A very simple illustration is found in the theorem, stating the existence of an unambiguously determined decimal representation for any natural number. The special character of this method — which HILBERT and BERNAYS [5]) called 'set theoretic method' —, however, was not clearly conceived before the publication of TARSKI's papers. To the field of semantics belong, besides the concepts of *truth* and of *meaning*, such fundamental notions of axiomatics as *model* and *categoricity* (as defined by O. VEBLEN).

It should be noted, that the concept of *logical consequence* likewise belongs to the field of semantics; whereas the proof theoretical notion of consequence does not coincide with the idea, we connect with the word 'consequence' in the 'ordinary' study of mathematics — as is shown by GÖDEL's theorem —, the following adequate definition of the 'ordinary' concept of logical consequence was given by TARSKI in semantical terms: a statement X is a locical consequence of a classe K of statements, if and only if any model of the class K is as well a model of the statement X.

In applying the semantical method, TARSKI considers mathematical entities as well as their properties and relations as extant independently of mathematical thought, as this is usual in non-intuitionistic mathematics. With regard to those entities, which belong to mathematical terminology, he adopts a more constructive point of view. This should not be considered as an inconsistency: the main intention of semantical research is an investigation into the requirements to be imposed upon mathematical terminology.

A very important result of TARSKI's researches, which is closely connected with GÖDEL's theorem — I need not dwell upon questions of priority — and with the analysis of the so-called semantical antinomies, should be mentioned here: a complete mathematical terminology is impossible; or, if stated more explicitly: if for a certain field of mathematical research a terminology T_0 is given, there are

1) notions, which according to their intention belong to this field of research, but are not capable of being defined by means of T_0;

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[5]) D. HILBERT und P. BERNAYS, "Grundlagen der Mathematik", 1. Bd, Berlin 1934.

2) true statements, which according to their intention belong to the field of research, the proofs of which are, however, not capable of being formulated by means of T_0.

Queer — but important, as it constitutes a refutation of a well-known assertion by KRONECKER — is the following result: there is a statement, belonging to the theory of real numbers, which is, however, not equivalent to any statement, belonging to the theory of natural numbers.

Returning to TARSKI's results concerning the incompleteness of any mathematical terminology T_0, I observe, that T_0 is supposed to agree in general structure with usual mathematical terminologies. This explains CHURCH's attempts [6]) to construct a complete mathematical terminology of divergent structure. Apart from this possibility of a complete mathematical terminology of a more or less unusual type — which is far from proved and which seems to me hardly probable —, we must, even if we accept a cantorist or a logicistic conception of mathematical existence, consider mathematical terminology, not as extant in a finished state, but as being in a constant state of growth. In order to formulate the results of mathematical research we need a series T_0, T_1, T_2, ... of terminologies of ever growing complication. For intuitionistic mathematics this need was indicated by HEYTING [7]).

I now come to my theme, that is, to the question, how far application of the semantical method may contribute to an elucidation of the principles of intuitionistic mathematics. Giving proofs for assertions to be made is, of course, out of the question within the modest scope of this paper, which is intended only to give some directives for further work on the question; I will, however, give an application 'per analogiam' of TARSKI's results in the consideration of intuitionistic mathematics.

In the first place a remark relating to a discussion, some years ago, between H. FREUDENTHAL and A. HEYTING on the occasion of the publication of a paper by the first [8]).

The problems concerning the interpretation of logical formulae can be solved only by applying the semantical method. In agreement with current use, which is also accepted by HEYTING, I consider a mathematical statement (or formula) or definition as belonging to mathematical terminology. A proof may then be considered as a series of statements or formulae, and consequently as belonging to mathematical terminology as well. Investigation into mathematical statements (formulae, definitions) and proofs therefore belongs to the field of metamathematics.

Mathematical method, on the contrary, is to the purpose, if questions concerning mathematical entities or their properties and relations are raised.

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[6]) A. CHURCH, "The Richard paradox", Am. math. monthly **41**, 1934.

[7]) A. HEYTING, "Die formalen Regeln der intuitionistischen Logik", Sitzungsber. d. Preuss. Akad. d. Wiss., Phys.-math. Kl., 1930.

[8]) H. FREUDENTHAL, "Zur intuitionistischen Deutung logischer Formeln"; A. HEYTING, "Bemerkungen zu dem Aufsatz von Herrn Freudenthal ...", Compos. math. **4**, 1936.

As soon, however, as we enter into the relations between mathematical entities (and their properties and relations) and mathematical constructions on the one side, and mathematical statements (formulae, definitions) and proofs on the other, we are to combine mathematical and metamathematical method, that is to say, we must apply the semantical method.

As an example I mention the interpretation of PEANO's axioms for arithmetics, which without any essential modification may be adopted by intuitionistic arithmetics; it should be observed that, as long as we do not allow definitions by recursion, only a very slight part of arithmetics can be derived from these axioms.

We now suppose an intuitive construction of the series of natural numbers. We then must show, that this series may be considered as a model for the axiom system; this may be done by applying TARSKI's methods, upon which, however, I will not dwell here.

Then we must justify the modes of inference, as adopted in intuitionistic mathematics. These modes of inference are best characterized by a method we owe to G. GENTZEN [9]); in this way we get away from the objections, which from the point of view of intuitionism may be raised against stating general logical laws. As a matter of fact, we do not state general logical laws; we only draw up inference schemes and these are justified, not in general, but only with regard to arithmetics.

For this purpose we must show: if the series of natural numbers is a model for a set of statements $X, Y, Z, \ldots$, while the statement U is obtained from the statements $X, Y, Z, \ldots$ by applying one of the inference schemes, then the series of natural numbers is a model for the statement U. This also may be done by applying TARSKI's method.

It is true that we must be prepared for technical and perhaps even for essential difficulties in realizing this programme.

Even of classical mathematics so far only rather elementary sections — e.g., the so-called logic of classes — have been submitted to elaborate semantical analysis.

From the point of view of intuitionism, only limited value may be attributed to the analysis of such elementary formalisms. The formalism in question should contain at least a not too trivial part of arithmetics.

So much about the semantical method and its application to the elucidation of intuitionistic mathematics in general. I now pass to a more special problem, namely the definition of the notion of spread, as given by BROUWER [10]):

[9]) G. GENTZEN, "Untersuchungen über das logische Schliessen", Math. Zs. **39**, 1934.

[10]) Quoted from BROUWER's Cambridge lectures on intuitionistic mathematics, which will be published by the Cambridge University Press. Originally both notions, of spread law and of spread, were introduced by BROUWER under the name of "Menge" in his "Begründung der Mengenlehre unabhängig vom logischen Satz vom ausgeschlossenen Dritten", 1. Teil, Kon. Akad. v. Wetensch., Amsterdam, Verhandelingen 1e Sectie, **12** (1918). Comp. also "Intuitionistische Mengenlehre", these Proceedings **23** (1920), pp.949—954. Later on, in his lectures, for the notion of spread BROUWER replaced the term "Menge" by "Mengenspezies".

"A *spread law* is an instruction, according to which, if again and again an arbitrary natural number is chosen as "index", each of those choices has as its predeterminate effect, depending also on the preceding choices, that either a certain "figure" (viz. either no thing or a mathematical entity) is generated, or that the choice is "sterilized", in which case the figures generated are destroyed and generation of any further figures is prevented, so that all following choices are sterilized likewise. The only condition to be satisfied is that after each non-sterilized initial sequence of $n-1>0$ choices, one natural number at least is available which, if chosen as n_{th} index, generates a figure.

The infinite sequences of figures generated according to a spread law by infinitely proceeding sequences of choices are, by virtue of this genesis, together with all infinite sequences equal to one of them, the elements of a species. This species is called *a spread*."

Such a spread law definition may be given the following symbolical form:

$$\left.\begin{array}{ll} a_1 = M'_1 & p_1 = M_1(\tau a_1) \\ a_{k+1} = M'_{k+1}(\tau a_1, \tau a_2, \ldots \tau a_k) & p_{k+1} = M_{k+1}(\tau a_1, \tau a_2, \ldots \tau a_{k+1}) \end{array}\right\}$$

The relation $\prec$ between a spread and its elements may be defined as follows:

$$\left.\begin{array}{l} X \subset M' \underset{\text{Df}}{=} x_1 \varepsilon M'_1 \,\&\, (k)(x_{k+1} \varepsilon M'_{k+1}(x_1, x_2, \ldots x_k)) \\ P \prec M \underset{\text{Df}}{=} (Ex)[X \subset M' \,\&\, (k)(p_k = M_k(x_1, x_2, \ldots x_k)) \end{array}\right\}$$

We have applied the following notations:

X is a series, the terms $x_1, x_2, \ldots$ of which are natural numbers;

P is a series, the terms $p_1, p_2, \ldots$ of which are objects;

$\alpha_1, \alpha_2, \ldots$ are classes, the elements of which are natural numbers;

$\tau\alpha$ is an element, arbitrarily chosen from α;

$M'_{k+1}(x_1, x_2, \ldots x_k)$ is a function, the argument values of which are natural numbers, whereas its function values are classes of natural numbers:

$M_{k+1}(x_1, x_2, \ldots x_{k+1})$ is a function, the argument values of which are natural numbers, whereas its function values are objects.

Any spread is determined by two progressions M' and M of functions: consequently the question arises, in which manner these progressions should be defined; it will be evident, that we should apply recursion procedures.

The adoption of the semantical point of view gives rise to two questions, namely

1. whether there is, for any formal definition, a corresponding spread in the sense of non-formalized intuitionistic mathematics;

2. whether there is, for any spread in the sense of non-formalized intuitionistic mathematics, a corresponding formal definition of the type, described before.

The first question may be answered in the affirmative without any hesitation.

In dealing with the second question, we should, however, be prepared to meet difficulties, which are analogous to the ones, underlying the negative results, obtained by TARSKI, GÖDEL, CHURCH and SKOLEM. If, in the definition of the progressions M' and M, we stick to recursions of a definite type, only part of the spread in the sense of non-formalized intuitionistic mathematics will be capable of being defined in the formal manner described above. In this connection we should ask, whether from the intuitionistic point of view only recursions of a certain definite type are to be admitted; in my opinion, we should rather admit an indefinite range of types of recursion.

Mathematics. — *Extension of* PEARSON's *Probability Distributions to two Variables.* II. By M. J. VAN UVEN. (Communicated by Prof. W. VAN DER WOUDE.)

(Communicated at the meeting of October 25, 1947.)

§ 2. Classification of the probability density functions.

The nature of the solution of eq. (1) depends mainly on the structure of the denominators **G** and **H**, especially on their having common factors or not.

So the density functions q will be classified according to the degree and the mutual divisibility of **G** and **H**. Thus we can distinguish:

I. **G** and **H** have *no* common factors, with the subdivision:

a) Both **G** and **H** indecomposable; b) **G** indecomposable, $\mathbf{H} = \mathbf{BC}$ (resp. $\mathbf{G} = \mathbf{AC}$, **H** indecomp.); c) **G** indecomp., $\mathbf{H} = \mathbf{B}^2$ (resp. $\mathbf{G} = \mathbf{A}^2$, **H** indecomp.); d) $\mathbf{G} = \mathbf{AC}$, $\mathbf{H} = \mathbf{BD}$; e) $\mathbf{G} = \mathbf{A}^2$, $\mathbf{H} = \mathbf{BD}$ (resp. $\mathbf{G} = \mathbf{AC}$, $\mathbf{H} = \mathbf{B}^2$); f) $\mathbf{G} = \mathbf{A}^2$, $\mathbf{H} = \mathbf{B}^2$; g) **G** indecomp., $\mathbf{H} = \mathbf{B}$ (resp., $\mathbf{G} = \mathbf{A}$, **H** indecomp.); h) $\mathbf{G} = \mathbf{AC}$, $\mathbf{H} = \mathbf{B}$ (resp. $\mathbf{G} = \mathbf{A}$, $\mathbf{H} = \mathbf{BC}$); i) $\mathbf{G} = \mathbf{A}^2$, $\mathbf{H} = \mathbf{B}$ (resp. $\mathbf{G} = \mathbf{A}$, $\mathbf{H} = \mathbf{B}^2$); j) $\mathbf{G} = \mathbf{A}$, $\mathbf{H} = \mathbf{B}$; k) **G** indecomp., $\mathbf{H} = 1$ (resp. $\mathbf{G} = 1$, **H** indecomp.); l) $\mathbf{G} = \mathbf{AC}$, $\mathbf{H} = 1$ (resp. $\mathbf{G} = 1$, $\mathbf{H} = \mathbf{BC}$); m) $\mathbf{G} = \mathbf{A}^2$, $\mathbf{H} = 1$ (resp. $\mathbf{G} = 1$, $\mathbf{H} = \mathbf{B}^2$); n) $\mathbf{G} = \mathbf{A}$, $\mathbf{H} = 1$ (resp. $\mathbf{G} = 1$, $\mathbf{H} = \mathbf{B}$).

II. Both **G** and **H** quadratic with one common factor:

a) $\mathbf{G} = \mathbf{AC}$, $\mathbf{H} = \mathbf{BC}$; b) $\mathbf{G} = \mathbf{A}^2$, $\mathbf{H} = \mathbf{AB}$ (resp. $\mathbf{G} = \mathbf{AB}$, $\mathbf{H} = \mathbf{B}^2$).

III. **G** and **H** quadratic and identical:

a) $\mathbf{G} \equiv \mathbf{H}$ indecomp.; b) $\mathbf{G} \equiv \mathbf{H} = \mathbf{AC}$; c) $\mathbf{G} \equiv \mathbf{H} = \mathbf{A}^2$.

IV. **G** quadratic, **H** linear factor of **G** (resp. **H** quadratic, **G** linear factor of **H**)

a) $\mathbf{G} = \mathbf{AC}$, $\mathbf{H} = \mathbf{C}$ (resp. $\mathbf{G} = \mathbf{C}$, $\mathbf{H} = \mathbf{BC}$); b) $\mathbf{G} = \mathbf{A}^2$, $\mathbf{H} = \mathbf{A}$ (resp. $\mathbf{G} = \mathbf{B}$, $\mathbf{H} = \mathbf{B}^2$).

V. **G** and **H** linear and identical: $\mathbf{G} \equiv \mathbf{H} = \mathbf{C}$.

VI. Both **G** and **H** of degree zero: $\mathbf{G} \equiv \mathbf{H} = 1$.

The above types will be submitted to condition (2):

$$\mathbf{H}\frac{\partial \mathbf{P}}{\partial y} - \mathbf{G}\frac{\partial \mathbf{Q}}{\partial x} = \frac{\mathbf{H}}{\mathbf{G}}\mathbf{P}\frac{\partial \mathbf{G}}{\partial y} - \frac{\mathbf{G}}{\mathbf{H}}\mathbf{Q}\frac{\partial \mathbf{H}}{\partial x},$$

of which the left member is a whole function. The demand that also the right member shall be whole therefore restricts the possibilities for **G**, **H**, **P**, **Q**.

In many cases it is simpler to integrate $\frac{\partial \log \varphi}{\partial x} = \frac{\mathbf{P}}{\mathbf{G}}$ directly over x and to determine the additive function of y from the shape $\frac{\mathbf{Q}}{\mathbf{H}}$ of $\frac{\partial \log \varphi}{\partial y}$.

Type I. As $\mathbf{G}$ and $\mathbf{H}$ have no common factors, condition (2) can only be satisfied by

$$\frac{\partial \mathbf{G}}{\partial y} = 0 \text{ (or } \mathbf{G} = \mathbf{G}_1), \qquad \frac{\partial \mathbf{H}}{\partial x} = 0 \text{ (or } \mathbf{H} = \mathbf{H}_2),$$

whence $\mathbf{H}\frac{\partial \mathbf{P}}{\partial y} - \mathbf{G}\frac{\partial \mathbf{Q}}{\partial x} = p_2\mathbf{H} - q_1\mathbf{G} = 0.$

As $\frac{\mathbf{G}}{\mathbf{H}} = \frac{p_2}{q_1} = \text{const.}$ has been excluded, there remains

$$p_2 = \frac{\partial \mathbf{P}}{\partial y} = 0 \text{ (or } \mathbf{P} = \mathbf{P}_1), \qquad q_1 = \frac{\partial \mathbf{Q}}{\partial x} = 0 \text{ (or } \mathbf{Q} = \mathbf{Q}_2).$$

So we obtain

$$\frac{\partial \log \varphi}{\partial x} = \frac{\mathbf{P}_1}{\mathbf{G}_1}, \qquad \frac{\partial \log \varphi}{\partial y} = \frac{\mathbf{Q}_2}{\mathbf{H}_2},$$

whence

$$\log \varphi = \int \frac{\mathbf{P}_1}{\mathbf{G}_1}\, dx + \int \frac{\mathbf{Q}_2}{\mathbf{H}_2}\, dy = \log \varphi_1(x) + \log \varphi_2(y)$$

and

$$\varphi(x, y) = \varphi_1(x) \cdot \varphi_2(y) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

Here x and y are mutually independent (their correlation coefficient is zero). According to the subdivision a) ... n) φ_1 and φ_2 are of various *Pearsonian* types.

Type II a) $\mathbf{G} = \mathbf{AC}$, $\mathbf{H} = \mathbf{BC}$.

Here condition (2) is reduced to

$$\mathbf{B}\left(\mathbf{C}\frac{\partial \mathbf{P}}{\partial y} - \mathbf{P}\frac{\partial \mathbf{C}}{\partial y}\right) - \mathbf{A}\left(\mathbf{C}\frac{\partial \mathbf{Q}}{\partial x} - \mathbf{Q}\frac{\partial \mathbf{C}}{\partial x}\right) = \frac{\mathbf{BC}}{\mathbf{A}}\mathbf{P}\frac{\partial \mathbf{A}}{\partial y} - \frac{\mathbf{AC}}{\mathbf{B}}\mathbf{Q}\frac{\partial \mathbf{B}}{\partial x}.$$

Since the left member is a whole function, the right member must also be whole.

Therefore:

$$\frac{\partial \mathbf{A}}{\partial y} = 0, \text{ or } \mathbf{A} = \mathbf{A}_1 \equiv a_0 + a_1 x \text{ and } \frac{\partial \mathbf{B}}{\partial x} = 0, \text{ or } \mathbf{B} = \mathbf{B}_2 \equiv b_0 + b_2 y.$$

We now get

$$p_2\mathbf{C} - c_2\mathbf{P} \equiv \lambda \mathbf{A}_1, \qquad q_1\mathbf{C} - c_1\mathbf{Q} \equiv \lambda \mathbf{B}_2,$$

whence

$$\mathbf{P} \equiv \frac{p_2}{c_2}\mathbf{C} - \frac{\lambda}{c_2}\mathbf{A}_1, \qquad \mathbf{Q} \equiv \frac{q_1}{c_1}\mathbf{C} - \frac{\lambda}{c_1}\mathbf{B}_2.$$

So putting

$$\mu_1 = \frac{p_2}{a_1 c_2}, \qquad \mu_2 = \frac{q_1}{b_2 c_1}, \qquad \mu_3 = \frac{-\lambda}{c_1 c_2},$$

$$\frac{\partial \log \varphi}{\partial x} = \frac{\mathbf{P}}{\mathbf{A}_1 \mathbf{C}} = \mu_1 \frac{a_1}{\mathbf{A}_1} + \mu_3 \frac{c_1}{\mathbf{C}}, \quad \frac{\partial \log \varphi}{\partial y} = \frac{\mathbf{Q}}{\mathbf{B}_2 \mathbf{C}} = \mu_2 \frac{b_2}{\mathbf{B}_2} + \mu_3 \frac{c_2}{\mathbf{C}},$$

whence

$$\log \varphi = \mu_1 \log \mathbf{A}_1 + \mu_2 \log \mathbf{B}_2 + \mu_3 \log \mathbf{C} + \log K_0, \text{ or}$$

Type II a) $\mathbf{G} = \mathbf{A}_1\mathbf{C}$, $\mathbf{H} = \mathbf{B}_2\mathbf{C}$: $\qquad \varphi = K_0 \mathbf{A}_1^{\mu_1} \mathbf{B}_2^{\mu_2} \mathbf{C}^{\mu_3}$ . . $(20)_{IIa}$[1])

Type II b) $\mathbf{G} = \mathbf{A}^2$, $\mathbf{H} = \mathbf{AB}$.

For condition (2) we can now write

$$\mathbf{B} \frac{\partial \mathbf{P}}{\partial y} - \mathbf{A} \frac{\partial \mathbf{Q}}{\partial x} + \mathbf{Q} \frac{\partial \mathbf{A}}{\partial x} = 2 \frac{\mathbf{B}}{\mathbf{A}} \mathbf{P} \frac{\partial \mathbf{A}}{\partial y} - \frac{\mathbf{A}}{\mathbf{B}} \mathbf{Q} \frac{\partial \mathbf{B}}{\partial x}.$$

Consequently

$$\frac{\partial \mathbf{A}}{\partial y} = 0, \text{ or } \mathbf{A} = \mathbf{A}_1 \equiv a_0 + a_1 x, \quad \frac{\partial \mathbf{B}}{\partial x} = 0, \quad \text{or } \mathbf{B} = \mathbf{B}_2 \equiv b_0 + b_2 y,$$

and $\mathbf{Q} \equiv q_0 + q_1 x + q_2 y \equiv \frac{q_1}{a_1} \mathbf{A}_1 - \frac{p_2}{a_1} \mathbf{B}_2$, or $q_0 = \frac{a_0 q_1 - b_0 p_2}{a_1}$, $q_2 = -\frac{b_2 p_2}{a_1}$.

Writing $\mathbf{P} \equiv p_0 + p_1 x + p_2 y$ in the form $\mathbf{P} \equiv \mu_1 a_1 \mathbf{A}_1 + a_1 \mathbf{D}_2$ (whence $p_2 = a_1 d_2$) and putting $\mu_2 = \frac{q_1}{a_1 b_2}$ (whence $\mathbf{Q} = \mu_2 b_2 \mathbf{A}_1 - d_2 \mathbf{B}_2$), we obtain

$$\frac{\partial \log \varphi}{\partial x} = \frac{\mathbf{P}}{\mathbf{A}_1^2} = \mu_1 \frac{d \log \mathbf{A}_1}{dx} - \frac{\partial}{\partial x}\left(\frac{\mathbf{D}_2}{\mathbf{A}_1}\right),$$

$$\frac{\partial \log \varphi}{\partial y} = \frac{\mathbf{Q}}{\mathbf{A}_1 \mathbf{B}_2} = \frac{\mu_2 b_2}{\mathbf{B}_2} - \frac{d_2}{\mathbf{A}_1} = y_2 \frac{d \log \mathbf{B}_2}{dy} - \frac{\partial}{\partial y}\left(\frac{\mathbf{D}_2}{\mathbf{A}_1}\right).$$

So

$$\log \varphi = \mu_1 \log \mathbf{A}_1 + \mu_2 \log \mathbf{B}_2 - \frac{\mathbf{D}_2}{\mathbf{A}_1} + \log K_0, \text{ or}$$

Type II b) $\mathbf{G} = \mathbf{A}_1^2$, $\mathbf{H} = \mathbf{A}_1\mathbf{B}_2$: $\qquad \varphi = K_0 \mathbf{A}_1^{\mu_1} \mathbf{B}_2^{\mu_2} e^{-\frac{\mathbf{D}_2}{\mathbf{A}_1}}$ . . $(20)_{IIb}$

To $\mathbf{D}_2$ any multiple of $\mathbf{A}_1$ (say $\varrho \mathbf{A}_1$) may be added (by which K_0 is multiplied by e^{ϱ}).

Type III a): $\mathbf{G} \equiv \mathbf{H}$ indecomposable.

Condition (2) now becomes

$$(p_2 - q_1)\,\mathbf{G} = \mathbf{P} \frac{\partial \mathbf{G}}{\partial y} - \mathbf{Q} \frac{\partial \mathbf{G}}{\partial x}.$$

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[1]) This density function has been derived by L. N. G. FILON and L. ISSERLISS, and published by K. PEARSON in his paper: Notes on Skew Frequency Surfaces. Biometrika vol. V (1923), p. 224.

1. If $p_2 - q_1 \neq 0$, the conic $\mathbf{G} = 0$ passes through the point of intersection of $\frac{\partial \mathbf{G}}{\partial x} = 0$ and $\frac{\partial \mathbf{G}}{\partial y} = 0$, so through its own centre. Then, however, it degenerates into a pair of straight lines, which implies that G is decomposable, contrary to the assumption. The only admissible solution, therefore, is:

2. $p_2 = q_1$, and $\frac{\mathbf{P}}{\frac{\partial \mathbf{G}}{\partial x}} = \frac{\mathbf{Q}}{\frac{\partial \mathbf{G}}{\partial y}} = \mu$ (const.), or $\mathbf{P} = \mu \frac{\partial \mathbf{G}}{\partial x}$, $\mathbf{Q} = \mu \frac{\partial \mathbf{G}}{\partial y}$;

this gives $\log \varphi = \mu \log \mathbf{G} + \log K_0$, or

Type III a): $\mathbf{G} \equiv \mathbf{H}$ indecomposable:

$$\varphi = K_0 \mathbf{G}^{\mu}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20_{\text{IIIa}}{}^{2})$$

This type can be subdivided into two classes (see § 3).

Type III b α): $\mathbf{G} \equiv \mathbf{H} = \mathbf{AC}$ ($\mathbf{A}$ and $\mathbf{C}$ real).

Here condition (2) leads to

$$p_2 - q_1 = \frac{a_2\mathbf{P} - a_1\mathbf{Q}}{\mathbf{A}} + \frac{c_2\mathbf{P} - c_1\mathbf{Q}}{\mathbf{C}},$$

whence

$$a_2\mathbf{P} - a_1\mathbf{Q} = \varrho\mathbf{A}, \qquad c_2\mathbf{P} - c_1\mathbf{Q} = \sigma\mathbf{C} \qquad (\varrho + \sigma = p_2 - q_1).$$

Putting

$$\mu_1 = \frac{\sigma}{a_1 c_2 - a_2 c_1}, \qquad \mu_3 = \frac{-\varrho}{a_1 c_2 - a_2 c_1},$$

we find

$$\mathbf{P} = \mu_1 a_1 \mathbf{C} + \mu_3 c_1 \mathbf{A}, \quad \mathbf{Q} = \mu_1 a_2 \mathbf{C} + \mu_3 c_2 \mathbf{A}.$$

So

$$\frac{\partial \log \varphi}{\partial x} = \frac{\mathbf{P}}{\mathbf{AC}} = \mu_1 \frac{a_1}{\mathbf{A}} + \mu_3 \frac{c_1}{\mathbf{C}}, \qquad \frac{\partial \log \varphi}{\partial y} = \frac{\mathbf{Q}}{\mathbf{AC}} = \mu_1 \frac{a_2}{\mathbf{A}} + \mu_3 \frac{c_2}{\mathbf{C}},$$

$$\log \varphi = \mu_1 \log \mathbf{A} + \mu_3 \log \mathbf{C} + \log K_0.$$

Hence

Type III b α)

$$\mathbf{G} \equiv \mathbf{H} = \mathbf{AC}: \qquad \varphi = \mathbf{K}_0 \mathbf{A}^{\mu_1} \mathbf{C}^{\mu_3}. \quad . \quad . \quad . \quad (20)_{\text{IIIb}}$$

Type III b β): $\mathbf{G} \equiv \mathbf{H} = \mathbf{AC}$ ($\mathbf{A}$ and $\mathbf{C}$ complex).

Starting from $\mathbf{G} \equiv g_{00} + 2g_{01}x + 2g_{02}y + g_{11}x^2 + 2g_{12}xy + g_{22}y^2$, we put

$$\Delta \equiv \begin{vmatrix} g_{00}, & g_{01}, & g_{02} \\ g_{10}, & g_{11}, & g_{12} \\ g_{20}, & g_{21}, & g_{22} \end{vmatrix}, \quad \Delta_{ij} = \frac{\partial \Delta}{\partial g_{ij}}, \quad \mathbf{J}_2 \equiv -\Delta_{00}y^2 + 2\Delta_{02}y - \Delta_{22}. \quad (21)$$

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2) This type has been studied by K. PEARSON: On Non-skew Frequency Surfaces, Biometrika vol. V (1923) p. 231.

Since $\mathbf{G} = \mathbf{AC}$, we have

$$\Delta = 0, \text{ so } g_{11}\Delta = \Delta_{00}\Delta_{22} - \Delta_{02}^2 = 0, \text{ and } \mathbf{J}_2 = -\Delta_{00}\left(y - \frac{\Delta_{02}}{\Delta_{00}}\right)^2.$$

For $\mathbf{G}$ we can write $\mathbf{G} = g_{11}(x - x_1)(x - x_2)$, where

$$x_1 = \frac{1}{g_{11}}\{-(g_{01} + g_{12}y) + \sqrt{\mathbf{J}_2}\}, \quad x_2 = \frac{1}{g_{11}}\{-(g_{01} + g_{12}y) - \sqrt{\mathbf{J}_2}\}.$$

$\mathbf{A}$ and $\mathbf{C}$ being complex, $\mathbf{J}_2$ is negative, so $\Delta_{00} > 0$. Putting $\mathbf{J}_2 = -\mathbf{I}_2^2$, we have

$$\mathbf{I}_2 = \sqrt{\Delta_{00}} \cdot \left(y - \frac{\Delta_{02}}{\Delta_{00}}\right) \text{ and } g_{11}(x_1 - x_2) = 2i\mathbf{I}_2.$$

Thus

$$g_{11}(x - x_1) = (g_{01} + g_{11}x + g_{12}y) - i\mathbf{I}_2 = \varrho e^{-i\theta},$$

$$g_{11}(x - x_2) = (g_{01} + g_{11}x + g_{12}y) + i\mathbf{I}_2 = \varrho e^{+i\theta},$$

$$\frac{x - x_1}{x - x_2} = e^{-2i\theta}, \qquad \cot\theta = \frac{g_{01} + g_{11}x + g_{12}y}{\mathbf{I}_2}.$$

Integrating

$$\frac{\partial \log\varphi}{\partial x} = \frac{\mathbf{P}}{\mathbf{G}} = \frac{p_0 + p_1 x + p_2 y}{g_{11}(x - x_1)(x - x_2)} \text{ over } x, \text{ we find}$$

$$\log\varphi = \frac{p_1}{2g_{11}}\log\mathbf{G} + \frac{(g_{11}p_0 - g_{01}p_1) - (g_{12}p_1 - g_{11}p_2)y}{2ig_{11}\sqrt{\Delta_{00}} \cdot \left(y - \frac{\Delta_{02}}{\Delta_{00}}\right)} \cdot \log\left(\frac{x - x_1}{x - x_2}\right) + \mathbf{f}(y).$$

As $\dfrac{\partial \log\varphi}{\partial y} = \dfrac{\mathbf{Q}}{\mathbf{H}} = \dfrac{\mathbf{Q}}{\mathbf{G}}$ must be algebraic, the factor of $\log\left(\dfrac{x - x_1}{x - x_2}\right)$ must be a constant, hence also

$$\omega(y) \equiv \frac{(g_{11}p_0 - g_{01}p_1) - (g_{12}p_1 - g_{11}p_2)y}{y - \dfrac{\Delta_{02}}{\Delta_{00}}} = \text{constant, namely}$$

$$\omega = -(g_{12}p_1 - g_{11}p_2).$$

So we arrive at

$$\log\varphi = \frac{p_1}{2g_{11}}\log\mathbf{G} + \frac{i(g_{12}p_1 - g_{11}p_2)}{2g_{11}\sqrt{\Delta_{00}}}\log\left(\frac{x - x_1}{x - x_2}\right) + \mathbf{f}(y),$$

and

$$\frac{\partial \log\varphi}{\partial y} = \frac{p_1}{2g_{11}} \cdot \frac{\dfrac{\partial \mathbf{G}}{\partial y}}{\mathbf{G}} + \frac{i(g_{12}p_1 - g_{11}p_2)}{2g_{11}\sqrt{\Delta_{00}}} \cdot \frac{\partial \log\left(\dfrac{x - x_1}{x - x_2}\right)}{\partial y} + \mathbf{f}'(y).$$

Since

$$\frac{\partial \log\left(\frac{x-x_1}{x-x_2}\right)}{\partial y} = \frac{\partial\{g_{11}(x-x_1)\}}{\partial y} - \frac{\partial\{g_{11}(x-x_2)\}}{\partial y} =$$

$$= \frac{g_{12}-i\sqrt{\Delta_{00}}}{g_{11}(x-x_1)} - \frac{g_{12}+i\sqrt{\Delta_{00}}}{g_{11}(x-x_2)} = \frac{(g_{12}-i\sqrt{\Delta_{00}})(x-x_2)-(g_{12}+i\sqrt{\Delta_{00}})(x-x_1)}{\mathbf{G}},$$

$f'(y)$ is zero. Hence

$$\log\varphi = \frac{p_1}{2g_{11}}\log\mathbf{G} + \frac{i(g_{12}p_1-g_{11}p_2)}{2g_{11}\sqrt{\Delta_{00}}} \times -2i\theta + \log K_0 =$$

$$= \frac{p_1}{2g_{11}}\log\mathbf{G} + \frac{g_{12}p_1-g_{11}p_2}{g_{11}\sqrt{\Delta_{00}}}\operatorname{arc\,cot}\frac{g_{01}+g_{11}x+g_{12}y}{\mathbf{I}_2} + \log K_0 =$$

$$= \frac{p_1}{2g_{11}}\log\mathbf{G} + \frac{g_{11}p_2-g_{12}p_1}{g_{11}\sqrt{\Delta_{00}}}\operatorname{arc\,tan}\frac{g_{01}+g_{11}x+g_{12}y}{\mathbf{I}_2} + \log K_0',$$

or, putting

$$\mu = \frac{p_1}{2g_{11}}, \quad \lambda = \frac{g_{11}p_2-g_{12}p_1}{g_{11}\sqrt{\Delta_{00}}},$$

$$\log\varphi = \mu\log\mathbf{G} + \lambda\operatorname{arc\,tan}\frac{g_{01}+g_{11}x+g_{12}y}{\sqrt{\Delta_{00}}\cdot\left(y-\frac{\Delta_{02}}{\Delta_{00}}\right)} + \log K_0',$$

whence

Type III b β) $\quad \mathbf{G} \equiv \mathbf{H} = \mathbf{AC}$ (**A** and **C** complex):

$$\varphi = K_0'\,\mathbf{G}^{\mu}\,e^{\lambda\operatorname{arc\,tan}\frac{g_{01}+g_{11}x+g_{12}y}{\sqrt{\Delta_{00}}\cdot\left(y-\frac{\Delta_{02}}{\Delta_{00}}\right)}} \quad . \quad . \quad . \quad . \quad (20)_{\mathrm{III\,b}\,\beta}$$

Type III c) $\quad \mathbf{G} \equiv \mathbf{H} = \mathbf{A}^2$.

Here condition (2) furnishes $p_2-q_1 = \dfrac{2(a_2\mathbf{P}-a_1\mathbf{Q})}{\mathbf{A}}$.

So

$$\mathbf{Q} \equiv q_0 + q_1x + q_2y = \frac{2a_2\mathbf{P}-(p_2-q_1)\mathbf{A}}{2a_1}, \text{ whence } q_1 = \frac{a_2p_1}{a_1} - \frac{p_2-q_1}{a_1},$$

or $$q_1 = \frac{2a_2p_1}{a_1} - p_2, \quad p_2-q_1 = \frac{2a_2p_1}{a_1} - 2q_1.$$

Putting for $\mathbf{P} \equiv p_0 + p_1x + p_2y$ $\mathbf{P} = \varrho\mathbf{A} + a_1\mathbf{D}_2$, whence $p_1 = \varrho a_1$, $p_2 = \varrho a_2 + a_1d_2$, and

$$q_1 = 2\varrho a_2 - (\varrho a_2 + a_1d_2) = \varrho a_2 - a_1d_2, \quad p_2-q_1 = 2a_1d_2,$$

we get $$\mathbf{Q} = \frac{a_2}{a_1}\varrho\mathbf{A} + a_2\mathbf{D}_2 - d_2\mathbf{A},$$

and with $$\mu = \frac{\varrho}{a_1} = \frac{p_1}{a_1^2},$$

$$\frac{\partial \log \varphi}{\partial x} = \frac{\mathbf{P}}{\mathbf{A}^2} = \frac{\mu a_1}{\mathbf{A}} + \frac{a_1 \mathbf{D}_2}{\mathbf{A}^2}, \quad \frac{\partial \log \varphi}{\partial y} = \frac{\mathbf{Q}}{\mathbf{A}^2} = \frac{\mu a_2}{\mathbf{A}} + \frac{a_2 \mathbf{D}_2}{\mathbf{A}^2} - \frac{d_1}{\mathbf{A}},$$

whence

$$\log \varphi = \mu \log \mathbf{A} - \frac{\mathbf{D}_2}{\mathbf{A}} + \log K_0.$$

Therefore

Type III c) $\mathbf{G} \equiv \mathbf{H} = \mathbf{A}^2$: $\varphi = K_0 \mathbf{A}^\mu e^{-\frac{\mathbf{D}_2}{\mathbf{A}}}$ . . . . . $(20)_{\text{IIIc}}$

To $\mathbf{D}_2$ any multiple of $\mathbf{A}$ may be added.

Type IV a) $\mathbf{G} = \mathbf{AC}$, $\mathbf{H} = \mathbf{C}$.

Condition (2) becomes

$$\mathbf{C} \frac{\partial \mathbf{P}}{\partial y} - \mathbf{AC} \frac{\partial \mathbf{Q}}{\partial x} = \frac{\mathbf{PC}}{\mathbf{A}} \frac{\partial \mathbf{A}}{\partial y} + \mathbf{P} \frac{\partial \mathbf{C}}{\partial y} - \mathbf{AQ} \frac{\partial \mathbf{C}}{\partial x},$$

whence $$\frac{\partial \mathbf{A}}{\partial y} = 0, \quad \text{or} \quad \mathbf{A} = \mathbf{A}_1 \equiv a_0 + a_1 x.$$

So (2) leads to $p_2\mathbf{C} - c_2\mathbf{P} = \mathbf{A}_1(q_1\mathbf{C} - c_1\mathbf{Q})$.

Consequently $q_1\mathbf{C} - c_1\mathbf{Q} = \varrho$ (const.), or

$$\mathbf{Q} = \frac{q_1}{c_1}\mathbf{C} - \frac{\varrho}{c_1} \quad \text{and} \quad \mathbf{P} = \frac{p_2}{c_2}\mathbf{C} - \frac{\varrho}{c_2}\mathbf{A}_1.$$

This gives

$$\frac{\partial \log \varphi}{\partial x} = \frac{\mathbf{P}}{\mathbf{A}_1\mathbf{C}} = \frac{p_2}{c_2} \cdot \frac{1}{\mathbf{A}_1} - \frac{\varrho}{c_2} \cdot \frac{1}{\mathbf{C}}, \quad \frac{\partial \log \varphi}{\partial y} = \frac{\mathbf{Q}}{\mathbf{C}} = \frac{q_1}{c_1} - \frac{\varrho}{c_1} \cdot \frac{1}{\mathbf{C}}.$$

So

$$\log \varphi = \frac{p_2}{a_1 c_2} \log \mathbf{A}_1 - \frac{\varrho}{c_1 c_2} \log \mathbf{C} + \mathrm{f}(y), \quad \frac{\partial \log \varphi}{\partial y} = -\frac{\varrho}{c_1} \frac{1}{\mathbf{C}} + \mathrm{f}'(y),$$

whence $$\mathrm{f}'(y) = \frac{q_1}{a_1} = -\lambda_2, \quad \mathrm{f}(y) = -\lambda_2 y + \log K_0$$

and, with $$\mu_1 = \frac{p_2}{a_1 c_2}, \quad \mu_3 = -\frac{\varrho}{c_1 c_2},$$

$$\log \varphi = \mu_1 \log \mathbf{A}_1 + \mu_3 \log \mathbf{C} - \lambda_2 y + \log K_0.$$

Therefore

Type IV a) $\mathbf{G} = \mathbf{A}_1\mathbf{C}$, $\mathbf{H} = \mathbf{C}$: $\varphi = K_0 \mathbf{A}_1^{\mu_1} \mathbf{C}^{\mu_3} e^{-\lambda_2 y}$ . . . $(20)_{\text{IVa}}$

Type IV b) $\mathbf{G} = \mathbf{A}^2$, $\mathbf{H} = \mathbf{A}$.

Here condition (2) is reduced to

$$\frac{\partial \mathbf{P}}{\partial y} - \mathbf{A} \frac{\partial \mathbf{Q}}{\partial x} = 2 \frac{\mathbf{P}}{\mathbf{A}} \cdot \frac{\partial \mathbf{A}}{\partial y} - \mathbf{Q} \frac{\partial \mathbf{A}}{\partial x}.$$

So $\frac{\partial \mathbf{A}}{\partial y} = 0$, or $\mathbf{A} = \mathbf{A}_1 \equiv a_0 + a_1 x$, and $p_2 - q_1 \mathbf{A}_1 = -a_1 \mathbf{Q}$, whence $\frac{\partial \mathbf{Q}}{\partial y} = 0$, or $\mathbf{Q} = \mathbf{Q}_1 \equiv q_0 + q_1 x$, and $p_2 - q_1(a_0 + a_1 x) = -a_1(q_0 + q_1 x)$, or $p_2 - q_1 a_0 + a_1 q_0 = 0$.

Writing $\mathbf{P} \equiv p_0 + p_1 x + p_2 y$ in the form

$$\mathbf{P} = \varrho \mathbf{A}_1 + a_1 \mathbf{D}_2,$$

we get $p_0 = \varrho a_0 + a_1 d_0$, $p_1 = \varrho a_1$, $p_2 = a_1 d_2$.

Further

$$\frac{\partial \log \varphi}{\partial x} = \frac{\mathbf{P}}{\mathbf{A}_1^2} = \frac{\varrho}{\mathbf{A}_1} + a_1 \frac{\mathbf{D}_2}{\mathbf{A}_1^2}, \quad \frac{\partial \log \varphi}{\partial y} = \frac{\mathbf{Q}_1}{\mathbf{A}_1} = \frac{-p_2 + q_1 \mathbf{A}_1}{a_1 \mathbf{A}_1} = -\frac{d_2}{\mathbf{A}_1} + \frac{q_1}{a_1}.$$

Hence

$$\log \varphi = \frac{\varrho}{a_1} \log \mathbf{A}_1 - \frac{\mathbf{D}_2}{\mathbf{A}_1} + \mathrm{f}(y), \quad \frac{\partial \log \varphi}{\partial y} = -\frac{d_2}{\mathbf{A}_1} + \mathrm{f}'(y),$$

so $$\mathrm{f}'(y) = \frac{q_1}{a_1} = -\lambda_2, \quad \text{or} \quad \mathrm{f}(y) = -\lambda_2 y + \log K_0.$$

Consequently, putting $\mu_1 = \frac{\varrho}{a_1}$,

$$\log \varphi = \mu_1 \log \mathbf{A}_1 - \frac{\mathbf{D}_2}{\mathbf{A}_1} - \lambda_2 y + \log K_0, \quad \text{and}$$

Type IV b) $\mathbf{G} = \mathbf{A}_1^2$, $\mathbf{H} = \mathbf{A}_1$: $\varphi = K_0 \mathbf{A}_1^{\mu_1} e^{-\frac{\mathbf{D}_2}{\mathbf{A}_1} - \lambda_2 y}$ . . . $(20)_{\mathrm{IV\,b}}$

Type V. $\mathbf{G} \equiv \mathbf{H} = \mathbf{C}$.

Condition (2) becomes

$$\mathbf{C} \frac{\partial \mathbf{P}}{\partial y} - \mathbf{C} \frac{\partial \mathbf{Q}}{\partial x} = \mathbf{P} \frac{\partial \mathbf{C}}{\partial y} - \mathbf{Q} \frac{\partial \mathbf{C}}{\partial x}, \text{ or } (p_2 - q_1)\mathbf{C} = c_2 \mathbf{P} - c_1 \mathbf{Q}.$$

Putting $\mathbf{P} = -\lambda_1 \mathbf{C} + \varrho_1$, $\mathbf{Q} = -\lambda_2 \mathbf{C} + \varrho_2$, we get

$(p_2 - q_1)\,\mathbf{C} = (-\lambda_1 c_2 + \lambda_2 c_1)\,\mathbf{C} + (c_2 \varrho_1 - c_1 \varrho_2)$, whence $\varrho_1 = \mu c_1$, $\varrho_2 = \mu c_2$.

So

$$\frac{\partial \log \varphi}{\partial x} = \frac{\mathbf{P}}{\mathbf{C}} = -\lambda_1 + \mu \frac{c_1}{\mathbf{C}}, \quad \frac{\partial \log \varphi}{\partial y} = \frac{\mathbf{Q}}{\mathbf{C}} = -\lambda_2 + \mu \frac{c_2}{\mathbf{C}}.$$

This gives

$$\log \varphi = -\lambda_1 x + \mu \log \mathbf{C} + \mathrm{f}(y), \quad \frac{\partial \log \varphi}{\partial y} = \frac{\mu c_2}{\mathbf{C}} + \mathrm{f}'(y), \text{ whence } \mathrm{f}'(y) = -\lambda_2,$$

or $$\mathrm{f}(y) = -\lambda_2 y + \log K_0.$$

Consequently

$$\log \varphi = -\lambda_1 x + \mu \log \mathbf{C} - \lambda_2 y + \log K_0,$$

or

Type V. $\mathbf{G} \equiv \mathbf{H} = \mathbf{C}$: $\varphi = K_0\, \mathbf{C}^{\mu}\, e^{-\lambda_1 x - \lambda_2 y}$ . . . . . (20)$_V$

Type VI. $\mathbf{G} \equiv \mathbf{H} = 1$.

Here $\frac{\partial \log \varphi}{\partial x} = p_0 + p_1 x + p_2 y$, $\log \varphi = p_0 x + \frac{1}{2} p_1 x^2 + p_2 xy + \mathrm{f}(y)$,

$$\frac{\partial \log \varphi}{\partial y} = p_2 x + \mathrm{f}'(y) \equiv q_0 + q_1 x + q_2 y.$$

Consequently

$$q_1 = p_2,\ \mathrm{f}'(y) = q_0 + q_2 y,\ \mathrm{f}(y) = q_0 y + \tfrac{1}{2} q_2 y^2 + \log K_0,$$

and

$$\log \varphi = \log K_0 + p_0 x + q_0 y + \tfrac{1}{2} p_1 x^2 + p_2 xy + \tfrac{1}{2} q_2 y^2 = \\ = \log K_0' - (\psi_{00} + 2\psi_{01} x + 2\psi_{02} y + \psi_{11} x^2 + 2\psi_{12} xy + \psi_{22} y^2),$$

whence

Type VI. $\mathbf{G} \equiv \mathbf{H} = 1$:

$\varphi = K_0' e^{-\Psi}$, where $\Psi \equiv \psi_{00} + 2\psi_{01} x + 2\psi_{02} y + \psi_{11} x^2 + 2\psi_{12} xy + \psi_{22} y^2$. (20)$_{VI}$

Summary of the different types:

I $\mathbf{G}$ and $\mathbf{H}$ have no common factors: $\varphi = \varphi_1(x)\, \varphi_2(y)$, where φ_1 and φ_2 are *Pearsonian* types.

II a) $\mathbf{G} = \mathbf{A}_1 \mathbf{C}$, $\mathbf{H} = \mathbf{B}_2 \mathbf{C}$, $\varphi = K_0 \mathbf{A}_1^{\mu_1} \mathbf{B}_2^{\mu_2} \mathbf{C}^{\mu_3}$.

II b) $\mathbf{G} = \mathbf{A}_1^2$, $\mathbf{H} = \mathbf{A}_1 \mathbf{B}_2$, $\varphi = K_0' \mathbf{A}_1^{\mu_1} \mathbf{B}_2^{\mu_2} e^{-\frac{\mathrm{D}}{\mathbf{A}_1}}$,
resp. $\mathbf{G} = \mathbf{A}_1 \mathbf{B}_2$, $\mathbf{H} = \mathbf{B}_2^2$, $\varphi = K_0' \mathbf{A}_1^{\mu_1} \mathbf{B}_2^{\mu_2} e^{-\frac{\mathrm{D}}{\mathbf{B}_2}}$.

III a) $\mathbf{G} \equiv \mathbf{H}$ indecomposable, $\varphi = K_0 \mathbf{G}^{\mu}$.

III b α) $\mathbf{G} \equiv \mathbf{H} = \mathbf{AC}$, $\mathbf{A}$ and $\mathbf{C}$ real, $\varphi = K_0 \mathbf{A}^{\mu_1} \mathbf{C}^{\mu_2}$.

III b β) $\mathbf{G} \equiv \mathbf{H} = \mathbf{AC}$, $\mathbf{A}$ and $\mathbf{C}$ complex, $\varphi = K_0' \mathbf{G}^{\mu} e^{\lambda \arctan \frac{g_{10} + g_{11} x + g_{12} y}{\left(y - \frac{\lambda_{00}}{\lambda_{00}}\right) \lambda_{00}}}$.

III c) $\mathbf{G} \equiv \mathbf{H} = \mathbf{A}^2$, $\varphi = K_0 \mathbf{A}^{\mu} e^{-\frac{\mathrm{D}}{\mathbf{A}}}$.

IV a) $\mathbf{G} = \mathbf{A}_1 \mathbf{C}$, $\mathbf{H} = \mathbf{C}$, $\varphi = K_0\, e^{-\lambda_2 y}\, \mathbf{A}_1^{\mu_1} \mathbf{C}^{\mu_2}$,
resp. $\mathbf{G} = \mathbf{C}$, $\mathbf{H} = \mathbf{B}_2 \mathbf{C}$, $\varphi = K_0\, e^{-\lambda_1 x}\, \mathbf{B}_2^{\mu_2} \mathbf{C}^{\mu_3}$.

IV b) $\mathbf{G} = \mathbf{A}_1^2$, $\mathbf{H} = \mathbf{A}_1$, $\varphi = K_0' \mathbf{A}_1^{\mu_1} e^{-\lambda_2 y - \frac{\mathrm{D}}{\mathbf{A}_1}}$,
resp. $\mathbf{G} = \mathbf{B}_2$, $\mathbf{H} = \mathbf{B}_2^2$, $\varphi = K_0' \mathbf{B}_2^{\mu_2} e^{-\lambda_1 x - \frac{\mathrm{D}}{\mathbf{B}_2}}$.

V $\mathbf{G} \equiv \mathbf{H} = \mathbf{C}$, $\varphi = K_0\, e^{-\lambda_1 x - \lambda_2 y}\, \mathbf{C}^{\mu}$.

VI $\mathbf{G} \equiv \mathbf{H} = 1$, $\varphi = K_0' e^{-\Psi}$,
$\Psi \equiv \psi_{00} + 2\psi_{01} x + 2\psi_{02} y + \psi_{11} x^2 + 2\psi_{12} xy + \psi_{22} y^2$.

As we shall see in the next section, only part of the functions here found have natural boundaries and so can be considered as probability functions in a proper sense.

§ 3. Standard forms for the probability density functions.

By a suitable choice of zero-point and scale we can reduce the general expressions for φ in § 2 to simple standard forms with a minimal number of parameters.

As to type I: $\varphi = \varphi_1(x) \cdot \varphi_2(y)$, we may refer to the standardization of PEARSON's types for one variable.

Type II a). $\mathbf{G} = \mathbf{A}_1\mathbf{C}$, $\mathbf{H} = \mathbf{B}_2\mathbf{C}$, $\varphi = K_0\mathbf{A}_1^{\mu_1}\mathbf{B}_2^{\mu_2}\mathbf{C}^{\mu_3}$.

Putting $\mathbf{A}_1 \equiv a_0 + a_1x = k_1X$, $\mathbf{B}_2 \equiv b_0 + b_2y = k_2Y$ $(k_1 > 0, k_2 > 0)$, we introduce new co-ordinates X and Y, acting only in the first quadrant.

With the abbreviations

$$l_0 = c_0 - \frac{c_1a_0}{a_1} - \frac{c_2b_0}{b_2}, \quad l_1 = \frac{c_1}{a_1}k_1, \quad l_2 = \frac{c_2}{a_2}k_2 \quad . \quad . \quad . \quad (22)$$

the form $\mathbf{C}$ passes into

$$\mathbf{L}(X, Y) \equiv l_0 + l_1X + l_2Y \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

So we get

$$\varphi = K_0' X^{\mu_1} Y^{\mu_2} (l_0 + l_1X + l_2Y)^{\mu_3}, \quad . \quad . \quad . \quad . \quad . \quad (24)$$

or with

$$\alpha_1 = \mu_1 + 1, \quad \alpha_2 = \mu_2 + 1, \quad \alpha_3 = \mu_3 + 1, \quad . \quad . \quad . \quad . \quad (25)$$

$$\varphi = K_0' X^{\alpha_1 - 1} Y^{\alpha_2 - 1} (l_0 + l_1X + l_2Y)^{\alpha_3 - 1} \quad . \quad . \quad . \quad (24\text{ bis})$$

For $\mathbf{R}$ and $\mathbf{S}$ we now find

$$\mathbf{R} = \alpha_1 l_0 + (\alpha_1 + \alpha_3) l_1 X + \alpha_1 l_2 Y, \quad \mathbf{S} = \alpha_2 l_0 + \alpha_2 l_1 X + (\alpha_2 + \alpha_3) l_2 Y. \quad (26)$$

Also the form $\mathbf{L}(X, Y)$ must be positive.

Provided that $l_0 \neq 0$, $l_1 \neq 0$, $l_2 \neq 0$, the lines $X = 0$, $Y = 0$ and $\mathbf{L} = 0$ form with the line I at infinity a complete fourside, of which the sides can act as bounding lines of the probability domain.

We require that on the boundary lines $\mathbf{G}\varphi$ and $\mathbf{H}\varphi$ shall be zero.

The boundary triangle can consist

α) of $X=0$, $Y=0$ and $\mathbf{L}=0$, provided that $\alpha_1>0$, $\alpha_2>0$, $\alpha_3>0$;

β) „ $X=0$, $Y=0$ „ I , „ „ $\alpha_1>0$, $\alpha_2>0$, $\alpha_1+\alpha_2+\alpha_3<0$;

γ) „ $X=0$, $\mathbf{L}=0$ „ I , „ „ $\alpha_1>0$, $\alpha_3>0$, $\alpha_1+\alpha_2+\alpha_3<0$;

δ) „ $Y=0$, $\mathbf{L}=0$ „ I , „ „ $\alpha_2>0$, $\alpha_3>0$, $\alpha_1+\alpha_2+\alpha_3<0$.

Only the 4 triangles formed by the fourside can constitute the contour of the probability domain. These triangles must lie in the first quadrant.

We have moreover

$$\hat{X} = \frac{-\alpha_1}{\alpha_1 + \alpha_2 + \alpha_3} \cdot \frac{l_0}{l_1}, \quad \hat{Y} = \frac{-\alpha_2}{\alpha_1 + \alpha_2 + \alpha_3} \cdot \frac{l_0}{l_2} \quad . \quad . \quad . \quad (27)$$

and

$$\mathbf{L}(\hat{X}, \hat{Y}) = \frac{\alpha_3}{\alpha_1 + \alpha_2 + \alpha_3} \cdot l_0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

We choose k_1 and k_2 in such a way, that l_0, l_1 and l_2 have the same absolute value, by which all the coefficients of $\mathbf{L}$ can be made ± 1.

Thus we obtain the following cases:

α) Contour $X=0$, $Y=0$, $\mathbf{L}=0$; $a_1>0$, $a_2>0$, $a_3>0$; hence (see (27) and (28))

$$l_0=+1,\ l_1=-1,\ l_2=-1,$$

and

$$\mathbf{L}_\alpha=1-X-Y \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)_\alpha$$

β) Contour $X=0$, $Y=0, I$; $a_1>0$, $a_2>0$, $a_1+a_2+a_3<0$; hence

$$l_0=+1,\ l_1=+1,\ l_2=+1$$

and

$$\mathbf{L}_\beta=1+X+Y \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)_\beta$$

γ) Contour $X=0$, $\mathbf{L}=0, I$; $a_1>0$, $a_3>0$, $a_1+a_2+a_3<0$; hence

$$l_0=-1,\ l_1=-1,\ l_2=+1$$

and

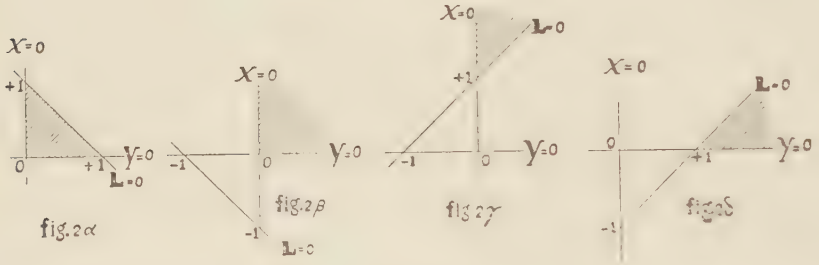
$$\mathbf{L}_\gamma=-1-X+Y. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)_\gamma$$

δ) Contour $Y=0$, $\mathbf{L}=0, I$; $a_2>0$, $a_3>0$, $a_1+a_2+a_3<0$; hence

$$l_0=-1,\ l_1=+1,\ l_2=-1$$

and

$$\mathbf{L}_\delta=-1+X-Y. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)^\delta$$



The cases $l_0=0$, $l_1=0$ and $l_2=0$ lead to functions wanting artificial boundaries.

Consequently

$$\left.\begin{aligned}
&\alpha)\ \varphi=K_0'\ X^{\alpha_1-1}\ Y^{\alpha_2-1}(1-X-Y)^{\alpha_3-1},\ a_1>0,\ a_2>0,\ a_3>0,\\
&\beta)\ \varphi=K_0'\ X^{\alpha_1-1}\ Y^{\alpha_2-1}(1+X+Y)^{\alpha_3-1},\ a_1>0,\ a_2>0,\ a_3<-a_1-a_2,\\
&\gamma)\ \varphi=K_0''\ X^{\alpha_1-1}\ Y^{\alpha_2-1}(-1-X+Y)^{\alpha_3-1},\ a_1>0,\ a_3>0,\ a_2<-a_1-a_3,\\
&\delta)\ \varphi=K_0''\ X^{\alpha_1-1}\ Y^{\alpha_2-1}(-1+X-Y)^{\alpha_3-1},\ a_2>0,\ a_3>0,\ a_1<-a_2-a_3.
\end{aligned}\right\} \quad (30)_{\mathrm{IIa}}$$

Treating these cases separately, we get, putting

$$a_1+a_2+a_3=\beta,$$

α)

$$\hat{X}=\frac{a_1}{\beta},\quad \hat{Y}=\frac{a_2}{\beta}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)_{\mathrm{IIa}\,\alpha}$$

$$K_0' = \frac{\Gamma(\beta)}{\Gamma(\alpha_1)\,\Gamma(\alpha_2)\,\Gamma(\alpha_3)};\ \mathrm{R} = \alpha_1 - (\alpha_1+\alpha_3)X - \alpha_1 Y,\ \mathrm{S} = \alpha_2 - \alpha_2 X - (\alpha_2+\alpha_3)Y;$$

$$\hat{\mathrm{G}} = \frac{\alpha_1\,\alpha_3}{\beta\,(\beta+1)},\ \hat{\mathrm{H}} = \frac{\alpha_2\,\alpha_3}{\beta\,(\beta+1)};\ \delta \equiv r_1\,s_2 - r_2\,s_1 = \alpha_3\,\beta.$$

Eqq. (10′) and (10″) now give

$$\left.\begin{aligned} m^{2,0} = \frac{-s_2\hat{\mathrm{G}}}{\delta} = \frac{\alpha_1(\alpha_2+\alpha_3)}{\beta^2(\beta+1)},\ m^{1,1} = \frac{+r_2\hat{\mathrm{H}}}{\delta} = \frac{+s_1\hat{\mathrm{G}}}{\delta} = \frac{-\alpha_1\,\alpha_2}{\beta^2(\beta+1)},\\ m^{0,2} = \frac{-s_2\hat{\mathrm{H}}}{\delta} = \frac{\alpha_2(\alpha_1+\alpha_3)}{\beta^2(\beta+1)};\end{aligned}\right\} (32)_{\mathrm{IIa}\,\alpha}$$

$$\gamma = -\sqrt{\frac{\alpha_1\,\alpha_2}{(\alpha_1+\alpha_3)(\alpha_2+\alpha_3)}},\quad 1-\gamma^2 = \frac{\alpha_3(\alpha_1+\alpha_2+\alpha_3)}{(\alpha_1+\alpha_3)(\alpha_2+\alpha_3)}.\quad (32')_{\mathrm{IIa}\,\alpha}$$

The probability distribution here considered is that of the estimates of the *a priori* probabilities (round the *a posteriori* probabilities $f_1 = \frac{\alpha_1}{\beta}$, $f_2 = \frac{\alpha_2}{\beta}$, $f_3 = 1 - f_1 - f_2 = \frac{\alpha_3}{\beta}$ as most probable values) with the trinomial distribution

$$dW(p_1, p_2) = C_0\,p_1^{\alpha_1}\,p_2^{\alpha_2}\,p_3^{\alpha_3} \qquad (p_3 = 1-p_1-p_2).$$

It is the extension of PEARSON's type I for both variables.

β)
$$\hat{X} = \frac{\alpha_1}{-\beta},\quad \hat{Y} = \frac{\alpha_2}{-\beta};\ \ldots\ldots\ (31)_{\mathrm{IIa}\,\beta}$$

$$K_0' = \frac{\Gamma(-\alpha_3+1)}{\Gamma(\alpha_1)\,\Gamma(\alpha_2)\,\Gamma(-\beta+1)};\ \mathrm{R} = \alpha_1 + (\alpha_1+\alpha_3)X + \alpha_1 Y,\ \mathrm{S} = \alpha_2 + \alpha_2 X +$$

$$+ (\alpha_2+\alpha_3)\,Y;\ \hat{\mathrm{G}} = \frac{\alpha_1\cdot -\alpha_3}{(-\beta)(-\beta-1)},\ \hat{\mathrm{H}} = \frac{\alpha_2\cdot -\alpha_3}{(-\beta)(-\beta-1)};\ \delta = (-\alpha_3)(-\beta);$$

$$m^{2,0} = \frac{-(\alpha_2+\alpha_3)\cdot\alpha_1}{(-\beta)^2(-\beta-1)},\ m^{1,1} = \frac{+\alpha_1\,\alpha_2}{(-\beta)^2(-\beta-1)},\ m^{0,2} = \frac{-(\alpha_1+\alpha_3)\cdot\alpha_2}{(-\beta)^2(-\beta-1)};\quad (32)_{\mathrm{IIa}\,\beta}$$

$$\gamma = +\sqrt{\frac{\alpha_1\,\alpha_2}{(-\alpha_1-\alpha_3)(-\alpha_2-\alpha_3)}},\quad 1-\gamma^2 = \frac{(-\alpha_3)(-\beta)}{(-\alpha_1-\alpha_3)(-\alpha_2-\alpha_3)}.\quad (32')_{\mathrm{IIa}\,\beta}$$

γ)
$$\hat{X} = \frac{\alpha_1}{-\beta},\quad \hat{Y} = \frac{-\alpha_2}{-\beta};\ \ldots\ldots\ (31)_{\mathrm{IIa}\,\gamma}$$

$$K_0'' = \frac{\Gamma(-\alpha_2+1)}{\Gamma(\alpha_1)\,\Gamma(\alpha_3)\,\Gamma(-\beta+1)};\ \mathrm{R} = -\alpha_1 - (\alpha_1+\alpha_3)X + \alpha_1 Y,\ \mathrm{S} = -\alpha_2 - \alpha_2 X +$$

$$+ (\alpha_2+\alpha_3)\,Y;\ \hat{\mathrm{G}} = \frac{\alpha_1\,\alpha_3}{(-\beta)(-\beta-1)},\ \hat{\mathrm{H}} = \frac{-\alpha_2\,\alpha_3}{(-\beta)(-\beta-1)};\ \delta = \alpha_3(-\beta);$$

$$m^{2,0}=\frac{-(\alpha_2+\alpha_3)\,\alpha_1}{(-\beta)^2(-\beta-1)},\quad m^{1,1}=\frac{\alpha_1\cdot(-\alpha_2)}{(-\beta)^2(-\beta-1)},\quad m^{0,2}=\frac{-\alpha_2\cdot(\alpha_1+\alpha_3)}{(-\beta)^2(-\beta-1)};\quad (32)_{\text{IIa}\gamma}$$

$$\gamma=+\sqrt{\frac{\alpha_1(-\beta+\alpha_1+\alpha_3)}{(\alpha_1+\alpha_3)(-\beta+\alpha_1)}},\quad 1-\gamma^2=\frac{\alpha_3(-\beta)}{(\alpha_1+\alpha_3)(-\beta+\alpha_1)}.\quad (32')_{\text{IIa}\gamma}$$

δ)
$$\hat{X}=\frac{-\alpha_1}{-\beta},\quad \hat{Y}=\frac{\alpha_2}{-\beta};\quad .\ .\ .\ .\ .\ .\ .\quad (31)_{\text{IIa}\delta}$$

$$K_0''=\frac{\Gamma(-\alpha_1+1)}{\Gamma(\alpha_2)\,\Gamma(\alpha_3)\,\Gamma(-\beta+1)};\quad \mathbf{R}=-\alpha_1+(\alpha_1+\alpha_3)X-\alpha_1 Y,\quad \mathrm{S}=-\alpha_2+\alpha_2 X-$$

$$-(\alpha_2+\alpha_3)\,Y;\quad \hat{\mathbf{G}}=\frac{-\alpha_1\cdot\alpha_3}{(-\beta)(-\beta-1)},\quad \hat{\mathbf{H}}=\frac{\alpha_2\,\alpha_3}{(-\beta)(-\beta-1)};\quad \delta=\alpha_3(-\beta);$$

$$m^{2,0}=\frac{-\alpha_1\cdot(\alpha_2+\alpha_3)}{(-\beta)^2(-\beta-1)},\quad m^{1,1}=\frac{-\alpha_1\cdot\alpha_2}{(-\beta)^2(-\beta-1)},\quad m^{0,2}=\frac{-(\alpha_1+\alpha_3)\cdot\alpha_2}{(-\beta)^2(-\beta-1)};\quad (32)_{\text{IIa}\delta}$$

$$\gamma=+\sqrt{\frac{\alpha_2(-\beta+\alpha_2+\alpha_3)}{(\alpha_2+\alpha_3)(-\beta+\alpha_2)}},\quad 1-\gamma^2=\frac{\alpha_3(-\beta)}{(\alpha_2+\alpha_3)(-\beta+\alpha_2)}.\quad (32')_{\text{IIa}\delta}$$

Mathematics. — *Inequalities concerning polynomials in the complex domain.* By N. G. DE BRUIJN. (Communicated by Prof. W. VAN DER WOUDE.)

(Communicated at the meeting of November 29, 1947.)

In this paper inequality theorems for polynomials will be obtained by means of one and the same underlying method which uses theorems on the location of the roots of polynomials.

The method can be illustrated by the following proof for S. BERNSTEINs theorem [1]: "If $P(z)$ and $Q(z)$ are polynomials satisfying $|P(z)| \leq |Q(z)|$, $Q(z) \neq 0$ for any z in the upper half-plane or on the real axis, then we have $|P'(z)| \leq |Q'(z)|$ for those values of z". Proof: If λ is a complex number, $|\lambda| > 1$, then all the roots of $P(z) - \lambda Q(z)$ lie in the lower half-plane. Now, by the well-known GAUSS–LUCAS theorem, it follows that $P'(z) - \lambda Q'(z)$ has its roots in the same domain and consequently $P'(z) - \lambda Q'(z) \neq 0$ for z in the closed upper half plane. Since this is true for any λ whose modulus exceeds unity, the assertion follows.

The simple idea on which this proof is based yields some surprising results if we use some other theorems on the location of roots. In section 1 of this paper, we use the general form of the GAUSS–LUCAS theorem. In section 2 a theorem of SZEGÖ is shown to lead to a result which includes a theorem of SCHAAKE and VAN DER CORPUT and which leads to a simple proof of a conjecture of P. ERDÖS, recently proved by P. D. LAX. Section 3 is based on GRACE's Apolarity Theorem. In section 4, which stands apart from the other sections more or less, we consider an inequality of ZYGMUND for polynomials, in the special case of functions which have no roots inside the unit circle.

1. We first prove a direct generalisation of the BERNSTEIN theorem mentionned in the introduction.

Theorem 1. *Let R be a convex region in the z-plane and let B be its boundary* [2]*. Let $P(z)$ and $Q(z)$ be polynomials; suppose that the roots of $Q(z)$ belong to $R + B$, and that the degree of P does not exceed that of Q. Now if $|P(z)| \leq |Q(z)|$ for z on B, then we have $|P'(z)| \leq |Q'(z)|$ for z on B.*

Proof. Let D denote the complement of $R + B$. Since $Q(z) \neq 0$ for $z \in D$, the inequality $|P| \leq |Q|$ for $z \in B$ implies that $|P| \leq |Q|$ for $z \in B + D$. Consequently, if $|\lambda| > 1$, the roots of $P(z) - \lambda Q(z)$ belong to R. Now, by the GAUSS–LUCAS theorem, the roots of $P'(z) - \lambda Q'(z)$ also belong to R. From this the assertion follows.

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[1] BERNSTEIN [1] p. 56. Bracketed numbers refer to the bibliography at the end.

[2] B may contain the point $z = \infty$.

The above result was obtained by S. BERNSTEIN for the case that B is the unit circle [3]. His proof does not depend on a direct application of the GAUSS–LUCAS theorem and does not admit the generalization obtained here.

BERNSTEIN's result is a generalization of the well-known theorem: "If $|P(z)| \leq 1$ for $|z| \leq 1$, then $|P'(z)| \leq n$ for $|z| \leq 1$". This is obtained by specializing $Q(z) = z^n$. Analogous results may be obtained for general convex domains, in virtue of Theorem 1.

Without any difficulty we can prove the following generalisation of Theorem 1:

Theorem 2. *Let R be a convex region in the z-plane, B its boundary, and S a simply connected region in the w-plane. Let $P(z)$ and $Q(z)$ be polynomials, the degree of P not exceeding that of Q, and suppose that the roots of $Q(z)$ belong to $R + B$. Now if $w = P(z)/Q(z) \in S$ for any $z \in B$, then we have $P'(z)/Q'(z) \in S$ for those values of z.*

2. By *circular domain* we denote a domain in the z-plane whose image on the z-sphere is either a closed region or an open region bounded by a circle. For instance, the point sets $|z| \geq 1$, $|z| > 1$, $|z| < 1$, $Re\, z \geq 0$ are circular domains.

We shall use the following theorem of G. SZEGÖ [4]:

Theorem 3. *If the polynomial $P(z)$ of degree n has no roots in the circular domain C [5], and if $\xi \in C$, then we have*

$$(\xi - z)\, P'(z) + n\, P(z) \neq 0 \text{ for } z \in C. \quad . \quad . \quad . \quad . \quad . \quad (1)$$

We directly infer

Theorem 4. *Let C be a circular domain in the z-plane, and S an arbitrary point set in the w-plane. If the polynomial $P(z)$ of degree n satisfies $P(z) = w \in S$ for any $z \in C$, then we have, for any $z \in C$ and any $\xi \in C$*

$$\frac{\xi}{n} P'(z) + P(z) - \frac{z\, P'(z)}{n} \in S. \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Proof. If the number λ does not belong to S, we have $P(z) \neq \lambda$ for $z \in C$. Applying theorem 3 to the polynomial $P(z) - \lambda$ we infer that $(\xi - z)P'(z) + n\,P(z) \neq n\lambda$ for $z \in C$, $\xi \in C$ and any λ which does not belong to S. This proves (2).

We notice that a special case of theorem 4 was proved by SCHAAKE and VAN DER CORPUT [6], who assumed that C is the unit circle (an unessential restriction) but also that S is a convex domain. A number of old and new

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[3] BERNSTEIN [2].

[4] SZEGÖ [1], p. 33.

[5] We adopt the convention that $z = \infty$ is a root of $P(z)$ if the coefficient of z^n vanishes.

[6] SCHAAKE and VAN DER CORPUT [1], p. 350, Satz 20.

results concerning polynomials and trigonometric polynomials were derived from this special case by these authors. We now give an application where S is not convex.

Theorem 5 (ERDÖS–LAX [7]). *If the polynomial $P(z)$ of degree n satisfies $|P(z)| \leq 1$ for $|z| \leq 1$ and if $P(z)$ has no roots in $|z| \leq 1$, then $|P'(z)| \leq \frac{1}{2} n$ for $|z| \leq 1$* [8]).

Proof. Take for C the region $|z| < 1$ and for S the set $0 < |w| < 1$. Now (2) expresses, if $|z| < 1$, that the interior of a circle with radius $P'(z)/n$ completely belongs to S. Since the maximum radius of such a circle is $\frac{1}{2}$, the result follows.

It is however neither difficult to prove this result by the SCHAAKE and VAN DER CORPUT theorem, taking for S the region $|w| < 1$, inferring that $|P'(z)/n| + |P(z) - zP'(z)/n| \leq 1$ for $|z| \leq 1$ and noticing that from $P(z) \neq 0$ $(|z| \leq 1)$ it follows that $|P'(z)/n| \geq |P(z) - zP'(z)/n|$ [9]).

3. We shall now expose some consequences of J. H. GRACE's theorem on the roots of polynomials [10]).

Theorem 6 (Grace's Apolarity Theorem). *If $n \geq 1$, and*

$$P(z) = a_0 + \binom{n}{1} a_1 z + \binom{n}{2} a_2 z^2 + \ldots + \binom{n}{n} a_n z^n, \quad . \quad . \quad (3)$$

$$Q(z) = b_0 + \binom{n}{1} b_1 z + \binom{n}{2} b_2 z^2 + \ldots + \binom{n}{n} b_n z^n, \quad . \quad . \quad (4)$$

and if $P(z)$ has no roots in a circular domain C which contains all the roots of $Q(z)$, then we have

$$\{P, Q\} = a_0 b_n - \binom{n}{1} a_1 b_{n-1} + \binom{n}{2} a_2 b_{n-2} + \ldots + (-1)^n \binom{n}{n} a_0 b_n \neq 0 \quad (5)$$

We can put this in a different form by taking $Q(z) = (z - z_1) \ldots (z - z_n)$.

Theorem 6a. *Let $f(z_1, z_2, \ldots, z_n)$ be a linear combination of the elementary symmetric functions of $z_1, \ldots, z_n$:*

$$f(z_1, \ldots, z_n) = a_0 + a_1 \Sigma z_1 + a_2 \Sigma z_1 z_2 + \ldots + a_\mu \Sigma z_1 \ldots z_\mu + a_n z_1 \ldots z_n \quad (6)$$

so that, if $P(z)$ is given by (3):

$$f(z, z, \ldots, z) = P(z).$$

Now if $f(z, \ldots, z)$ (considered as a polynomial of degree n [11])*) has no roots in the circular domain C, then for $z_1 \in C, \ldots, z_n \in C$ we have $f(z_1, \ldots, z_n) \neq 0$.*

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[7]) LAX [1].

[8]) Of course, several alterations of this theorem are possible by replacing signs $\leq$ by $<$. The same remark applies to Theorems 8, 9 and 10.

[9]) In LAX [1], p. 511, a similar argument is used.

[10]) GRACE [1]; SZEGÖ [1]; PÓLYA–SZEGÖ [2], Abschnitt V, Aufg. 145.

[11]) Hence $a_n = 0$ would imply that $z = \infty$ is a root.

From this we deduce, in the same way as Theorem 4 was derived from Theorem 3:

Theorem 7. *Let C be a circular domain in the z-plane and S an arbitrary point-set in the w-plane. Suppose that $f(z_1, \ldots, z_n)$ is of the type (6) and satisfies $f(z, \ldots, z) = w \in S$ for any $z \in C$. Then we have, for $z_1 \in C, \ldots, z_n \in C$:*

$$f(z_1, \ldots, z_n) \in S.$$

Since $f(z, \ldots, z, \xi) = P(z) + (\xi - z)\, P'(z)/n$, Theorem 4 is a special case of this one.

SCHAAKE and VAN DER CORPUT's paper again contains the result of Theorem 7 for the special case that S is convex. Their proof is based on Theorem 11 below.

In the following theorem, a direct consequence of Theorem 6, it is convenient to restrict ourselves to the case that C is the unit circle. Application of Theorem 6 to $P(z) - a$ and $z^n Q(-\xi/z)$ leads to

Theorem 8. *Let S be a point-set, let $P(z)$ and $Q(z)$ be given by (3) and (4), and suppose that $Q(z) \neq 0$ for $|z| < 1$, $b_0 = 1$, and*

$$P(z) \in S \text{ for } |z| \leqq 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Now putting

$$PQ(z) = a_0 b_0 + \binom{n}{1} a_1 b_1 z + \ldots + \binom{n}{n} a_n b_n z^n, \quad . \quad . \quad (8)$$

we have

$$PQ(z) \in S \text{ for } |z| \leqq 1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

Conversely, if the numbers $1 = b_0, b_1, \ldots, b_n$ are such that (9) holds for any S and for any polynomial $P(z)$ satisfying (7), we have $Q(z) \neq 0$ for $|z| < 1$. This immediately follows from a theorem of SZEGÖ [12] which covers the case that S is the set $w \neq 0$.

This remark leads to the following consideration on FÉJER sums: The FÉJER sums of the polynomial $P(z)$, viz.

$$s_k(z) = \frac{1}{k+1}\left\{(k+1)a_0 + k\binom{n}{1} a_1 z + (k-1)\binom{n}{2} a_2 z^2 + \ldots + \binom{n}{k} a_k z^k\right\},$$

have the following well-known property: "If S is a convex domain and if $P(z) \in S$ for $|z| \leq 1$, then $s_k(z) \in S$ for $|z| \leq 1$". This need not hold for general point-sets S. That depends on the location of the roots of

$$(k+1)\, Q(z) = (k+1) + k\binom{n}{1} z + (k-1)\binom{n}{2} z^2 + \ldots + \binom{n}{k} z^k,$$

which may have roots inside the unit circle (e.g. if $n = 3$, $k = 1$).

The convexity of S may however be dropped if $k \geq n-1$, for then we have $(k+1)\, Q(z) = \{(k-n+1)z + k + 1\}\, (z+1)^{n-1}$. The case $k = n-1$ also follows from Theorem 4 since $s_{k-1}(z) = P(z) - zP'(z)/n$.

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[12] SZEGÖ [1], p. 50. It is sufficient to consider $P(z) = (z - \zeta)^n$, $|\zeta| > 1$.

As an application of Theorem 8 we give

Theorem 9. *Let S be a point-set, $P(z)$ a polynomial of degree n, and suppose that $P(z) \in S$ for $|z| \leq 1$. Then for $p > 1$, $|z| \leq 1$, $|\lambda| \leq 1$ we have*

$$\left[p^n P\left(\frac{z}{p}\right) - p^{-n} P(pz) - \lambda \left\{ P(pz) - P\left(\frac{z}{p}\right) \right\} \right] \Big/ (p^n - p^{-n}) \in S. \quad (10)$$

Proof. According to Theorem 8 it is sufficient to prove that for $p > 1$, $|\lambda| \leq 1$ the polynomial

$$(z+p)^n - (z+p^{-1})^n - \lambda \{ (zp+1)^n - (zp^{-1}+1)^n \} \quad . \quad . \quad (11)$$

has no roots in $|z| < 1$.

For $|z| \leq 1$, we have $|z+p| < |z+p^{-1}|$, hence $\varphi_1(z) = (z+p)^n - (z+p^{-1})^n \neq 0$ for $|z| \leq 1$. If we put $\varphi_2(z) = (zp+1)^n - (zp^{-1}+1)^n$, then we have $|\varphi_1(z)| = |\varphi_2(z)|$ for $|z| = 1$. It follows that $|\varphi_1(z)| \leq |\varphi_2(z)|$ for $|z| \leq 1$, consequently the polynomial (11), equalling $\varphi_1(z) - \lambda \varphi_2(z)$, has no roots in $|z| < 1$ if $|\lambda| \leq 1$.

The limit case $p \to 1$ leads back to theorem 4. Another special case of Theorem 9 was proved by SCHAEFFER and SZEGÖ ([1]); there S represents the region $|Re\, w| \leq 1$.

The following consequence [13]) of Theorem 8 is symmetric in P and Q.

Theorem 10. *If $P(z)$, $Q(z)$ and $PQ(z)$ are given by (3), (4) and (8), respectively, and if $|P(z)| \leq 1$, $|Q(z)| \leq 1$ for $|z| \leq 1$, then we have $|PQ(z)| \leq 1 - |b_0 - |a_0||$ for $|z| \leq 1$.*

Proof. Let λ satisfy $|\lambda| > 1$, then $Q(z) - \lambda \neq 0$ for $|z| < 1$. On applying Theorem 8 to $P(z)$ and $(Q(z) - \lambda)/(b_0 - \lambda)$ we obtain

$$|PQ(z) - \lambda a_0| \leq |b_0 - \lambda| \text{ for } |z| \leq 1. \quad . \quad . \quad . \quad . \quad (12)$$

An argument of continuity shows that this holds for $|\lambda| = 1$ also. We can choose a special λ_0 with modulus 1 such that $|b_0 - \lambda_0| = 1 - |b_0|$. It follows that

$$|PQ(z)| \leq |\lambda_0 a_0| + 1 - |b_0| = 1 - \{ |b_0| - |a_0| \}.$$

By interchanging the roles of P and Q the result follows.

GRACE's theorem also supplies a proof for the following theorem of SCHAAKE and VAN DER CORPUT [14]), which they showed to lead to Theorem 7 (for S convex).

Theorem 11. (SCHAAKE–VAN DER CORPUT). *If $f(z_1, \ldots, z_n)$ is of the type (6), and if we put*

$$\lambda_n(z_1, \ldots, z_n) = \frac{1}{n} \sum_{\mu=0}^{n-1} \binom{n}{\mu}^{-1} \Sigma z_1 z_2 \ldots z_\mu,$$

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[13]) Communicated by Mr. T. A. SPRINGER.

[14]) SCHAAKE and VAN DER CORPUT [1] Satz 17, p. 343 and Satz 18, p. 345.

then we have the identity

$$f(z_1, \ldots, z_n) = \sum_p \lambda_n \left(\frac{z_1}{p}, \ldots, \frac{z_n}{p} \right) f(p, p, \ldots, p), \quad . \quad . \quad (13)$$

where p runs through the n-th roots of $z_1 z_2 \ldots z_n$.

Furthermore

$$\sum_p \lambda_n \left(\frac{z_1}{p}, \ldots, \frac{z_n}{p} \right) = 1, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

and if $|z_1| = |z_2| = \ldots = |z_n| = 1$ we have

$$\lambda_n (z_1 p^{-1}, \ldots, z_n p^{-1}) \geqq 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

Proof. The relations (13) and (14) are easily verified; the difficulty lies in proving that $\lambda_n \geqq 0$ if all z_i have the modulus 1. Putting $z_i = p \zeta_i$, we have to establish that

$$\lambda_n (\zeta_1, \ldots, \zeta_n) \geqq 0 \text{ if } |\zeta_1| = \ldots = |\zeta_n| = 1, \zeta_1 \zeta_2 \ldots \zeta_n = 1.$$

Putting $b_\mu = \Sigma \zeta_1 \zeta_2 \ldots \zeta_\mu$ we find $\overline{b}_\mu = b_{n-\mu}$, consequently λ_n is real. It remains to be shown that λ_n cannot be negative. Taking $P(z) = z + z^2 + \ldots + z^{n-1} + \delta z^n$ and $Q(z) = z^m - b_1 z^{m-1} + \ldots = (z - \zeta_1) \ldots (z - \zeta_n)$, we obtain for the expression (5):

$$\{P, Q\} = \binom{n}{1}^{-1} b_1 + \binom{n}{2}^{-1} b_2 + \ldots + \binom{n}{n-1}^{-1} b_{n-1} + \delta b_n =$$
$$= n \lambda_n (\zeta_1, \ldots, \zeta_n) + \delta - 1.$$

Now if $\delta > 1$, $P(z)$ has no roots for $|z| \geqq 1$, according to a theorem of KAKEYA [15]), so that Theorem 6 yields $\{P, Q\} \neq 0$. It follows that λ_n cannot be negative.

4. In this concluding section we shall obtain an integral inequality related to an inequality of ZYGMUND (formule (18) below), generalizing the ERDÖS–LAX theorem. We deduce it from the following result which depends on SCHAAKE and VAN DER CORPUT's theorem (Theorem 11 above).

Theorem 12. *If $f(z_1, \ldots, z_n)$ is of the type (6), and if $\Phi(w)$ is a real and convex function of the complex variable w, i.e.*

$$\Phi(\alpha w_1 + \beta w_2) \leqq \alpha \Phi(w_1) + \beta \Phi(w_2) \text{ for } \alpha \geqq 0, \beta \geqq 0, \alpha + \beta = 1,$$

then we have, for $|z_1| \leq 1, \ldots, |z_n| \leq 1$

$$\int_0^{2\pi} \Phi \{ f(z_1 e^{i\theta}, \ldots, z_n e^{i\theta}) \} \, d\theta \leqq \int_0^{2\pi} \Phi \{ f(e^{i\theta}, \ldots, e^{i\theta}) \} \, d\theta. \quad . \quad (16)$$

Proof. Since $f(z_1 e^{i\theta}, \ldots, z_n e^{i\theta})$ is a linear function of z_1, the left hand side of (16) is a convex function of z_1. Consequently, its maximum for $|z_1| \leq 1$ is attained at the boundary $|z_1| = 1$. The same applies to $z_2, \ldots, z_n$, and hence it is sufficient to prove (16) for the case

$$|z_1| = \ldots = |z_n| = 1.$$

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[15]) Cf. PÓLYA–SZEGÖ [1], Abschn. III, Aufg. 22.

By theorem 11 we then have

$$f(z_1 e^{i\theta}, \ldots, z_n e^{i\theta}) = \sum_p \lambda_n (z_1 p^{-1}, \ldots, z_n p^{-1}) f(p e^{i\theta}, \ldots, p e^{i\theta}),$$

where p runs through the n-th roots of $z_1 \ldots z_n$, and where the λ_n satisfy (14) and (15). Since $\Phi(w)$ is convex we have

$$\Phi\{f(z_1 e^{i\theta}, \ldots, z_n e^{i\theta}\} \leqq \sum_p \lambda_n (z_1 p^{-1}, \ldots, z_n p^{-1}) \Phi\{f(pe^{i\theta}, \ldots, pe^{i\theta})\}.$$

On integrating, and using (14), we obtain (16).

Now let $P(z)$ be a polynomial of degree n and let $f(z_1, \ldots, z_n)$ be such that $f(z, \ldots, z) = P(z)$. Take $z_1 = z_2 \ldots = z_{n-1} = 1$, $z_n = e^{i\eta}$, where η is a real number, and $\Phi(w) = |w|^p$ $(p \geq 1)$. Since $f(z, \ldots, z, \xi) = P(z) + (\xi - z)P'(z)/n$, Theorem 12 gives

$$\int_0^{2\pi} |P(e^{i\vartheta}) - e^{i\vartheta} P'(e^{i\vartheta})/n + e^{i(\eta+\vartheta)} P'(e^{i\vartheta})/n|^p \, d\theta \leqq \int_0^{2\pi} |P(e^{i\theta})|^p \, d\theta.$$

Putting $P(e^{i\vartheta}) - e^{i\vartheta} P'(e^{i\vartheta})/n = A(\theta)$, $e^{i\vartheta} P'(e^{i\theta})/n = B(\theta)$, we obtain

$$\int_0^{2\pi} d\theta \int_0^{2\pi} |A(\theta) + B(\theta) e^{i\eta}|^p \, d\eta \leqq 2\pi \int_0^{2\pi} |P(e^{i\theta})|^p \, d\theta. \quad . \; . \; (17)$$

ZYGMUND's inequality [16])

$$\int_0^{2\pi} \left| \frac{P'(e^{i\theta})}{n} \right|^p d\theta \leqq \int_0^{2\pi} |P(e^{i\theta})|^p \, d\theta \qquad (p \geq 1) \quad . \; . \; . \; (18)$$

can be derived from (17), by the formula (valid for any real value of p)

$$\int_0^{2\pi} |a + b e^{i\eta}|^p \, d\eta \geqq 2\pi \text{ Max } \{|a|^p, |b|^p\}. \quad . \; . \; . \; . \; . \; (19)$$

Owing to the symmetry with respect to a and b it is sufficient to prove (19) for $a \geq b > 0$. Then it follows by

$$2\pi |\varphi(0)|^2 \leqq \int_0^{2\pi} |\varphi(e^{i\eta})|^2 \, d\eta, \text{ where } \varphi(z) = (a + bz)^{\frac{1}{2}p}.$$

Our present aim is to investigate how ZYGMUND's result can be refined if we suppose that $P(z)$ has no roots inside the unit circle. In that case we have, by Theorem 4, $A(\theta) + \xi B(\theta) \neq 0$ for $|\xi| < 1$, so that

$$|B(\theta)| \leqq |A(\theta)| \qquad (0 \leqq \theta \leqq 2\pi). \quad . \; . \; . \; . \; . \; . \; (20)$$

For $|a| \geq |b|$ we have

$$\int_0^{2\pi} |a + b e^{i\eta}| \, d\eta \geqq |b|^p \int_0^{2\pi} |1 + e^{i\eta}|^p \, d\eta. \qquad (p \geqq 0). \quad . \; . \; (21)$$

It is sufficient to prove this for $b = 1$, $a > 1$. In that case (21) follows from $|a + e^{i\eta}| \geq |1 + e^{i\eta}|$ (η real).

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[16]) ZYGMUND [1].

From (17), (20) and (21) we infer

$$\int_0^{2\pi} |B(\theta)|^p \, d\theta \cdot \int_0^{2\pi} |1+e^{i\eta}|^p \, d\theta \leqq 2\pi \int_0^{2\pi} |P(e^{i\theta})|^p \, d\theta,$$

and we obtain

Theorem 13. *If the polynomial $P(z)$ of degree n has no roots for $|z|<1$, then we have, for $p \geq 1$,*

$$\int_0^{2\pi} \left| \frac{P'(e^{i\theta})}{n} \right|^p d\theta \leqq C_p \int_0^{2\pi} |P(e^{i\theta})|^p \, d\theta$$

where $C_p = 2\pi / \int_0^{2\pi} |1+e^{i\eta}|^p \, d\eta = 2^{-p} \sqrt{\pi}\, \Gamma(\tfrac{1}{2}p+1) / \Gamma(\tfrac{1}{2}p+\tfrac{1}{2})$.

It is easily seen that the sign of equality holds if $P(z) = \alpha + \beta z^n$, $|\alpha| = |\beta|$. It can also be shown that the sign $<$ holds otherwise.

The case $p = 2$ was obtained by P. D. LAX (LAX [1]), whereas $p \to \infty$ leads to the ERDÖS–LAX theorem.

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Mathematics. — *Sur les intégrales multiples dans les corps évalués et algébriquement-fermés.* By F. LOONSTRA. (Communicated by Prof. J. G. VAN DER CORPUT.)

(Communicated at the meeting of September 27, 1947.)

Nous supposons que le corps T satisfasse aux conditions suivantes:

1. T est évalué non-archimédien;
2. T est algébriquement-fermé, c'est à dire que chaque polynome de l'anneau $T[x]$ se compose de facteurs linaires;
3. T est complet.

Les éléments a avec $|a| \leqq 1$ s'appellent des éléments „entiers".

Nous considérons un nombre fini quelconque de suites de polynomes

$$g_{11}(x), g_{12}(x), \ldots, g_{1n}(x), \ldots$$
$$g_{21}(x), g_{22}(x), \ldots, g_{2n}(x), \ldots$$
$$\ldots\ldots\ldots\ldots\ldots$$
$$g_{s1}(x), g_{s2}(x), \ldots, g_{sn}(x), \ldots,$$

où

$$g_{ki}(x) = x^{n_{ki}} + c_{ki,1}\, x^{n_{ki,1}} + c_{ki,2}\, x^{n_{ki,2}} + \ldots + c_{ki,\mu}\, x^{n_{ki,\mu}} + c_{ki,\mu+1}$$
$$(k = 1, 2, \ldots, s;\ n_{ki} > n_{ki,1} > \ldots > n_{ki,\mu}),$$

avec des exposants croissants et de sorte qu'aucun des polynomes considérés ne possède des racines multiples. En outre nous supposons que les coëfficients soient entiers et $|c_{ki,\mu+1}| = 1 (k = 1, 2, \ldots, s)$ [1]).

En vertu de 2. le polynome $g_{ki}(x)$ a n_{ki} racines

$$\alpha_{ki,1}, \alpha_{ki,2}, \ldots, \alpha_{ki,n_{ki}},$$

pour lesquelles on a

$$|\alpha_{ki,1}| = |\alpha_{ki,2}| = \ldots = |\alpha_{ki,n_{ki}}| = |c_{ki,\mu+1}|^{\frac{1}{n_{ki}}} = 1.$$

Nous supposons ensuite $|n_{ki}| = 1$ et

$$n_{ki} \to \infty,\ n_{ki,l} \to \infty\ (l = 1, 2, \ldots, \mu),\ (n_{ki} - n_{ki,1}) \to \infty \text{ pour } i \to \infty.$$

Alors on a

$$\left|\frac{n_{ki} - n_{ki,l}}{n_{ki}}\right| = \left|1 - \frac{n_{ki,l}}{n_{ki}}\right| \leqq \max\left(1, \left|\frac{n_{ki,l}}{n_{ki}}\right|\right) = \max(1, |n_{ki,l}|) = 1,$$

parce que les $n_{ki,l}$ sont des nombres naturels, c'est à dire $|n_{ki,l}| \leqq 1$ $(l = 1, 2, \ldots, \mu)$.

En vertu de la relation $|\alpha_{ki,l}| = 1$ pour chaque k à part les racines des polynomes $g_{ki}(x)$ se trouvent sur le „cercle d'unité".

---

[1]) Nous supprimerons dans la suite l'adjonction $k = 1, 2, \ldots, s$.

Avec les racines $\alpha_{ki,1}, \alpha_{ki,2}, \ldots, \alpha_{ki,n_{ki}}$ des polynomes g_{ki} nous considérons maintenant des suites d'éléments

$$\left.\begin{array}{l} \alpha_1 + \varrho_1 \alpha_{1i,1},\ \alpha_1 + \varrho_1 \alpha_{1i,2}, \ldots, \alpha_1 + \varrho_1 \alpha_{1i,n_{1i}} \\ \alpha_2 + \varrho_2 \alpha_{2i,1},\ \alpha_2 + \varrho_2 \alpha_{2i,2}, \ldots, \alpha_2 + \varrho_2 \alpha_{2i,n_{2i}} \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ \alpha_s + \varrho_s \alpha_{si,1},\ \alpha_s + \varrho_s \alpha_{si,2}, \ldots, \alpha_s + \varrho_s \alpha_{si,n_{si}} \end{array}\right\} \quad \ldots \quad (1)$$

avec α_k, ϱ_k sont des éléments constants de T. Nous appelons de telles suites des cercles discrets d'un centre α_k et d'un rayon $|\varrho_k|$; en effet, la „distance" du centre α_k aux éléments $\alpha_k + \varrho_k \alpha_{ki,l}$ est $|\varrho_k \alpha_{ki,l}| = |\varrho_k|$. Nous indiquerons les suites (1) par $\alpha_k + \varrho_k g_k$.

Maintenant nous définirons l'intégrale de l'ordre s d'une fonction $f(x_1, x_2, \ldots, x_s)$ des variables indépendants $x_1, x_2, \ldots, x_s$ le long des cercles $(\alpha_k; \varrho_k)$; nous supposons que $f(x_1, \ldots, x_s)$ soit défini pour tous les éléments de T, qui appartiennent aux cercles discrets $\alpha_k + \varrho_k g_k$.

Définition: Si la limite de l'expression

$$\frac{1}{n_{1i} \cdot n_{2i} \cdot \ldots \cdot n_{si}} \sum_{\substack{1 \leqq l_1 \leqq n_{1i} \\ \ldots\ldots\ldots \\ 1 \leqq l_s \leqq n_{si}}} f(\alpha_1 + \varrho_1 \alpha_{1i,l_1},\ \alpha_2 + \varrho_2 \alpha_{2i,l_2}, \ldots,\ \alpha_s + \varrho_s \alpha_{si,l_s}),\ i \to \infty$$

existe, appellerons cette limite l'intégrale de l'ordre s de $f(x_1, \ldots, x_s)$ le long des cercles discrets $\alpha_k + \varrho_k g_k$ et nous indiquerons cette intégrale par

$$\int_{\alpha_1 + \varrho_1 g_1}^{x_1} \int_{\alpha_2 + \varrho_2 g_2}^{x_2} \ldots \int_{\alpha_s + \varrho_s g_s}^{x_s} f(x_1, x_2, \ldots, x_s)$$

Si cette limite est indépendante des $g_{ki}(x)$, nous écrirons brièvement

$$\int_{\alpha_1, \varrho_1}^{x_1} \int_{\alpha_2, \varrho_2}^{x_2} \ldots \int_{\alpha_s, \varrho_s}^{x_s} f(x_1, x_2, \ldots, x_s)\ ^{2})$$

Théorème I: Pour un polynome quelconque $P(x_1, x_2, \ldots, x_s)$ de $T[x_1, \ldots, x_s]$ on a

$$\int_{\alpha_1, \varrho_1}^{x_1} \ldots \int_{\alpha_s, \varrho_s}^{x_s} P(x_1, x_2, \ldots, x_s) = P(\alpha_1, \alpha_2, \ldots, \alpha_s);$$

en outre on aura démontré que l'intégrale est indépendante des $g_{ki}(x)$. Nous démontrerons la validité du théorème pour

$$P(x_1, x_2, \ldots, x_s) = c\, x_1^{k_1} x_2^{k_2} \ldots x_s^{k_s},$$

---

2) Cette définition est une généralisation de celle de L. H. SCHNIRELMAN, Bulletin de l'Académie des Sciences de l'URSS, Série Math., 1938, p. 487.

aussi bien que pour la somme de deux polynomes

$$\int_{\alpha_1,\varrho_1}^{x_1}\cdots\int_{\alpha_s,\varrho_s}^{x_s} c\,x_1^{k_1}x_2^{k_2}\ldots x_s^{k_s} = \lim \frac{c}{n_{1i}\,n_{2i}\ldots n_{si}}$$

$$\{(\alpha_1+\varrho_1\alpha_{1i,1})^{k_1}(\alpha_2+\varrho_2\alpha_{2i,1})^{k_2}\ldots(\alpha_s+\varrho_s\alpha_{si,1})^{k_s}+$$

$$+\ldots+(\alpha_1+\varrho_1\alpha_{1i,n_{1i}})^{k_1}(\alpha_2+\varrho_2\alpha_{2i,n_{2i}})^{k_2}\ldots(\alpha_s+\varrho_s\alpha_{si,n_{si}})^{k_s}\}$$

$$= \lim \frac{c}{n_{1i}\,n_{2i}\ldots n_{si}}\{n_{1i}\,n_{2i}\ldots n_{si}\,\alpha_1^{k_1}\alpha_2^{k_2}\ldots\alpha_s^{k_s}+$$

$$+k_1\,n_{2i}\,n_{3i}\ldots n_{si}\,\varrho_1\,\alpha_1^{k_1-1}\alpha_2^{k_2}\ldots\alpha_s^{k_s}\sum_{l=1}^{n_{1i}}\alpha_{1i,l}+$$

$$+k_2\,n_{1i}\,n_{3i}\ldots n_{si}\,\varrho_2\,\alpha_1^{k_1}a_2^{k_2-1}a_3^{k_3}\ldots a_s^{k_s}\sum_{l=1}^{n_{2i}}\alpha_{2i,l}+$$

$$+\ldots+k_s\,n_{1i}\,n_{2i}\ldots n_{s-1,i}\,\varrho_s\,\alpha_1^{k_1}\alpha_2^{k_2}\ldots\alpha_{s-1}^{k_{s-1}}\alpha_s^{k_s-1}\sum_{l=1}^{n_{si}}\alpha_{si,l}$$

$$+\varrho_1^2\,n_{2i}\,n_{3i}\ldots n_{si}\,\frac{k_1(k_1-1)}{1\cdot 2}\,\alpha_1^{k_1-2}\alpha_2^{k_2}\ldots\alpha_s^{k_s}\sum_{l=1}^{n_{1i}}\alpha_{1i,l}^2+$$

$$+\ldots+n_{1i}\,n_{2i}\ldots n_{s-1,i}\,\alpha_1^{k_1}\alpha_2^{k_2}\ldots\alpha_{s-1}^{k_{s-1}}\varrho_s^{k_s}\sum_{l=1}^{n_{si}}\alpha_{si,l}^{k_s}\}.$$

Nous avons supposé que $n_{ki}\to\infty$ et $n_{ki}-n_{ki,1}\to\infty$ pour $i\to\infty$. Appliquons maintenant la propriété, que pour un polynome

$$\varphi(x)=(x-\alpha_1)(x-\alpha_2)\ldots(x-\alpha_n)=$$

$$=x^n-\sigma_1\,x^{n-1}+\sigma_2\,x^{n-2}-\ldots+(-1)^n\,\sigma_n$$

$$\sigma_1=\alpha_1+\alpha_2+\ldots+\alpha_n$$

$$\sigma_2=\alpha_1\,\alpha_2+\ldots+\alpha_{n-1}\,\alpha_n$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\alpha_n=\alpha_1\cdot\alpha_2\cdot\ldots\cdot\alpha_n$$

$$\alpha_1^2+\alpha_2^2+\ldots+\alpha_n^2=\sigma_1^2-2\sigma_2$$

et en général $\alpha_1^k+\alpha_2^k+\ldots+\alpha_n^k$ est un polynome en $\sigma_1,\sigma_2,\ldots,\sigma_k$ sans terme constant. Alors nous supposons que les n_{ki}, $n_{ki}-n_{ki,1}$ soient pour chaque $k=1,2,\ldots,s$ si grands que les coëfficients de

$$x^{n_{1i}-1},\,x^{n_{1i}-2},\ldots,x^{n_{1i}-k_1}\text{ de }g_{1i}$$

$$x^{n_{2i}-1},\,x^{n_{2i}-2},\ldots,x^{n_{2i}-k_2}\text{ de }g_{2i}$$

$$\ldots\ldots\ldots\ldots\ldots$$

$$x^{n_{si}-1},\,x^{n_{si}-2},\ldots,x^{n_{si}-k_s}\text{ de }g_{si}$$

seront zéro, alors de même

$$\sum_h\alpha_{1i,h}=0,\ \sum_h\alpha_{1i,h}^2=0,\ldots,\sum_h\alpha_{1i,h}^{k_1}=0$$

$$\sum_h\alpha_{2i,h}=0,\ \sum\alpha_{2i,h}^i=0,\ldots,\sum\alpha_{2i,h}^{k_2}=0$$

$$\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\sum_h\alpha_{si,h}=0,\ \sum_h\alpha_{si,h}^2=0,\ldots,\sum_h\alpha_{si,h}^{k_s}=0,$$

on a $$\int_{\alpha_1,\varrho_1}^{x_1}\dots\int_{\alpha_s,\varrho_s}^{x_s} c\,x_1^{k_1}x_2^{k_2}\dots x_s^{k_s}=c\,\alpha_1^{k_1}\alpha_2^{k_2}\dots\alpha_s^{k_s}.$$

Si on a

$$\int_{\alpha_1,\varrho_1}^{x_1}\dots\int_{\alpha_s,\varrho_s}^{x_s}P(x_1,x_2,\dots,x_s)=P(a_1,a_2,\dots,a_s)\text{ et}$$

$$\int_{\alpha_1,\varrho_1}^{x_1}\dots\int_{\alpha_s,\varrho_s}^{x_s}Q(x_1,x_2,\dots,x_s)=Q(a_1,a_2,\dots,a_s),$$

dans lequel $P(x_1,\dots,x_s)$ et $Q(x_1,\dots,x_s)$ sont des polynomes quelconques, il ne s'agit plus que de démontrer

$$\int_{\alpha_1,\varrho_1}^{x_1}\dots\int_{\alpha_s,\varrho_s}^{x_s}\{P(x_1,\dots,x_s)+Q(x_1,\dots,x_s)\}=P(a_1,\dots,a_s)+Q(a_1,\dots,a_s).$$

$$\int_{\alpha_1,\varrho_1}^{x_1}\dots\int_{\alpha_s,\varrho_s}^{x_s}\{P(x_1,\dots,x_s)+Q(x_1,\dots,x_s)\}=$$

$$=\lim\frac{1}{n_{1i}n_{2i}\dots n_{si}}\Sigma\{P(a_1+\varrho_1a_{1i,1},\dots)+Q(a_1+\varrho_1a_{1i,1},\dots)\}=$$

$$=\lim\frac{1}{n_{1i}n_{2i}\dots n_{si}}\Sigma P(a_1+\varrho_1a_{1i,1},\dots)+\lim\frac{1}{n_{1i}n_{2i}\dots n_{si}}\Sigma(a_1+\varrho_1a_{1i,1},\dots)=$$

$$=P(a_1,a_2,\dots,a_s)+Q(a_1,a_2,\dots,a_s).$$

Par là nous avons démontré que pour un polynome $P(x_1,\dots,x_s)$ on a

$$\int_{\alpha_1,\varrho_1}^{x_1}\dots\int_{\alpha_s,\varrho_s}^{x_s}P(x_1,x_2,\dots,x_s)=P(a_1,a_2,\dots,a_s).$$

Théorème II:

$$I=\int_{0,\varrho_1}^{x_1}\dots\int_{0,\varrho_s}^{x_s}\frac{x_1x_2\dots x_s}{(x-x_{10})(x-x_{20})\dots(x-x_{s0})}=\begin{cases}1 & \text{pour } |x_{k0}|<|\varrho_k|.\\ & k=1,2,\dots,s.\\ 0 & \text{pour } |x_{k0}|>|\varrho_k|.\\ & k=1,2,\dots,s.\end{cases}$$

$$I=\lim\frac{1}{n_{1i}\dots n_{si}}\left\{\frac{a_{1i,1}a_{2i,1}\dots a_{si,1}\varrho_1\varrho_2\dots\varrho_s}{(a_{1i,1}-x_{10})(a_{2i,1}-x_{20})\dots(a_{si,1}-x_{s0})}+\right.$$

$$\left.+\dots+\frac{a_{1i,n_{1i}}\dots a_{si,n_{si}}\varrho_1\varrho_2\dots\varrho_s}{(a_{1i,n_{1i}}-x_{10})\dots(a_{si,n_{si}}-x_{s0})}\right\}$$

$$=\lim\frac{\frac{n_{1i}-n_{1i,1}}{n_{1i}}\cdot\frac{n_{2i}-n_{2i,1}}{n_{2i}}\dots\frac{n_{si}-n_{si,1}}{n_{si}}\left(\frac{x_{10}}{\varrho_1}\right)^{n_{1i,1}}\dots\left(\frac{x_{s0}}{\varrho_s}\right)^{n_{si,1}}c_{1i,1}c_{2i,1}\dots c_{si,1}+}{\left\{\left(\frac{x_{10}}{\varrho_1}\right)^{n_{1i}}+c_{1i,1}\left(\frac{x_{10}}{\varrho_1}\right)^{n_{1i,1}}+\dots+c_{1i,\mu+1}\right\}}$$

$$\frac{\dots+c_{1i,\mu+1}c_{2i,\mu+1}\dots c_{si,\mu+1}}{\dots\left\{\left(\frac{x_{s0}}{\varrho_s}\right)^{n_{si}}+c_{si,1}\left(\frac{x_{s0}}{\varrho_s}\right)^{n_{si,1}}+\dots+c_{si,\mu+1}\right\}}.$$

Supposons que $|x_{10}|<|\varrho_1|,\dots,|x_{s0}|<|\varrho_s|$ et faisons attention à ce que

$$|c_{1i,\mu+1}|=|c_{2i,\mu+1}|=\dots=|c_{si,\mu+1}|=1\text{ et }|c_{ki,l}|\leqq 1\ (l=1,2,\dots,\mu),$$

on peut déterminer les exposants de $\frac{x_{10}}{\varrho_1}, \frac{x_{20}}{\varrho_2}, \ldots, \frac{x_{s0}}{\varrho_s}$ de telle façon que la valeur de ces termes est tout au plus ε $(0<\varepsilon<1)$; on s'aperçoit que la valeur du numérateur et du dénominateur est 1, de sorte que $I=1$ pour $|x_{k0}|<|\varrho_k|$. Si $|x_{k0}|>|\varrho_k|$ les premiers termes des facteurs du dénominateur prédominent; parce que $n_{ki}\to\infty$, la limite de la fraction est zéro, alors $I=0$ pour $|x_k|>|\varrho_k|$.

Théorème III:

$$I=\int_{0,\varrho_1}^{x_1}\cdots\int_{0,\varrho_s}^{x_s}\frac{1}{(x-x_{10})(x-x_{20})\ldots(x-x_{s0})}=\begin{cases}0 \text{ pour } |x_{k0}|<|\varrho_k|.\\ \dfrac{(-1)^s}{x_{10}x_{20}\ldots x_{s0}} \text{ pour } |x_{k0}|>|\varrho_k|.\end{cases}$$

Démonstration:

$$I=\lim\frac{1}{n_{1i}n_{2i}\ldots n_{si}}\left\{\frac{1}{(\varrho_1\alpha_{1i,1}-x_{10})(\varrho_2\alpha_{2i,1}-x_{20})\ldots(\varrho_s\alpha_{si,1}-x_{s0})}+\ldots\right\}$$

$$=\lim\frac{1}{n_{i1}n_{i2}\ldots n_{si}}\cdot\frac{(-1)^s}{\varrho_1\varrho_2\ldots\varrho_s}\left\{\frac{1}{\left(\frac{x_{10}}{\varrho_1}-\alpha_{1i,1}\right)\left(\frac{x_{20}}{\varrho_2}-\alpha_{2i,1}\right)\ldots\left(\frac{x_{s0}}{\varrho_s}-\alpha_{si,1}\right)}+\ldots\right\}$$

$$\lim\frac{1}{n_{i1}n_{i2}\ldots n_{is}}\cdot\frac{(-1)^s}{\varrho_1\varrho_2\ldots\varrho_s}\cdot\frac{n_{1i}n_{2i}\ldots n_{si}\left(\frac{x_{10}}{\varrho_1}\right)^{n_{1i}-1}\left(\frac{x_{20}}{\varrho_1}\right)^{n_{2i}-1}\ldots\left(\frac{x_{s0}}{\varrho_s}\right)^{n_{si}-1}+}{\left\{\left(\frac{x_{10}}{\varrho_1}\right)^{n_{1i}}+c_{1i,1}\left(\frac{x_{10}}{\varrho_1}\right)^{n_{1i,1}}+\ldots+c_{1i,\mu+1}\right\}}$$

$$\frac{\ldots+n_{1i,\mu}n_{2i,\mu}\ldots n_{si,\mu}\left(\frac{x_{10}}{\varrho_1}\right)^{n_{1i,\mu}-1}\ldots\left(\frac{x_{s0}}{\varrho_s}\right)^{n_{si,\mu}-1}c_{1i,\mu}\ldots c_{si,\mu}}{\ldots\left\{\left(\frac{x_{s0}}{\varrho_s}\right)^{n_{si}}+c_{si,1}\left(\frac{x_{s0}}{\varrho_s}\right)^{n_{si,1}}+\ldots\right\}}.$$

Si $|x_{k0}|<|\varrho_k|$ on peut faire correspondre à tout ε $(0<\varepsilon<1)$ des exposants suffisamment grands pour que la valeur du numérateur $\leqq\varepsilon$, tandis que la valeur du dénominateur est de 1, c'est à dire $I=0$ pour $|x_{k0}|<|\varrho_k|$. Pour $|x_{k0}|>|\varrho_k|$ nous écrivons

$$I=\lim\frac{(-1)^s}{x_{10}x_{20}\ldots x_{s0}}\cdot$$

$$\cdot\frac{\left(\frac{x_{10}}{\varrho_1}\right)^{n_{1i}}\left(\frac{x_{20}}{\varrho_2}\right)^{n_{2i}}\ldots\left(\frac{x_{s0}}{\varrho_s}\right)^{n_{si}}+\frac{n_{1i,1}}{n_{1i}}\left(\frac{x_{10}}{\varrho_1}\right)^{n_{1i,1}}\left(\frac{x_{20}}{\varrho_2}\right)^{n_{2i}}\ldots\left(\frac{x_{s0}}{\varrho_s}\right)^{n_{si}}+\ldots}{\left\{\left(\frac{x_{10}}{\varrho_1}\right)^{n_{1i}}+\ldots\right\}\ldots\left\{\left(\frac{x_{s0}}{\varrho_s}\right)^{n_{si}}+\ldots\right\}}$$

$$=\lim\frac{(-1)^s}{x_{10}x_{20}\ldots x_{s0}}\cdot$$

$$\cdot\frac{1+\frac{n_{1i,1}}{n_{1i}}\left(\frac{x_{10}}{\varrho_1}\right)^{n_{1i,1}-n_{1i}}+\ldots}{\left\{1+c_{1i,1}\left(\frac{x_{10}}{\varrho_s}\right)^{n_{1i,1}-n_{1i}}+\ldots\right\}\ldots\left\{1+c_{si,1}\left(\frac{x_{s0}}{\varrho_s}\right)^{n_{si,i}-n_{si}}+\ldots\right\}}.$$

Pour $n_{1i}, n_{1i,1}, \ldots$ suffisamment grand la valeur de la dernière fraction est égale à 1, c'est à dire toute la fraction se rapproche de $\frac{(-1)^s}{x_{10}x_{20}\ldots x_{so}}$, alors $I = \frac{(-1)^s}{x_{10}x_{20}\ldots x_{so}}$ pour $|x_{k0}| > |\varrho_k|$.

Théorème IV: Soit $f(x_1, x_2, \ldots, x_s)$ une série infinie

$$\Sigma a_{m_1 m_2 \ldots m_s} x_1^{m_1} x_2^{m_2} \ldots x_s^{m_s},$$

convergeant pour $|x_l| < |\varrho_l|$ $(l = 1, 2, \ldots, s)$ et soit en outre

$$|\alpha_l| < |\varrho_{l_0}| < |\varrho_l|$$

$(l = 1, 2, \ldots, s)$, alors nous démontrerons

$$\int_{\alpha_1, \varrho_{10}}^{x_1} \ldots \int_{\alpha_s, \varrho_{s0}}^{x_s} f(x_1, x_2, \ldots, x_s) = f(\alpha_1, \alpha_2, \ldots, \alpha_s).$$

Pour la démonstration il suffit d'expliquer qu'il est permis d'échanger la limite $f_{m_1 m_2 \ldots m_s}(x_1, \ldots, x_s) \to f(x_1, \ldots, x_s)$ et l'intégration.

$$\int_{\alpha_1, \varrho_{10}}^{x_1} \ldots \int_{\alpha_s, \varrho_{s0}}^{x_s} f(x_1, x_2, \ldots, x_s) = \int_{\alpha_1, \varrho_{10}}^{x_1} \ldots \int_{\alpha_s, \varrho_{s0}}^{x_s} A(x_1, x_2, \ldots, x_s) + \int_{\alpha_1, \varrho_{10}}^{x_1} \ldots \int_{\alpha_s, \varrho_{s0}}^{x_s} R(x_1, x_2, \ldots, x_s),$$

où

$$A(x_1, \ldots, x_s) = \sum_{\substack{0 \leqq m_1 \leqq M_1 \\ \cdots\cdots \\ 0 \leqq m_s \leqq M_s}} a_{m_1 m_2 \ldots m_s} x_1^{m_1} x_2^{m_2} \ldots x_s^{m_s},$$

$$R(x_1, \ldots, x_s) = \sum_{m_i > M_i} a_{m_1 m_2 \ldots m_s} x_1^{m_1} x_2^{m_2} \ldots x_s^{m_s},$$

$$\int_{\alpha_1, \varrho_{10}}^{x_1} \ldots \int_{\alpha_s, \varrho_{s0}}^{x_s} R(x_1, x_2, \ldots, x_s) =$$

$$= \lim \frac{1}{n_{1i} n_{2i} \ldots n_{si}} \Big\{ a_{M_1 M_2 \ldots M_s} (\alpha_1 + \varrho_{10} \alpha_{1i,1})^{M_1} (\alpha_2 + \varrho_{20} \alpha_{1i,1})^{M_2} \ldots (\alpha_s + \varrho_{s0} \alpha_{si,1})^{M_s} +$$

$$+ \ldots + a_{M_1 M_2 \ldots M_s} (\alpha_1 + \varrho_{10} \alpha_{1i,n_{1i}})^{M_1} (\alpha_2 + \varrho_{20} \alpha_{2i,n_{2i}})^{M_2} \ldots + \ldots \Big\}$$

Nous ajoutons: si nous avons une suite dénombrable d'éléments avec la propriété qu'il n'y a qu'un nombre fini d'éléments dont la valeur $\geq \varepsilon$, toute série formée de ces éléments converge à la même limite. Remarquons que $|\alpha_{ki,l}| = 1, |\alpha_l| < |\varrho_{l0}|$; il s'ensuit que la valeur des premiers $n_{1i} n_{2i} \ldots n_{si}$ termes est $|a_{M_1 M_2 \ldots M_s}| |\varrho_{10}|^{M_1} \ldots |\varrho_{s0}|^{M_s}$, de sorte que la valeur de la somme est

$$\max \left(|a_{M_1 M_2 \ldots M_s} \varrho_{10}^{M_1} \ldots \varrho_{s0}^{M_s}|, |a_{M_1+1, M_2 \ldots M_s} \varrho_{10}^{M_1+1} \ldots \varrho_{s0}^{M_s}|, \ldots \right).$$

Comme la série converge pour $x_1 = \varrho_{10}, \ldots, x_s = \varrho_{s0}$, il est possible de choisir $M_1, M_2, \ldots, M_s$ de telle façon, que

$$|a_{M_1 + p_1, M_2 + p_2, \ldots, M_s + p_s} \varrho_{10}^{M_1 + p_1} \ldots \varrho_{s0}^{M_s + p_s}| < \varepsilon$$

pour tous les $p_1, p_2, \ldots, p_s$, c'est à dire

$$\left| \int_{\alpha_1, \varrho_{10}}^{x_1} \cdots \int_{\alpha_s, \varrho_{s0}}^{x_s} f(x_1, x_2, \ldots, x_s) - \int_{\alpha_1, \varrho_{10}}^{x_1} \cdots \int_{\alpha_s, \varrho_{s0}}^{x_s} A(x_1, \ldots, x_s) \right| < \varepsilon,$$

de sorte que

$$\sum_{m_i} \int_{\alpha_1, \varrho_{10}}^{x_1} \cdots \int_{\alpha_s, \varrho_{s0}}^{x_s} a_{m_1 m_2 \ldots m_s} x_1^{m_1} \cdot x_2^{m_2} \ldots x_s^{m_s}$$

converge, et possède la limite

$$\int_{\alpha_1, \varrho_{10}}^{x_1} \cdots \int_{\alpha_s, \varrho_{s0}}^{x_s} f(x_1, x_2, \ldots, x_s).$$

Alors on a

$$\int_{\alpha_1, \varrho_{10}}^{x_1} \cdots \int_{\alpha_s, \varrho_{s0}}^{x_s} f(x_1, x_2, \ldots, x_s) = f(a_1, a_2, \ldots, a_s) \text{ pour } |a_k| < |\varrho_{k0}| < |\varrho_k|.$$

Une conséquence immédiate est

Théorème V:

$$\int_{\alpha_1, \varrho_{10}}^{x_1} \cdots \int_{\alpha_s, \varrho_{s0}}^{x_s} f(x_1, x_2, \ldots, x_s)(x_1 - a_1)(x_2 - a_2) \ldots (a_s - a_s) = 0.$$

Théorème VI:

$$I = \int_{0, \varrho_{10}}^{x_1} \cdots \int_{0, \varrho_{s0}}^{x_s} \frac{f(x_1, x_2, \ldots, x_s)\, x_1\, x_2 \ldots x_s}{(x_1 - x_{10})(x_2 - x_{20}) \ldots (x_s - x_{s0})} =$$

$$= \begin{cases} f(x_{10}, \ldots, x_{s0}) \text{ pour } |x_{k0}| < |\varrho_{k0}| < |\varrho_k| \\ 0 \text{ pour } |x_{k0}| > |\varrho_{k0}|, \end{cases}$$

où $f(x_1, x_2, \ldots, x_s) = \Sigma\, a_{m_1 m_2 \ldots m_s} x^{m_1} \ldots x_s^{m_s}$ converge pour $|x_k| < |\varrho_k|$. Nous intégrons terme à terme

$$\int_{0, \varrho_{10}}^{x_1} \cdots \int_{0, \varrho_{s0}}^{x_s} \frac{a_{m_1 m_2 \ldots m_s} x_1^{m_1 + 1} \ldots x_s^{m_s + 1}}{(x_1 - x_{10})(x_1 - x_{20}) \ldots (x_s - x_{s0})} =$$

$$= a_{m_1 m_2 \ldots m_s} \int_{0, \varrho_{10}}^{x_1} \cdots \int_{0, \varrho_{s0}}^{x_s} \left\{ \frac{(x_1^{m_1+1} - x_{10}^{m_1+1})(x_2^{m_2+1} - x_{20}^{m_2+1}) \ldots (x_s^{m_s+1} - x_{s0}^{m_s+1})}{(x_1 - x_{10})(x_2 - x_{20}) \ldots (x_s - x_{s0})} + \right.$$

$$\left. + \frac{P}{(x_1 - x_{10})(x_2 - x_{20}) \ldots (x_s - x_{s0})} \right\},$$

où P contient des termes, contenant un nombre de x_i à la puissance $m_i + 1$ et pour le reste $x_{k0}^{m_k + 1}$. Alors la première partie est

$$a_{m_1 m_2 \ldots m_s} (x_1^{m_1} + x_1^{m_1 - 1} x_{10} + \ldots + x_{10}^{m_1}) \ldots (x_s^{m_s} + x_s^{m_s - 1} x_{s0} + \ldots + x_{s0}^{m_s}),$$

où l'intégrale est identique (selon théorème I) à $a_{m_1 m_2 \dots m_s} x_{10}^{m_1} x_{20}^{m_2} \dots x_{s0}^{m_s}$. Nous démontrerons de la dernière intégrale qu'elle est zéro, et pour cela nous ne choisirons qu'un des termes existants

$$\int_{0,\varrho_{10}}^{x_1} \dots \int_{0,\varrho_{s0}}^{x_s} \frac{x_1^{m_1+1} x_2^{m_2+1} \dots x_{k-1}^{m_{k-1}+1} x_{k0}^{m_k+1} \dots x_{s0}^{m_s+1}}{(x_1-x_{10}) \dots (x_s-x_{s0})} =$$

$$= x_{k0}^{m_k-1} x_{k+1,0}^{m_{k+1}+1} \dots x_{s0}^{m_s+1} \int_{0,\varrho_{10}}^{x_1} \dots \int_{0,\varrho_{s0}}^{x_s} \frac{x_1^{m_1+1} x_2^{m_2+1} \dots x_{k-1}^{m_{k-1}+1}}{(x_1-x_{10})(x_2-x_{20}) \dots (x_s-x_{s0})}$$

$$\int_{0,\varrho_{10}}^{x_1} \dots \int_{0,\varrho_{s0}}^{x_s} \frac{x_1^{m_1+1} \dots x_{k-1}^{m_{k-1}+1}}{(x_1-x_{10}) \dots (x_s-x_{s0})} =$$

$$= \lim \frac{1}{n_{1i} n_{2i} \dots n_{si}} \left\{ \frac{a_{1i,1}^{m_1+1} \dots a_{k-1,i,1}^{m_{k-1}+1} \varrho_{10}^{m_1+1} \dots \varrho_{k-1,0}^{m_{k-1}+1}}{(a_{1i,1}\varrho_{10}-x_{10}) \dots (a_{si,1}\varrho_{s0}-x_{s0})} + \dots \right\}$$

$$= \lim \frac{1}{n_{1i} n_{2i} \dots n_{si}} \cdot \frac{1}{\varrho_{10} \varrho_{20} \dots \varrho_{s0}} \cdot$$

$$\cdot \frac{n_{1i} n_{2i} \dots n_{s-1,i} \varrho_{s0}^{m_{k-1}+1} \left(\frac{x_{10}}{\varrho_{10}}\right)^{n_{1i}-1} \left(\frac{x_{20}}{\varrho_{20}}\right)^{n_{2i}-1} \dots \left(\frac{x_{s0}}{\varrho_{s0}}\right)^{n_{si}-1} A + \dots}{\left\{\left(\frac{x_{10}}{\varrho_{10}}\right)^{n_{1i}} + c_{1i,1}\left(\frac{x_{10}}{\varrho_{10}}\right)^{n_{1i,1}} + \dots + c_{1i,\mu+1}\right\} \dots \left\{\left(\frac{x_{s0}}{\varrho_{s0}}\right)^{n_{si}} + \dots + c_{si,u+1}\right\}}$$

où A est une somme de produits des $c_{ki,n}$ sans terme constant; nous pouvons éviter le calcul laborieux d'autres termes du numérateur, si nous observons que tous les termes contiennent des puissances de $\frac{x_{10}}{\varrho_{10}}, \frac{x_{20}}{\varrho_{20}}, \dots, \frac{x_{s0}}{\varrho_{s0}}$, pendant que la valeur de tous les coëfficients $\leqq 1$. Pour $|x_{k0}| < |\varrho_k|$ on peut rendre les exposants de $\frac{x_{k0}}{\varrho_k}$ si grands, que pour chaque $\varepsilon > 0$ la valeur du numérateur devient $< \varepsilon$; parce que la valeur du dénominateur est précisément 1, la limite de la fraction est 0 pour $|x_{k0}| < |\varrho_k|$. Pour $|x_{k0}| > |\varrho_k|$ on peut démontrer d'une manière analogue à celle du théorème III, que les termes, qui sont contenus en P ne contribuent pas à l'intégrale sauf le dernier (que nous étudierons plus tard). L'intégrale produit donc:

$$\int_{0,\varrho_0}^{x_1} \dots \int_{0,\varrho_s}^{x_s} \frac{a_{m_1 m_2 \dots m_s} x_1^{m_1+1} \dots x_s^{m_s+1}}{(x_1-x_{10})(x_2-x_{20}) \dots (x_s-x_{s0})} = a_{m_1 m_2 \dots m_s} x_{10}^{m_1} x_{20}^{m_2} \dots x_{s0}^{m_s}$$

pour $|x_{k0}| < |\varrho_{k0}| < |\varrho_k|$,

alors

$$\int_{0,\varrho_0}^{x_1} \dots \int_{0,\varrho_s}^{x_s} f(x_1, x_2, \dots, x_s) = f(x_{10}, x_{20}, \dots, x_{s0}) \text{ pour } |x_{k0}| < |\varrho_{k0}| < |\varrho_k|.$$

Pour $|x_{k0}|>|\varrho_{k0}|$ le dernier terme de P produit $-\dfrac{1}{x_{10}x_{20}\dots x_{s0}}$, de sorte que pour $|x_{k0}|>|\varrho_{k0}|$

$$\int_{0,\varrho_0}^{x_1}\dots\int_{0,\varrho_s}^{x_s}\frac{a_{m_1m_2\dots m_s}x_1^{m_1+1}\dots x_s^{m_s+1}}{(x_1-x_{10})(x_2-x_{20})\dots(x_s-x_{s0})}=$$

$$=a_{m_1m_2\dots m_s}x_{10}^{m_1}\dots x_{s0}^{m_s}-\frac{1}{x_{10}x_{20}\dots x_{s0}}x_{10}^{m_1+1}\dots x_{s0}^{m_s+1}=0$$

alors

$$\int_{0,\varrho_0}^{x_1}\dots\int_{0,\varrho_s}^{x_s}\frac{f(x_1,x_2,\dots,x_s)\,x_1\,x_2\dots x_s}{(x_1-x_{10})(x_2-x_{20})\dots(x_s-x_{s0})}=0 \text{ pour } |x_{k0}|>|\varrho_{k0}|.$$

Théorème VIa: De même on a en général

$$\int_{\alpha_1,\varrho_{10}}^{x_1}\dots\int_{\alpha_s,\varrho_{s0}}^{x_s}\frac{f(x_1,x_2,\dots,x_s)\,x_1\,x_2\dots x_s}{(x_1-x_{10})(x_2-x_{20})\dots(x_s-x_{s0})}=\begin{cases}f(x_{10},x_{20},\dots,x_{s0}) \text{ pour}\\ \qquad |x_{k0}-\alpha_k|<|\varrho_{k0}|<|\varrho_k|\\ 0 \text{ pour } |x_{k0}-\alpha_k|>|\varrho_{k0}|.\end{cases}$$

Théorème VII: Soit $f(x_1,x_2,\dots,x_s)=$

$$=\sum_{m_i}a_{m_1m_2\dots m_s}x_1^{m_1}x_2^{m_2}\dots x_s^{m_s}$$

convergeant pour $|x_k|<|\varrho_k|$; représentons les dérivés de $f(x_1,\dots,x_s)$ par $f^{(p_1,p_2,\dots,p_s)}(x_1,x_2,\dots,x_s)$, on a

$$f^{(p_1,p_2,\dots,p_s)}(x_{10},x_{20},\dots,x_{s0})=$$

$$=p_1!\,p_2!\dots p_s!\int_{0,\varrho_{10}}^{x_1}\dots\int_{0,\varrho_{s0}}^{x_s}\frac{f(x_1,x_2,\dots,x_s)\,x_1\,x_2\dots x_s}{(x_1-x_{10})^{p_1+1}\dots(x_s-x_{s0})^{p_s+1}}$$

pour $|x_{k0}|<|\varrho_{k0}|<|\varrho_k|$.

Démonstration: Soit

$$f(x_1,x_2,\dots,x_s)=\sum a_{m_1m_2\dots m_s}x_1^{m_1}x_2^{m_2}\dots x_s^{m_s},$$

alors

$$f^{(p_1,p_2,\dots,p_s)}(x_1,x_2,\dots,x_s)=$$

$$=\sum_{l_i}\frac{(p_1+l_1)!\,(p_2+l_2)!\dots(p_s+l_s)!}{l_1!\,l_2!\dots l_s!}a_{p_1+l_1,\dots p_s+l_s}x_1^{l_1}x_2^{l_2}\dots x_s^{l_s}.$$

Nous intégrons terme à terme de manière que

$$p_1!\,p_2!\dots p_s!\int_{0,\varrho_{10}}^{x_1}\dots\int_{0,\varrho_{s0}}^{x_s}\frac{a_{m_1m_2\dots m_s}x_1^{m_1+1}x_1^{m_2+1}\dots x_s^{m_s+1}}{(x_1-x_{10})^{p_1+1}(x_2-x_{20})^{p_2+1}\dots(x_s-x_{s0})^{p_s+1}},$$

et nous distinguons alors

1°. $m_i<p_i$ $(i=1,2,\dots,s)$; 2°. $m_i\geq p_i$ $(i=1,2,\dots,s)$; 3°. les autres cas.

1°. Pour $m_i < p_i$ nous nous servons de la décomposition suivante

$$\frac{x_1^{m_1+1}\, x_2^{m_2+1} \dots x^{m_s+1}}{(x_1-x_{10})^{p_1+1}(x_2-x_{20})^{p_2+1} \dots (x_s-x_{s0})^{p_s+1}} =$$

$$= \frac{A_{11}}{x_1-x_{10}} + \frac{A_{12}}{(x_1-x_{10})^2} + \dots + \frac{A_{1,p_1+1}}{(x_1-x_{10})^{p_1+1}} +$$

$$+ \frac{A_{21}}{x_2-x_{20}} + \dots + \frac{A_{2,p_2+1}}{(x_2-x_{20})^{p_2+1}} + \dots + \frac{B}{(x_1-x_{10})(x_2-x_{20})} + \dots +$$

$$+ \frac{C}{(x_1-x_{10})^{p_1+1} \dots (x_s-x_{s0})^{p_s+1}},$$

où les $A, B, \dots, C$, représentent des constants, qui sont éventuellement dépendants de $x_{10}, x_{20}, \dots, x_{s0}$. On peut s'assurer de la validité de cette décomposition en comparant le nombre des éléments inconnus dans les numérateurs au nombre des équations disponibles. Or, les intégrales produites par ces fractions aux numérateurs constants sont tous 0, selon la méthode du théorème III.

2°. Pour $m_i \geqq p_i$ $(i = 1, 2, \dots, s)$ nous avons à faire à des intégrales de la forme

$$\int_{0,\varrho_0}^{x_1} \dots \int_{0,\varrho_s}^{x_s} \frac{a_{p_1+h_1, p_2+h_2, \dots, p_s+h_s}\, x_1^{p_1+h_1+1} \dots x_s^{p_s+h_s+1}}{(x_1-x_{10})^{p_1+1}(x_2-x_{20})^{p_2+1} \dots (x_s-x_{s0})^{p_s+1}}.$$

En décomposant:

$$\frac{x_1^{p_1+h_1+1} \dots x_s^{p_s+h_s+1}}{(x_1-x_{10})^{p_1+1} \dots (x_s-x_{s0})^{p_s+1}} = x_1^{h_1} x_2^{h_2} \dots x_s^{h_s} + (p_1+1) x_1^{h_1-1} x_{10} x_2^{h_2} \dots x_s^{h_s} + \dots$$

$$+ \frac{(p_1+1)(p_1+2)\dots(p_1+h_1)(p_2+1)\dots(p_2+h_2)\dots(p_s+1)\dots(p_s+h_s)}{h_1!\, h_2! \dots h_s!} x_{10}^{h_1} x_{20}^{h_2} \dots x_{s0}^{h_s} +$$

$$+ \frac{P}{(x_1-x_{10})^{p_1+1}(x_2-x_{20})^{p_2+1} \dots (x_s-x_{s0})^{p_s+1}},$$

où P est un polynome dépendant de $x_1, x_2, \dots, x_s, x_{10}, \dots, x_{s0}$, pendant que les termes séparés de P ont la propriété que ou bien le degré en x_1 est plus petit que $p_1 + 1$, ou bien le degré en x_2 est plus petit que $p_2 + 1, \dots$, ou bien le degré en x_s est plus petit que $p_s + 1$. La contribution des premiers termes du dévelopment est selon théorème I

$$\frac{(p_1+h_1)!\,(p_2+h_2)! \dots (p_s+h_s)!}{h_1!\, h_2! \dots h_s!} x_{10}^{h_1} x_{20}^{h_2} \dots x_{s0}^{h_s}.$$

La contribution des termes de P est 0, comme on démontre d'une manière analogue comme celle en théorème VI. Alors il s'ensuit

$$p_1!\, p_2! \dots p_s! \int_{0,\varrho_{10}}^{x_1} \dots \int_{0,\varrho_{s0}}^{x_s} \frac{f(x_1, x_2, \dots, x_s)\, x_1\, x_2 \dots x_s}{(x_1-x_{10})^{p_1+1} \dots (x_s-x_{s0})^{p_s+1}} =$$

$$= \Sigma \frac{(p_1+h_1)!\,(p_2+h_2)! \dots (p_s+h_s)!}{h_1!\, h_2! \dots h_s!} x_{10}^{h_1} x_{20}^{h_2} \dots x_{s0}^{h_s} = f^{(p_1, p_2, \dots, p_s)}(x_{10}, x_{20}, \dots, x_{s0}).$$

Théorème VIII: Soit $f(x_1, x_2, \ldots, x_s)$ une fonction, admettant pour $|x_{k0} - a_k| < |\varrho_{k0}|$ $(k = 1, 2, \ldots, s)$ une représentation

$$f(x_{10}, x_{20}, \ldots, x_{s0}) = \int_{\alpha_1, \varrho_{10}}^{x_1} \cdots \int_{\alpha_s, \varrho_{s0}}^{x_s} \frac{f(x_1, x_2, \ldots x_s)\, x_1\, x_2 \ldots x_s}{(x_1 - x_{10})(x_2 - x_{20}) \ldots (x_s - x_{s0})}$$

tandis que $|f(x_1, x_2, \ldots, x_s)| < M$ sur les cercles (a_k, ϱ_{k0}), on a pour tout x_{k0} avec $|x_{k0} - a_k| < |\varrho_{k0}|$

$$|f(x_{10}, x_{20}, \ldots, x_{s0})| < M \text{ et } f^{(p_1, p_2, \ldots, p_s)}(x_{10}, x_{20}, \ldots, x_{s0}) < \frac{M}{|\varrho_{10}|^{p_1} |\varrho_{20}|^{p_2} \ldots |\varrho_{s0}|^{p_s}}.$$

Démonstration:

$$f(x_{10}, x_{20}, \ldots, x_{s0}) = \lim \frac{1}{n_{1i}\, n_{2i} \ldots n_{si}} \left\{ \frac{f(a_1 + \varrho_{10}\, a_{1i,1}, a_2 + \varrho_{20}\, a_{2i,1}, \ldots)\,(a_1 + \varrho_{10}\, a_{1i,1}) \ldots}{(a_1 + \varrho_{10}\, a_{1i,1} - x_{10}) \ldots (a_s + \varrho_{s0}\, a_{si,1} - x_{s0})} + \ldots \right\}.$$

Supposons que $\dfrac{f(a_1 + \varrho_{10}\, a_{1i,1}, \ldots)\,(a_1 + \varrho_{10}\, a_{1i,1}) \ldots}{(a_1 + \varrho_{10}\, a_{1i,1} - x_{10}) \ldots (a_s + \varrho_{s0}\, a_{si,1} - x_{s0})}$ ait la valeur maximale, alors en vertu de l'évaluation non-archimédienne de T, on a

$$\left| \frac{f(a_1 + \varrho_{10}\, a_{1i,1}, \ldots)\,(a_1 + \varrho_{10}\, a_{1i,1}) \ldots (a_s + \varrho_{s0}\, a_{si,1})}{n_{1i}\, n_{2i} \ldots n_{si}\,(a_1 + \varrho_{10}\, a_{1i,1} - x_{10}) \ldots (a_s + \varrho_{s0}\, a_{si,1} - x_{s0})} \right| < M$$

ou

$$\frac{|f(a_1 + \varrho_{10}\, a_{1i,1}, \ldots)\,(a_1 + \varrho_{10}\, a_{1i,1}) \ldots (a_s + \varrho_{s0}\, a_{si,1})|}{|n_{1i}| \ldots |n_{si}|\,|\varrho_{10}|\,|\varrho_{20}| \ldots |\varrho_{s0}|} < M$$

parce que

$$|a_k + \varrho_{k0}\, a_{ki,1} - x_{k0}| = |\varrho_{k0}\, a_{ki,1} + (a_k - x_{k0})| \leqq$$
$$\max\,(|\varrho_{k0}\, a_{ki,1}|, |a_k - x_{k0}|) = \max\,(|\varrho_{k0}|, |a_k - x_{k0}|);$$

maintenant $|a_k - x_{k0}| < |\varrho_{k0}|$, alors on a précisément

$$|a_k + \varrho_{k0}\, a_{ki,1} - x_{k0}| = |\varrho_{k0}|.$$

Pour le $(p_1, p_2, \ldots, p_s)$-ième dérivé on a

$$f^{(p_1, p_2, \ldots, p_s)}(x_{10}, x_{20}, \ldots, x_{s0}) =$$
$$= \lim \frac{p_1!\, p_2! \ldots p_s!}{n_{1i} \ldots n_{si}} \left\{ \frac{f(a_1 + \varrho_{10}\, a_{1i,1}, \ldots, a_s + \varrho_{s0}\, a_{si,1})(a_1 + \varrho_{10}\, a_{1i,1}) \ldots}{(a_1 + \varrho_{10}\, a_{1i,1} - x_{10})^{p_1 + 1} \ldots} + \ldots \right\}$$

$$|f^{(p_1, p_2, \ldots, p_s)}(x_{10}, x_{20}, \ldots, x_{s0})| =$$
$$= \lim \frac{|p_1!| \ldots |p_s!|\,|f(a_1 + \varrho_{10}\, a_{1i,1}, \ldots)\,(a_1 + \varrho_{10}\, a_{1i,1}) \ldots|}{|\varrho_{10}|^{p_1 + 1} \ldots |\varrho_{s0}|^{p_s + 1}} < \frac{M}{|\varrho_{10}|^{p_1} |\varrho_{20}|^{p_2} \ldots |\varrho_{s0}|^{p_s}}.$$

Il s'ensuit du théorème VIII:

Théorème IX: Soit $f(x_1, x_2, \ldots, x_s)$ une fonction, admettant pour $|x_{k0} - a_k| < |\varrho_{k0}|$ $(k = 1, 2, \ldots, s)$ une représentation

$$f(x_{10}, x_{20}, \ldots, x_{s0}) = \int_{\alpha_1, \varrho_{10}}^{x_1} \cdots \int_{\alpha_s, \varrho_{s0}}^{x_s} \frac{f(x_1, x_2, \ldots, x_s)\, x_1\, x_2 \ldots x_s}{(x_1 - x_{10})(x_2 - x_{20}) \ldots (x_s - x_{s0})}$$

alors $f(x_{10}, x_{20}, \ldots, x_{s0})$ possède un développement en une série suivant des puissances de $x_{k0} - a_k$, convergeant pour

$$|x_{k0} - a_k| < |\varrho_{k0}|, \quad (k = 1, 2, \ldots, s).$$

Mathematics. -- *Bericht über die verschiedenen Methoden zur Lösung eines Systems linearer Gleichungen mit reellen Koeffizienten.* III. By E. BODEWIG. (Communicated by Prof. J. G. VAN DER CORPUT.)

(Communicated at the meeting of September 27, 1947.)

Das Verfahren von SEIDEL.

SEIDEL (Lit. 11) löste die i-te Gleichung des Systems $\mathfrak{B}\,\mathfrak{x} = \mathfrak{s}$ nach x_i auf, wiederholte das System unendlich oft und setzte auf der rechten Seite jeder Gleichung alle schon bekannten, bisherigen Näherungswerte ein, wodurch er für die jeweilige Unbekannte auf der linken Seite einen neuen Näherungswert erhielt.

Bezeichnen wir also zur grösseren Deutlichkeit und gleichzeitig zur einfacheren Schreibweise die Komponenten $x_i^{(k+1)}$ der k + 1-ten Näherung mit X_i, die der k-ten Näherung jedoch wie früher mit $x_i^{(k)}$, so ist

$$\begin{aligned}
X_1 &= -(\quad * \quad b_{12}\,x_2^{(k)} + b_{13}\,x_3^{(k)} + \ldots + b_{1n}\,x_n^{(k)} - s_1)\\
X_2 &= -(b_{21}\,X_1 \quad * \quad + b_{23}\,x_3^{(k)} + \ldots + b_{2n}\,x_n^{(k)} - s_2)\\
X_3 &= -(b_{31}\,X_1 + b_{32}\,X_2 \quad * \quad + \ldots + b_{3n}\,x_n^{(k)} - s_3)\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
X_n &= -(b_{n1}\,X_1 + b_{n2}\,X_2 + b_{n3}\,X_3 + \ldots \quad * \quad - s_n)
\end{aligned}$$

Hierbei setzt man etwa $\mathfrak{x}' = \mathfrak{e}_1 =$ der ersten Spalte der Einheitsmatrix, d.h. $x'_2 = x'_3 = \ldots = x'_n = 0$.

Die Konvergenz des Verfahrens ist schwer zu beurteilen. Man kann aber jedenfalls so viel sagen, dass sie leichter eintreten wird als bei den bisher besprochenen Iterationsverfahren. Denn während letztere nur bei jedem n-ten Schritt den Näherungsvektor wechselt, benutzt die SEIDELsche Methode bei *jedem* Schritt die neuesten Approximationswerte.

Ausserdem haben E. TREFFTZ und v. MISES (Lit. 7) folgenden Satz bewiesen:

Satz. Ist $\mathfrak{B}$ symmetrisch und definit, so konvergiert das SEIDELsche Verfahren monoton.

Dabei nennen wir $\mathfrak{B}$ definit, wenn die zugehörige quadratische Form mit $b_{ik}\,b_{ki}$

$$Q = \Sigma\,\Sigma\,b_{ik}\,x_i\,x_k$$

es ist, d.h. wenn sie für alle reellen x entweder nur Werte grösser als Null (positiv definit) oder kleiner als Null (negativ definit) annehmen kann.

Der *Beweis* verläuft so, dass man die Gleichungen des Systems $\mathfrak{B}\,\mathfrak{x} = \mathfrak{s}$

darstellen kann als die n partiellen Ableitungen $\partial F/\partial x_i = 0$, $i = 1, \ldots, n$ einer quadratischen Form

$$F = Q - 2 \sum_{i}^{n} s_i x_i.$$

Der Lösungsvektor $\mathfrak{x}$ des Systems gibt also einen Extremalwert der Funktion F an. Wenn nun Q definit positiv ist, so ist für hinreichend grosse x_i der Wert von F positiv, weil die quadratischen Bestandteile in F überwiegen über die linearen. Daher wächst F ausserhalb eines gewissen Bereiches G monoton mit jedem x_i. Ein Extremwert kann daher nur innerhalb G liegen. Da aber das System $\mathfrak{B}\,\mathfrak{x} = \mathfrak{s}$ nur eine einzige Lösung hat, so kann es auch nur einen einzigen Extremwert von F geben, und dieser muss nach dem Vorhergehenden ein Minimum sein.

Ist man nun bei der Iteration bis zu der Stelle

$$X_1, X_2, \ldots, X_{i-1}, x_i^{(k)}, x_{i+1}^{(k)}, \ldots, x_n^{(k)}$$

gekommen und ersetzen wir nunmehr $x_i^{(k)}$ durch X_i, so ändert sich F nur in den Gliedern, welche x_i enthalten, d.h. es vergrössert sich um

$$(X_i^2 - x_i^{(k)2}) + 2\,(\sum_{j=1}^{i-1} b_{ij} X_j + \sum_{j=i+1}^{n} b_{ij} X_j - s_i)\,(X_i - x_i^{(k)}).$$

Die erste Klammer aber ist nach den Gleichungen für das Iterationsverfahren gleich $-X_i$, daher obiger Zuwachs von F gleich

$$(X_i - x_i^{(k)})\,(X_i + x_i^{(k)} - 2\,X_i) = -(X_i - x_i^{(k)})^2 < 0.$$

F nimmt also monoton ab mit jedem Iterationsschritt, d.h. die Näherungswerte nähern sich monoton der Lösung $\mathfrak{x}$. Ferner kann die Änderung von F absolut genommen nicht stets grösser als eine feste Grösse sein, da F ein endliches Minimum hat. Die Änderung von F muss daher mit jedem Iterationsschritt monoton gegen 0 gehen, d.h. $X_i - x_i^{(k)} \to 0$, d.h. das Verfahren konvergiert und nach dem Vorhergehenden monoton.

Anwendung auf nichtsymmetrische Gleichungssysteme. Normalisierung. Um das Verfahren auch auf nichtsymmetrische Gleichungssysteme anwenden zu können und dabei die Konvergenz sicher zu stellen, bildet MISES aus dem unsymmetrischen System $\mathfrak{A}\,\mathfrak{x} = \mathfrak{r}$ durch Multiplikation mit der transponierten Matrix $\mathfrak{A}'$ das symmetrische System

$$\mathfrak{A}'\,\mathfrak{A}\,\mathfrak{x} = \mathfrak{A}'\,\mathfrak{r}, \quad \ldots\ldots\ldots \quad \text{(III, 10)}$$

dessen Matrix $\mathfrak{M} = \mathfrak{A}'\,\mathfrak{A}$ symmetrisch ist. Denn es ist $\mathfrak{M}' = \mathfrak{M}$. Ordnet man ferner die zu $\mathfrak{M}$ gehörige quadratische Form $\Sigma\, m_{ij} x_i x_j$ in bestimmter Weise, so folgt

$$(\sum_{1}^{n} a_{1j} x_j)^2 + (\sum_{1}^{n} a_{2j} x_j)^2 + \ldots + (\Sigma\, a_{nj} x_j)^2$$

und, da dieser Ausdruck stets positiv ist für reelle x, so ist $\mathfrak{M}$ positiv definit.

Satz. Um das Verfahren auf beliebige Systeme $\mathfrak{A}\,\mathfrak{x} = \mathfrak{r}$ anwenden zu können, normalisiere man sie durch Multiplikation mit der transponierten Matrix $\mathfrak{A}'$: $\mathfrak{A}'\,\mathfrak{A}\,\mathfrak{x} = \mathfrak{A}'\,\mathfrak{r}$. Für das neue System konvergiert das Verfahren bei beliebigem Anfangsvektor $\mathfrak{x}'$.

Anzahl der Operationen der Normalisierung (III, 10). Wenn damit auch das SEIDELsche Verfahren auf beliebige Systeme anwendbar gemacht worden ist, so dass die Konvergenz in jedem Falle eintritt, so ist damit noch nichts über die Zweckmässigkeit einer solchen Transformation gesagt. Dazu muss man erst ein Urteil über die Anzahl der Operationen, welche diese Transformation erfordert, haben.

Nun erfordert wegen der Symmetrie von $\mathfrak{M} = \mathfrak{A}'\,\mathfrak{A}$ die i-te Zeile von $\mathfrak{M}$: $n(n-i+1)$ Multiplikationen und $(n-1)\,(n-i+1)$ Additionen, die ganze Matrix $\mathfrak{M}$ also

$$n \sum_{1}^{n} i = \tfrac{1}{2} n^2 (n+1) \text{ Multiplikationen und}$$

$$(n-1) \sum_{1}^{n} i = \tfrac{1}{2} n (n^2-1) \text{ Additionen.}$$

Anderseits erfordert die rechte Seite $\mathfrak{A}'\,\mathfrak{r}$: n^2 Multiplikationen und $n(n-1)$ Additionen. Insgesamt erfordert daher die Transformation (10): $\tfrac{1}{2} n^2 (n+3)$ Multiplikationen und $\tfrac{1}{2} n (n-1)\,(n+3)$ Additionen.

Ergebnis. Die Normalisierung (10) eines Gleichungssystems erfordert für sich allein schon $1\tfrac{1}{2}$ mal so viel Operationen wie die gesamte Lösung des Gleichungssystems nach dem Verfahren von GAUSS. Die Normalisierung nach MISES ist daher völlig unökonomisch, denn nach ihrer Ausführung hat man das neue System auch noch aufzulösen.

Das Schema von MORRIS.

Der Stand, in welchem MISES die SEIDELsche Methode wie überhaupt die von ihm besprochenen Methoden zurückliess, war insofern nicht ganz befriedigend, als zu jeder rechnerischen Methode ein Schema ausgearbeitet werden muss, damit die Arbeit möglichst ökonomisch vor sich geht. Das gilt kaum irgendwo so sehr wie bei den linearen Gleichungen, wo die Rechnung wegen der hohen Zahl von Unbekannten und der grossen Zahl von Iterationen leicht unübersichtlich wird.

Diesen Schlusstein unter die SEIDELsche Methode hat MORRIS (Lit. 8) gesetzt, ohne freilich zu wissen, dass das Prinzip der von ihm angegebenen Methode von SEIDEL herrührte und durch MISES ausführlich besprochen worden war. Wir gehen auf das Schema aber nicht ein, weil die SEIDELsche Methode überhaupt zu langsam konvergiert, woran auch das Schema nichts ändern kann, und eine andere Methode bei weitem den Vorrang verdient, sowohl hinsichtlich der Uebersichtlichkeit als Konvergenz.

Die Relaxationsmethode von SOUTHWELL.

Wir haben die SEIDELsche Methode nicht deshalb so ausführlich besprochen, weil wir sie für zweckmässig hielten — das ist sie wegen ihrer langsamen Konvergenz auch mit dem Schema von MORRIS nicht —, sondern weil sie identisch ist mit der Relaxationsmethode, welche SOUTHWELL in einer Reihe von weit bekannt gewordenen Arbeiten auf die verschiedensten Fragestellungen der mathematischen Physik angewandt hat, darunter dann auch auf die Lösung von linearen Gleichungen. In diesem Punkte *fällt die Relaxationsmethode also mit derjenigen von* SEIDEL. Er selbst sagt darüber in seinem Buche (Lit. 12, p. 241): „For other failings in regard to references the author can only offer the excuse of Dr. JOHNSON: Time and again it has happened that devices found necessary or convenient which *at first were thought to be of our own invention* proved later to be special applications of theorems and methods *already known*" und in der Fussnote zu p. 241: „Thus the relaxation process as applied to „normalized" equations is in essence identical with „SEIDELs *process*"."

Dass ihm die alte Arbeit SEIDELs aus dem Jahre 1877 entgangen ist, ist aber nicht die Hauptsache, denn dasselbe passierte auch MORRIS und überhaupt trifft man es heutzutage ja öfter an, dass längst veröffentlichte Methoden aufs neue entdeckt und als neu ausgegeben werden. Indessen hätte ihm der grundlegende Bericht von MISES von 1929, wo ja die Methode SEIDELs ausführlich besprochen wird und zuerst auch ein Konvergenzkriterium aufgestellt wird, doch bekannt sein müssen und ebenso die Arbeit seines Landsmannes MORRIS.

Es kommt hinzu, dass SOUTHWELL in denselben Fehler wie MISES verfällt, eine Methode auch dann anwenden zu wollen, wenn sie ihrem Wesen nach auf jenen Fall gar nicht anwendbar ist. Um daher die Allgemeinheit der Methode zu zeigen, und ihre Anwendbarkeit auf beliebige Gleichungssysteme, nimmt er genau wie von MISES die Normalisierung des Gleichungssystems vor, dabei ebenso wie MISES übersehend, dass die dazu nötigen Operationen schon zur Lösung des gesamten Systems mehr als hinreichen würden.

Auch nimmt er die Normalisierung nur deshalb vor, um die Symmetrie der Gleichungen zu erreichen, im Glauben, dass schon die Symmetrie zur Konvergenz ausreichen würde, was bestimmt falsch ist. Den wahren Grund der Normalisierung haben TREFFTZ und MISES mit ihrem obrigem Satz angegeben.

Ergebnis. 1. Die Relaxations-Methode ist identisch mit der Methode von SEIDEL.

2. Die Normalisierung der Gleichungen ist ein „Kunstfehler".

Die Schnelligkeit der Konvergenz der Iterationsmethoden. Um über die Konvergenz-Schnelligkeit der Iterationsmethoden ein Urteil zu gewinnen, nehmen wir erst zwei *Zahlenbeispiele*, und zwar, um nicht die Verhältnisse

allzu ungünstig für die Iterationsmethoden erscheinen zu lassen, solche mit nur drei Unbekannten, also Fälle, in denen selbst die Vertreter iterativer Methoden die direkte Auflösung bevorzugen würden.

Das System
$$\begin{aligned} 1{,}137\,x + 0{,}981\,y + 0{,}873\,z &= 5{,}718 \\ 0{,}981\,x + 1{,}241\,y + 0{,}735\,z &= 5{,}668 \\ 0{,}873\,x + 0{,}735\,y + 1{,}353\,z &= 6{,}402 \end{aligned}$$

hat die Lösung $(x, y, z) = (1, 2, 3)$. Das System ist aus einer Ausgleichung entstanden, besitzt also die Matrix $\mathfrak{M} = \mathfrak{A}'\mathfrak{A}$, so dass die Methode von SEIDEL konvergiert. Ebenso konvergiert die Methode von MISES (III, 9) denn die grösste Wurzel der Gleichungsmatrix $\mathfrak{M}$ hat die obere Schranke: $\max (m_{i1} + m_{i2} + m_{i3}) < 3$. Das c von (III, 9) ist also gleich $-1/3$.

Nimmt man nun als Ausgangsvektor $\mathfrak{x}' = (0, 0, 0)$, so erhält man nach SEIDEL bei der ersten Iteration $\mathfrak{x}'' = (5{,}0;\ 0{,}6;\ 1{,}16)$ und als 10-te Iteration: $\mathfrak{x}^{(11)} = (1{,}12;\ 1{,}95;\ 2{,}95)$, also nach 10 Iterationen mit all ihren Rechnungen nicht einmal eine Dezimale!

Derselbe Ausgangsvektor $\mathfrak{x}'$ gibt nach MISES als erste Iteration $\mathfrak{x}'' = (1{,}9;\ 1{,}9;\ 2{,}1)$, also einen wesentlich besseren Näherungswert als SEIDEL. Dieser Vorsprung hält aber nicht an, denn die 10-te Iteration gibt $\mathfrak{x}^{(11)} = (1{,}47;\ 1{,}74;\ 2{,}78)$, also bedeutend schlechter als bei SEIDEL. Ich habe nicht ausgerechnet, wieviel Iterationen man durchführen müsste, um wenigstens die Ganzen der drei Lösungen, also eine einzige Stelle, zu bestimmen. *Zu welchen Resultaten man also kommt, wenn man etwa* 10 *oder* 20 *Unbekannten hat, braucht wohl nicht näher ausgeführt zu werden.*

Die gewöhnliche Iterationsmethode, wo also $d_i = -1/a_{ii}$, divergiert wegen (III, 7 und 8) überhaupt.

Von seiten der Vertreter iterativer Methoden wird man einwenden, das Beispiel sei „ungünstig gewählt" (vgl. dazu die Bemerkung im Vorwort). Das Beispiel ist aber in Wirklichkeit einer Arbeit über iterative Methoden entnommen (CESARI, Lit. 3). Wir nehmen aber noch ein zweites, „günstigeres" Beispiel, das derselben Arbeit entstammt, nämlich:

$$\begin{aligned} 0{,}734841\,x + 0{,}262827\,y - 0{,}013905\,z &= 1{,}218780 \\ 0{,}262827\,x + 1{,}052633\,y - 0{,}337503\,z &= 1{,}355584 \\ -0{,}013905\,x - 0{,}337503\,y + 1{,}331937\,z &= 3{,}306900 \end{aligned}$$

Mit dem Ausgangsvektor $\mathfrak{x}' = (0, 0, 0)$ bekommt man mittels der drei Methoden: der Iteration in gewöhnlichem Sinne, der von MISES und der von SEIDEL die folgenden Ergebnisse

	Iteration			MISES			SEIDEL		
$\mathfrak{x}''$	1,66	1,2878	2,4828	0,7	0,8	1,95	1,66	0,87	2,72
$\mathfrak{x}^{(7)}$	1,0066	1,9909	2,9953	1,07	1,93	2,97	1,0003	1,99986	2,99997
$\mathfrak{x}^{(8)}$	1,0031	1,9969	2,9978	1,05	1,96	2,980			
$\mathfrak{x}^{(9)}$	1,0011	1,9985	2,9992	1,034	1,97	2,987			
$\mathfrak{x}^{(10)}$	1,0005	1,9995	2,9996	1,024	1,982	2,992			
$\mathfrak{x}^{(11)}$	1,0002	1,9998	2,9999	1,016	1,988	2,995			

Wiederum ist die Methode von MISES *die schlechteste und die von* SEIDEL *die beste.* Immerhin ist selbst bei SEIDEL der Aufwand an Rechnung noch bedeutend, ein Vielfaches derjenigen von GAUSS. Während nämlich letzterer das symmetrische System nach (I, 7a') mit insgesamt 16 Multiplikationen und 10 Additionen auf theoretisch beliebig viele Dezimalen löst, braucht SEIDEL für 3—4 Dezimalen zur 6-maligen Anwendung seines Schemas: 36 Multiplikationen und 36 Additionen. Hinzu kommen noch die Multiplikationen, die gebraucht werden, um die Diagonalkoeffizienten gleich 1 zu machen, also 9 an der Zahl. Insgesamt haben wir somit 45 Mult. und 36 Additionen, also das Drei- bis Vierfache von GAUSS, abgesehen noch von der Zahl der Dezimalen, die bei GAUSS mit dem gleichen Rechenaufwand auf 10 Dezimalen erhöht werden kann, ohne dass sich die unvermeidlichen Abrungsfehler bemerkbar machen.

Die allgemeine Iteration von CESARI.

CESARI (Lit. 3) hat ein allgemeines Iterationsverfahren angegeben, das dadurch von Interesse ist, dass sich ihm die verschiedensten Iterationsmethoden unterordnen. Zur Lösung des Systems

$$\mathfrak{A}\,\mathfrak{x} = \mathfrak{r}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(III, 11)}$$

wähle man zwei $n \times n$-Matrizen $\mathfrak{B}$ und $\mathfrak{C}$ derart, dass

$$\mathfrak{B} + \mathfrak{C} = c\,\mathfrak{A}, \text{ wo } \mathfrak{C} \neq \mathfrak{D}, \quad \det \mathfrak{B} \neq 0, \quad c = \text{Zahl}.$$

Dann setzt man

$$(\mathfrak{B} + t\,\mathfrak{C})\,\mathfrak{x}(t) = c\,\mathfrak{r} + (t-1)\,\mathfrak{C}\,\mathfrak{v}, \quad . \quad . \quad . \quad . \quad \text{(III, 12)}$$

wo der Vektor $\mathfrak{v}$ beliebig ist.
Der Lösungsvektor $\mathfrak{x}(t)$ geht in den Lösungsvektor $\mathfrak{x}$ von (11) über für $t = 1$, d.h.

$$\mathfrak{x}(1) = \mathfrak{x}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(III, 13)}$$

Da die Lösung des Systems (12) Determinanten erfordert, ist $\mathfrak{x}(t)$ *eine rationale (Vektor-)Funktion von* t. Sind die (komplexen) Wurzeln von $\det(\mathfrak{B} + t\,\mathfrak{C}) = 0$ dem absoluten Betrage nach geordnet gleich

$$t_1, t_2, \ldots, t_n,$$

wo

$$|t_1| \leq |t_2| \leq \ldots \leq |t_n|,$$

so konvergiert $(\mathfrak{B} + t\,\mathfrak{C})^{-1}$ in eine Potenzreihe von t entwickelt sicherlich, wenn $|t| < |t_n|$, also konvergiert auch

$$\mathfrak{x}(t) = \mathfrak{x}(0) + t\,\mathfrak{x}'(0) + \tfrac{1}{2}t^2\,\mathfrak{x}''(0) + \ldots \text{ für } |t| < |t_n| \; . \quad \text{(III, 14)}$$

Setzt man also in dieser Reihe $t = 1$, so bekommt man nach (13) $\mathfrak{x}$. Zur Konvergenz von $\mathfrak{x} = \mathfrak{x}(1)$ ist also notwendig, dass die Wurzeln von $\det(\mathfrak{B} + t\,\mathfrak{C}) = 0$ absolut grösser als 1 sind.

Um daher $\mathfrak{x} = \mathfrak{x}(1)$ iterativ zu berechnen, hat man die Teilsummen der Reihe (14) zu bestimmen, also die Koeffizienten. Dazu differenziert man

(12) mehrmals nach t und setzt $t = 0$. Dann wird die nullte, erste usw. Ableitung:

$$\begin{aligned}
&\mathfrak{B}\,\mathfrak{x}(0) = c\,\mathfrak{r} - \mathfrak{C}\,\mathfrak{v}\\
&\mathfrak{C}\,\mathfrak{x}(0) + \mathfrak{B}\,\mathfrak{x}(0) = \mathfrak{C}\,\mathfrak{v}\\
&\mathfrak{C}\,\mathfrak{x}'(0) + \tfrac{1}{2}\,\mathfrak{B}\,\mathfrak{x}''(0) = 0\\
&\quad\cdots\cdots\cdots\cdots\\
&\frac{1}{(n-1)!}\,\mathfrak{C}\,\mathfrak{x}^{(n-1)}(0) + \frac{1}{n!}\,\mathfrak{B}\,\mathfrak{x}^{(n)}(0) = 0.
\end{aligned}$$

Durch Summieren bekommt man, wenn $\mathfrak{x}_n$ die n-te Teilsumme von (14) für $t = 1$ ist

$$\left.\begin{aligned}
&\mathfrak{B}\,\mathfrak{x}_0 = c\,\mathfrak{r} - \mathfrak{C}\,\mathfrak{v}\\
&\mathfrak{B}\,\mathfrak{x}_1 = c\,\mathfrak{r} - \mathfrak{C}\,\mathfrak{x}_0\\
&\quad\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&\mathfrak{B}\,\mathfrak{x}_n = c\,\mathfrak{r} - \mathfrak{C}\,\mathfrak{x}_{n-1}.\ \text{wo}\ \mathfrak{x}_n = \mathfrak{x}(0) + \mathfrak{x}'(0) + \ldots + \frac{1}{n!}\,\mathfrak{x}^{(n)}(0)
\end{aligned}\right\}\quad \text{(III, 15)}$$

Somit haben wir:

Satz. Der Iterationsprozess (15) konvergiert bei beliebigen $\mathfrak{v}$ dann und nur dann, wenn die Wurzeln t_i von $\det(\mathfrak{B} + t\,\mathfrak{C}) = 0$ absolut grösser als 1 sind.

Dieser Satz ist hervorgegangen aus einem Iterationsverfahren von PICONE, in welchem speziell $\mathfrak{v} = \mathfrak{v}$ war. Sein Schüler CESARI hat das Verfahren dann auf beliebige $\mathfrak{v}$ ausgedehnt.

Die andern bekannten Iterationsverfahren bekommt man aus diesem durch Spezialisierung. So entsteht das Verfahren der gewöhnlichen *Iteration*, also dasjenige bei überwiegender Diagonale (III, 7), wenn man oben in (III, 15)

$\mathfrak{v} = \mathfrak{v}$, $c = 1$, $\mathfrak{B} = \mathfrak{D}(a_{ii}) =$ Diagonalmatrix mit den Elementen a_{ii} setzt, also $\mathfrak{C} = \mathfrak{A} - \mathfrak{B}$. Dann wird in der Tat

$$a_{ii}\,x_{0,i} = r_i,\quad a_{ii}\,x_{k+1,i} = r_i - \sum_{k+1}^{n} a_{ik}\,x_{k,i}.$$

Für die anderen Untersuchungen CESARIS ist noch *die Methode von* MISES im Falle einer definiten Matrix $\mathfrak{A}$ wichtig. Sie entsteht, wenn in (III, 15) gesetzt wird:

$$\mathfrak{B} = E,\quad \mathfrak{C} = c\,\mathfrak{A} - E,\quad \mathfrak{v} = \text{beliebig}.$$

Sind nun λ_i die Wurzeln von $\det(\mathfrak{A} - \lambda E) = 0$, also kurz von $\mathfrak{A}$, und t_i die Wurzeln von $\det(\mathfrak{B} + t\,\mathfrak{C}) = 0 = \det(c\,t\,\mathfrak{A} - (t-1)E)$, so können die λ_i so numeriert werden, dass

$$t_i = 1 : (1 - c\,\lambda_i),\quad \cdots\cdots\quad \text{(III, 16)}$$

und, da zur Konvergenz des Prozesses (15) alle $|t_i| > 1$ sein müssen, so muss c so gewählt werden, dass

$$|1 - c\,\lambda_i| < 1.$$

Dies ist genau die Bedingung (III, 9) von MISES, wenn man c durch $-c$ ersetzt.

Untersuchung über die Schnelligkeit der Konvergenz. Wir hatten oben an Beispielen gesehen, wie langsam die Iterationsverfahren konvergieren. Es erhebt sich die Frage, wovon die Schnelligkeit der Konvergenz eigentlich abhängt. CESARI beschränkt seine Betrachtungen auf (positiv) definite Gleichungsmatrizen, um einen besseren Ueberblick über die Verhältnisse zu bekommen.

Sei also von jetzt an $\mathfrak{A}$ *eine positiv definite Matrix,* deren Wurzeln daher alle positiv sind: $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n > 0$. Wendet man die Methode von MISES an, so muss c so gewählt werden, dass $|1 - c\lambda_i| < 1$ für alle i, also $2 > c\lambda_i > 0$, d.h.: Damit (15) konvergiert, ist notwendig, dass $1/c > \frac{1}{2}\lambda_1$. Für $c = \lambda_1^{-1}$ konvergiert das Verfahren also sicher. Dann werden aber auch alle t_i positiv, weil

$$t_i = 1 : (1 - c\lambda_i) = 1 : (1 - \lambda_i/\lambda_1) > 0.$$

Da nun die λ_i nach abnehmender Grösse geordnet sind, so gilt für die t_i dasselbe. Das kleinste t_i, nämlich t_n, ist also gegeben durch

$$t_n = 1 : (1 - \lambda_n/\lambda_1). \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(III, 17)}$$

Nun ist aber t_n nach (14) der Konvergenzradius von der Reihe (14). Je näher dieser also an 1 liegt, umso schlechter konvergiert (14) für $t = 1$ offenbar. t_n aber ist umso kleiner, je kleiner λ_n/λ_1 ist. Somit haben wir den

Satz von CESARI. Für eine definite Matrix $\mathfrak{A}$ konvergiert das Verfahren von MISES umso schlechter, je grösser λ_1/λ_n ist, d.h. je mehr die Wurzeln von $\mathfrak{A}$ auseinander liegen.

In der Praxis muss zu einer befriedigenden Konvergenz $\lambda_1/\lambda_n \leq 3$ sein.

Obwohl obiger Satz nur für das Verfahren von MISES gilt, kann man doch wohl annehmen, dass *dieselben Verhältnisse auch für die übrigen Iterationsverfahren gelten,* umso mehr als sie alle Spezialfälle des Verfahrens von CESARI sind.

Methode zur Konvergenzbeschleunigung.

Will man für eine definite Matrix die Konvergenz irgendeines Iterationsverfahrens beschleunigen, so transformiere man $\mathfrak{A}\,\mathfrak{x} = \mathfrak{r}$ durch Multiplikation mit einer Matrix $\mathfrak{N}$ auf $\mathfrak{N}\,\mathfrak{A}\,\mathfrak{x} = \mathfrak{N}\,\mathfrak{r}$ und wähle $\mathfrak{N}$ so, dass die Wurzeln μ_i der Matrix $\mathfrak{M} = \mathfrak{N}\,\mathfrak{A}$ näher zusammenliegen als die von $\mathfrak{A}$. Denn der obige Satz von CESARI ist zwar nur abgeleitet für die Iteration von MISES. Wir dürfen aber annehmen, dass sich sonst die Sache nicht viel anders verhält. Man muss also dafür sorgen, dass $\mu_1/\mu_n < \lambda_1/\lambda_n$, m.a.W., dass der Quotient

$$q = \mu_1/\mu_n : \lambda_1/\lambda_n$$

möglichst klein wird.

Bei beliebigem $\mathfrak{N}$ hat man aber keine Sätze darüber, wie sich die

Wurzeln von $\mathfrak{N}\,\mathfrak{A}$ zu denen von $\mathfrak{A}$ verhalten. Man hat nur den Satz von CAYLEY, dass die Wurzeln μ_i von $f(\mathfrak{A})$, wenn $f(x)$ *ein Polynom* ist, gleich $\mu_i = f(\lambda_i)$ sind.

Man wählt daher oben $\mathfrak{N} = f(\mathfrak{A})$ und bekommt die Gleichungsmatrix $\mathfrak{M} = \mathfrak{A}\,f(\mathfrak{A}) \equiv F(\mathfrak{A})$.

Zu ihr gehört das Polynom $x \cdot f(x) = F(x)$. Man wählt nun $f(x)$ so, dass

$$f(x) > 0 \text{ im Intervall } (\lambda_1, \lambda_n) \ldots \ldots \text{(III, 18)}$$

Da alle $\lambda_i > 0$ sind, so ist in jenem Intervall auch $F(x) = xf(x)$ positiv, d.h. die Zahlen $F(\lambda_i) > 0$. Nun sind aber die $F(\lambda_i)$ die Wurzeln von $F(\mathfrak{A})$ und das Produkt aller charakteristischen Wurzeln einer Matrix ist gleich der Determinante der Matrix. Demnach ist $\det F(\mathfrak{A}) > 0$. Ferner ist

$$\det f(\mathfrak{A}) = \frac{\det F(\mathfrak{A})}{\det \mathfrak{A}} \neq 0.$$

Das System

$$f(\mathfrak{A})\,\mathfrak{y} = 0 \ldots \ldots \ldots \text{(III, 19)}$$

hat daher keine nichttriviale Lösung. Somit haben wir bisher das Ergebnis:

Das System

$$F(\mathfrak{A})\,\mathfrak{x} = f(\mathfrak{A})\,\mathfrak{r}, \ldots \ldots \ldots \text{(III, 20)}$$

wo $F(\mathfrak{A}) \equiv \overline{\mathfrak{A}}$ gemäss (18) gewählt ist und daher lauter positive Wurzeln $F(\lambda_i)$ hat, hat eine positive Determinante und ist wegen (19) inhomogen.

Nunmehr muss $F(x)$ *zweckmässig gewählt* werden. Die Wurzeln $\lambda_i' = F(\lambda_i)$ von $F(\mathfrak{A})$ sollen ein kleineres Abstandsverhältnis haben als die λ_i selber, d.h. es soll

$$q = \frac{\lambda'_{\max}}{\lambda'_{\min}} : \frac{\lambda_{\max}}{\lambda_{\min}} = \frac{\lambda'_{\max}}{\lambda'_{\min}} : \frac{\lambda_1}{\lambda_n} \quad \ldots \ldots \text{(III, 21)}$$

zwischen 0 und 1 liegen und möglichst klein sein unter Berücksichtigung von (18). Wenn nun

$$F(x) \text{ im Intervall } (\lambda_n, \lambda_1) \text{ monoton} \ldots \ldots \text{(III, 22)}$$

wäre, so wäre

$$\left.\begin{aligned} \lambda'_{\max} &= F(\lambda_1),\ \lambda'_{\min} = F(\lambda_n) \text{ bzw.} \\ \lambda'_{\max} &= F(\lambda_n),\ \lambda'_{\min} = F(\lambda_1), \end{aligned}\right\} \ldots \ldots \text{(III, 23)}$$

je nachdem ob $F(x)$ steigt oder fällt im Intervall.

Da sich beim Polynom $F(x)$ sowohl als bei $f(x)$ bei der Quotientenbildung ein Faktor heraushebt, so können wir den Leitkoeffizienten gleich ± 1 setzen. — Wir haben ferner als einzige Schätzung für

$$\left.\begin{aligned} \lambda'_{\max} &= \max F(x) \\ \lambda'_{\min} &= \min F(x) \end{aligned}\right\} \text{ in } (\lambda_n, \lambda_1).$$

Diese Werte werden natürlich nur ausnahmsweise erreicht. Denn z.B. wird das max $F(x)$ nicht gerade ein λ_i zur Abszisse haben.

Nehmen wir nun zunächst $f(x)$ *linear:* $f(x) = b \pm x$.

I. *Fall.* $f(x) = b - x$. Wegen (18) muss dann $b > \lambda_1$, etwa $b = \lambda_1 + \alpha$, wo $\alpha > 0$. Das Maximum von $F(x)$ liegt bei $x = \frac{1}{2}(\lambda_1 + \alpha)$. $F(x)$ kann aber nur dann monoton in (λ_n, λ_1) sein, wenn das Maximum rechts von λ_1 liegt, d.h. wenn $\alpha \geqq \lambda_1$. Dann wird $F(x) = 2\lambda_1 x - x^2$. Wir haben dann Fall (23[1]), und es ist $F(\lambda_1) = \lambda_1^2$, $F(\lambda_n) = \lambda_n\ (2\lambda_1 - \lambda_n)$ und $q = \lambda_1/(2\lambda_1 - \lambda_n) > \frac{1}{2}$. *Die Konvergenzverbesserung ist also nicht besonders wirkungsvoll.*

Wir versuchen, die Konvergenz zu verbessern, indem wir die Forderung der Monotonie fallen lassen. Dann ist $\max F(x) = \frac{1}{4}(\lambda_1 + \alpha)^2$. Für $\min F(x)$ jedoch erhält man zwei Werte, zwischen denen man entscheiden muss. Dies ist aber ohne Kenntnis von λ_1 und λ_n unmöglich. Jedoch fallen beide Werte zusammen, wenn $\alpha = \lambda_n$. Wir bekommen also

$$F(x) = (\lambda_1 + \lambda_n)x - x^2.$$

$$\max F(x) = \tfrac{1}{4}(\lambda_1 + \lambda_n)^2,\ \min F(x) = \lambda_1 \lambda_n,$$

daher

$$q = \tfrac{1}{4}\left(1 + \frac{\lambda_n}{\lambda_1}\right)^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(III, 24)}$$

Für grosses λ_1/λ_n ist also $q \approx \frac{1}{4}$.

Ein grosses Intervall (λ_n, λ_1) *wird somit gewissermassen auf den vierten Teil komprimiert.* Ist jedoch λ_n mit λ_1 vergleichbar, so nähert sich q rasch der 1, und das Verfahren wird *zwecklos.*

Nun kennt man freilich λ_1 und λ_n meistens nicht und muss sich daher mit Abschätzungen begnügen. Die bekannteste solche ist

$$\lambda_1 \leqq S = \max \sum_{k=1}^{n} |a_{ik}|, \quad i = 1, \ldots, n. \quad . \quad . \quad \text{(III, 25)}$$

Diese Schätzung ist aber so roh, dass man ruhig auch $\lambda_1 + \lambda_n$ durch S ersetzen kann, also schreiben kann

$$F(x) = Sx - x^2 = (S - x)x. \quad . \quad . \quad . \quad . \quad \text{(III, 26)}$$

Auch dann ist $q \approx \frac{1}{4}$.

In manchen Fällen wird S ein viel zu grosser Wert sein, und es ist möglich, dass die *Spur* s wesentlich kleiner ist. Dann setzt man

$$\alpha = \lambda_2 + \ldots + \lambda_{n-1},$$

also

$$F(x) = sx - x^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(III, 26')}$$

Auf die Matrix $F(\mathfrak{A})$ in (20) kann man die Transformation (26) oder (26') nochmals anwenden, natürlich nunmehr mit dem *neuen* S oder s. Dann wird die Gesamtkompression der beiden Transformationen schwächer als $1/16$, also etwa $1/10$.

Die Transformation (26) oder (26') kostet auf der linken Seite von (20): $2n^3$ Multiplikationen, auf der rechten Seite n^2 Multipl. im ganzen also $2n^3 + n^2$ Multiplikationen.

II. *Fall.* $f(x) = b + x$. Dies erweist sich bei näherer Durchführung als unmöglich.

Man könnte nun $f(x)$ von höherem Grade ansetzen. CESARI gibt z.B.

für $F(x)$ eine Funktion vom Grade 3 und vom Grade 4 und bekommt im ersten Falle, bzw. zweiten Falle, dass $q \approx \frac{1}{9}$ bzw. $q \approx \frac{1}{16}$. Der zweite Fall kostet aber $4n^3 + n^2$ Multiplikationen, also fast genau doppelt so viel wie zwei Transformationen (26), welche ja zusammen fast ebenso wirksam sind. Diese Transformationen sind also der Transformation (26) nur dann überlegen, wenn $\lambda_1/\lambda_n \approx 1$ ist.

Ergebnis. 1. Der Gedankengang CESARI's gestattet, durch die Transformation (III, 26 oder 26') die Wurzeln einer definiten Matrix $\mathfrak{A}$ in ein kleineres Intervall zusammenzudrängen und dadurch die Konvergenz jedes Iterationsverfahrens zu erhöhen.

2. Dieser Kunstgriff ist am wirksamsten, wenn die Wurzeln von $\mathfrak{A}$ weit auseinander liegen, so dass irgendein Iterationsverfahren überhaupt nicht (oder nur äusserst langsam) konvergiert. Dann kann nach Anwendung von ein oder zwei Transformationen die Konvergenz erreicht werden. — Liegen aber die Wurzeln von $\mathfrak{A}$ nahe beinander, so ist der Kunstgriff wirkungslos.

3. Jede solche Transformation kostet freilich $2n^3 + n^2$ Multiplikationen, also etwa 6 mal so viel wie die gesamte Auflösung des Systems nach der GAUSSschen Methode. Wobei man noch zu berücksichtigen hat, dass bei einem iterativen Verfahren die Rechnung nach jenen Transformationen überhaupt erst beginnt.

Als *Beispiel* nehmen wir das erste obige Gleichungssystem. Hier ist das Maximum der Spaltensummen $S = 3$. Wir setzen aber, da wir λ_n nicht kennen und auch keine Abschätzung dafür haben, in der Funktion (III, 26) $S = 3{,}3$ und bekommen

$$f(\mathfrak{A}) = 3{,}3\,E - \mathfrak{A} = \begin{pmatrix} 2{,}163 & -0{,}981 & -0{,}873 \\ -0{,}981 & 2{,}059 & 0{,}735 \\ -0{,}873 & -0{,}735 & 1{,}947 \end{pmatrix}.$$

Die Anwendung von (20) liefert dann das zweite obige Gleichungssystem. Die beigefügten Aufstellungen geben eine Uebersicht über den Grad der Konvergenzverbesserung, der in der Tat beträchtlich ist.

	Ohne Transformation			Mit Transformation		
	Iterat.	MISES	SEIDEL	Iterat.	MISES	SEIDEL
$\mathfrak{x}''$	—	1,9	5,0	1,66	0,7	1,66
	—	1,9	0,6	1,29	0,8	0,87
	—	2,1	1,16	2,48	1,95	2,72
$\mathfrak{x}^{(7)}$	—	—	—	1,0066	1,07	1,0003
	—	—	—	1,991	1,93	1,99986
	—	—	—	2,995	2,97	2,99997
$\mathfrak{x}^{(11)}$	—	1,47	1,72			
	—	1,74	1,95			
	—	2,78	2,95			

Mathematics. — *On the construction of simple perfect squared squares.* By C. J. BOUWKAMP. (Communicated by Prof. J. G. VAN DER CORPUT.)

(Communicated at the meeting of November 29, 1947.)

First of all, I have to apologize for having misled the readers of these Proceedings with regard to the construction of a simple perfect squared square as given in "paper A" by the authors of the preceding note. Indeed, my criticism of their method has proved wholly unjustifiable. Therefore, I have to withdraw part of section 8 of my paper III. Nevertheless, the discussion there given, upon appropriate changes, remains of value. To make the following independent of all previous results, it is worth while here to reproduce figs 12 and 13 of my paper III.

Consider the "rotor" network of fig. 12, with terminals A_1, A_2, A_3. Let its wires have unit conductance, and let currents $87a$, $87b$ leave the network at A_2, A_3, respectively. The current entering at A_1 must then be $87\ (a+b)$. The complete set of currents is uniquely determined, and is shown in fig. 12. The currents are integral linear combinations of a and b. Without lack of generality, we may suppose a and b to be integers, subject to $0<b<a$.

This network is a generalization of the "polar" networks treated before, in so far that now more than two terminals are present. It corresponds to a *squared polygon* of angles $\pi/2$ and $3\pi/2$. For example, the rotor network of fig. 12, in action, corresponds to a squared polygon P the dimensions of which are shown in fig. 13.

The vertical left side of the polygon may be considered as the terminal A_1, and the remaining vertical boundaries at the right correspond to A_3, A_2. The current flows horizontally from left to right. The ingoing current $87(a+b)$ is equal to the left vertical side, the two outgoing currents $87a$, $87b$ are equal to the other vertical boundaries.

The typical corner elements C_1, C_2 (shaded in fig. 13) have sides $27a—8b$, $8a+35b$, respectively. It must be noted that the situation of fig. 13 is possible only if $27a—8b < 49\ (a—b)$ and $8a+35b < 87b$; thus $41/22<a/b<13/2$. Otherwise at least one of the corner elements is too large. If the inequality above is not fulfilled, it is impossible to draw in fig. 13 the rectangle R which is important in the further construction.

If the skew-symmetrical rotor network of fig. 12 is replaced by its reflection (leaving the currents at A_1, A_2, A_3 invariant), the new squared polygon P' will have the same shape as the old one P; this follows from the triad symmetry of the rotor. The set of currents in the reflected rotor are easily found from those in fig. 12. We could also have interchanged

a and b, without reflecting the rotor; we prefer, however, the former method, in order to have always $a > b$ in the following.

Let us now re-consider fig. 13. By varying the ratio a/b we change the shape of the polygon P. The rectangle R becomes a square if and only if

$$87b - (8a+35b) = 49(a-b) - (27a-8b);$$

that is, if $10a = 31b$. Therefore, let us take $a = 31$, $b = 10$. It is then easily verified that of the elements of the polygons P and P' no two are equal. Moreover, the polygon P' can be brought into such position with respect to P that it overlaps P in the latter's corner elements $C_1 = 757$,

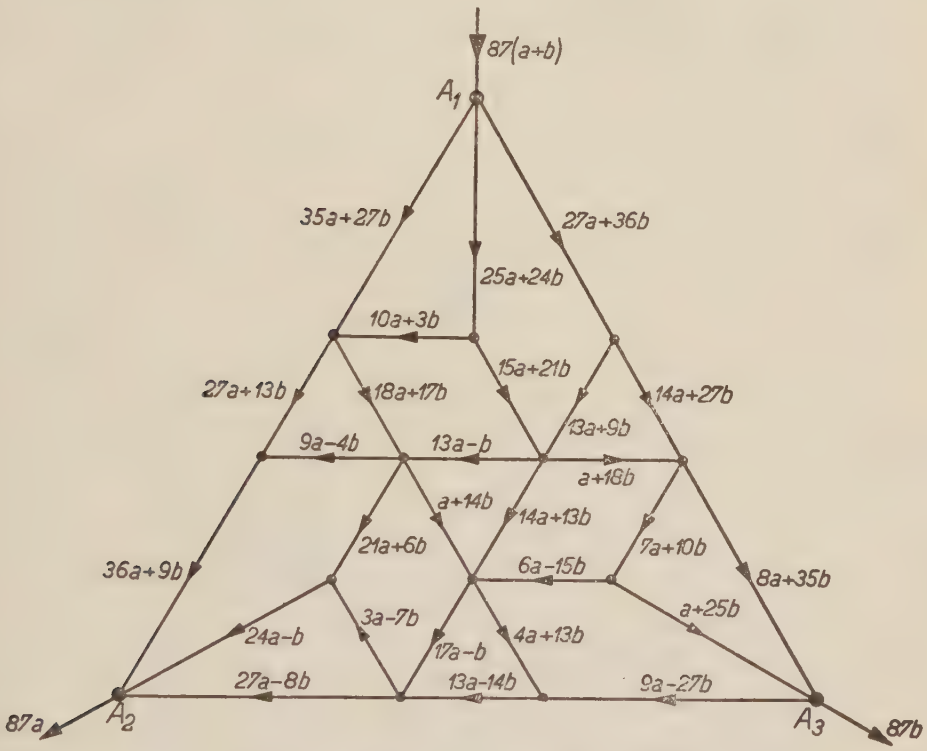


Fig. 12. Currents in a typical rotor network.

$C_2 = 598$. *Now, the important point is that these overlapped parts need not coincide with the corner elements of* P', which are of sides $27a-9b = 747$ and $9a+36b = 639$. Consequently, if the full-drawn parts of the boundaries of C_1 and C_2 are removed, the two systems of elements of P and P' fit together. The additional squares, 1901 and 1940, at the left-upper and the right-lower side, respectively, together with the square $R = 272$ at the middle then complete the square, which in fact is a *simple* perfect squared square of order 55. Upon turning the square over an angle π and codifying it, we obtain the solution given by the authors of the

preceding note; the elements there distinguished by an asterisk are those of P'.

Attention may be drawn to a second solution, obtained when the roles of P and P' are interchanged. In that case the shape of P' is fixed by $a = 91$, $b = 10$ — in order that the corresponding rectangle R' shall be a square ($R' = 762$). The final result is a simple perfect squared square of code

(5739, 3555, 2860, 4022) (695, 2165) (2153, 2097) (1003, 3019)
(1152, 2016) (56, 1516, 525) (2209) (3433, 2306) (1677)
(1462, 3573) (749, 767) (81, 947, 649) (1307, 871, 762, 18) (866)
(1127, 1179) (2111) (109, 653) (436, 544) (1813) (2891, 1669)
(1617, 1305) (1197) (312, 2105, 85) (2020, 5562) (1222, 2376)
(4113) (583, 3542) (2959).

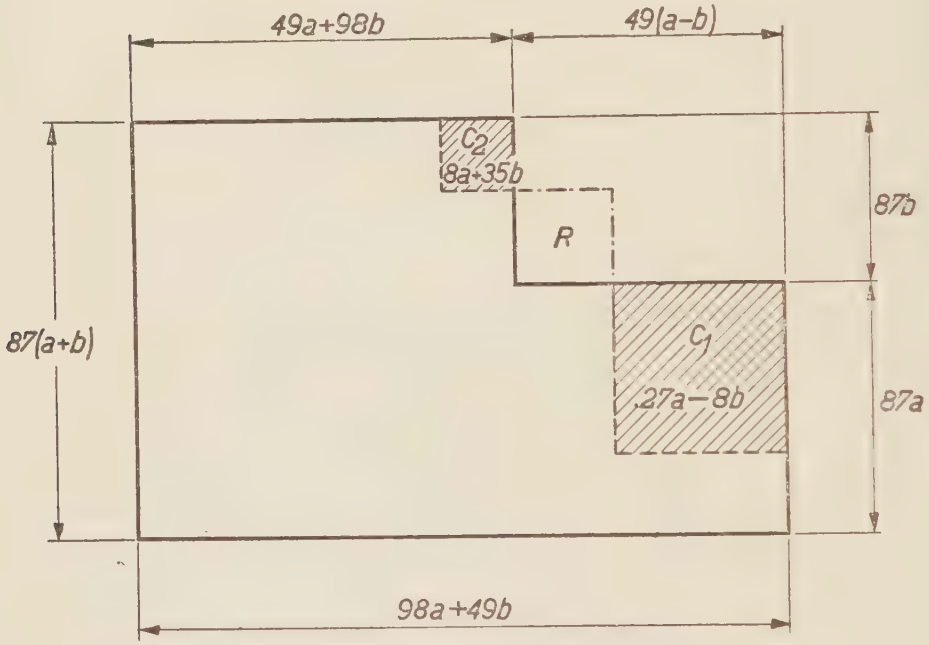


Fig. 13. Dimensions of the polygon corresponding to the rotor of fig. 12.

In conclusion, I am much indebted to Mr TUTTE and co-workers for calling attention to my incorrect interpretation of their construction. Moreover, Mr TUTTE kindly pointed out to me an error in my list of squared rectangles. The very last squaring of the list in my paper II ($C = 1176$) is unprimed as if it were perfect. Actually, it contains two equal elements of side 7. Consequently, there are not 214 but 213 simple perfect squared rectangles of order 13; this number is in complete agreement with Mr TUTTE's unpublished results, as he kindly informed me. In this con-

nection table I of paper I should read as follows (of course, corresponding corrections have to be made in the text):

TABLE I.

Numbers of squared rectangles of different type, and of order less than 14. Only the trivial imperfections are excluded (the latter show equal elements lying aside, and thus belong to the compound type).

Type \ Order	9	10	11	12	13	Totaal
Simple, perfect	2	6	22	67	213	310
Simple, imperfect	1	0	0	9	34	44
Trivially compound, perfect	0	4	16	60	194	274
Non-trivially compound, perfect . .	0	0	0	0	1	1
Trivially comp., non-trivially imperfect	0	2	2	2	20	26
Non-triv. compound, non-triv. imperf.	0	0	0	0	2	2
Perfect	2	10	38	127	408	585
Non-trivially imperfect	1	2	2	11	56	72
Total	3	12	40	138	464	657

Eindhoven, October 1947.

Natuurkundig Laboratorium der N.V. Philips' Gloeilampenfabrieken.

Mathematics. — *A simple perfect square.* By R. L. BROOKS, C. A. B. SMITH, A. H. STONE and W. T. TUTTE. (Communicated by Prof. J. G. VAN DER CORPUT.)

(Communicated at the meeting of November 29, 1947.)

In our paper "The Dissection of Rectangles into Squares" (Duke Mathematical Journal, vol. 7 (1940), pp. 312—340) we gave a method (pp. 332—334) for the construction of "simple perfect squares". (A "perfect square" is a dissection of a square into a finite number of non-overlapping smaller squares, called *elements*, no two of which are equal. A perfect square is "*simple*" if no proper subset of the elements, containing more than one element, forms a rectangle.)

In a recent paper in these Proceedings ("On the Dissection of Rectangles into Squares, III", vol. 50, pp. 72—78), C. J. BOUWKAMP has criticized this method, attempting to prove that it fails for the "rotor" we gave as an example (Fig. 11 in our paper), and suggesting that it is unlikely ever to succeed.

However, the method does in fact succeed for the given rotor. To clear up the misunderstanding we give the resulting simple perfect square below, in BOUWKAMP's very concise notation. We attach asterisks to the bracketted expressions corresponding to a particular one of the two "congruent squared polygons" involved.

(1940, 1206, 967, 1355), (239, 728), (734, 711), (340, 1015), (393, 675), (23, 517, 171), (1165, 785, 747)*, (564), (493, 1197), (263, 254), (36, 317, 211), (443, 304)*, (9, 281), (272), (380, 405)*, (704), (22, 209, 639)*, (139, 187)*, (977, 568)*, (543, 444)*, (396)*, (1901), (99, 707, 34)*, (673)*, (409, 801)*, (1386)*, (193, 1187)*, (994)*.

The elements not contained in either of the two polygons are those of sides 1940, 1901 and 272. Two elements (of sides 757 and 598) have been omitted from the polygon not distinguished by an asterisk.

Apparently Dr. BOUWKAMP has overlooked the fact that the overlapping regions of the two "squared polygons" need not coincide with elements in *both* polygons. It is sufficient that each such overlapping region should coincide with an element (which is then omitted) in *one* polygon. Our exposition of the method was probably over-condensed: we thought that our statement (p. 334) that the resulting perfect square was "uncrossed" would make our meaning clear.

It may be remarked, in conclusion, that the "simple uncrossed perfect square" just described does show one symptom of its symmetrical origin; one diagonal of the "central" element (of side 272) lies on one main diagonal of the complete square. By more complicated methods it is possible to remove even this feature. We hope that a full description of these methods will be published later by one of us (W.T.T.).

Trinity College, Cambridge, England.

Mathematics. — *On the figure of four projective spaces* $[n_1-1]$, $[n_2-1]$, $[n_3-1]$ *and* $[n_4-1]$ *in a* $[n-1]$, *where* $n_1+n_2+n_3+n_4=2n$. II. By G. H. A. GROSHEIDE F.WZN. (Communicated by Prof. J. A. SCHOUTEN.)

(Communicated at the meeting of September 27, 1947.)

11. Using a common phrase we can summarize the contents of this section as follows:

(GP) : *We suppose that A, B, C and D are spaces in general position.*

An exact examination of the meaning of the expression "in general position" shows that our assumption contains the following suppositions.

If $[n_{i_1}-1]$, $[n_{i_2}-1]$, $[n_{i_3}-1]$, $[n_{i_4}-1]$ is an arbitrary permutation of the four spaces A, B, C, D for which

$$n-n_{i_1}-n_{i_2}=n_{i_3}+n_{i_4}-n\geqq 0$$

then

(GP_1) : the projective space of lowest dimension containing both $[n_{i_1}-1]$ and $[n_{i_2}-1]$ is $(n_{i_1}+n_{i_2}-1)$-dimensional.

(GP_2) : the projective space of highest dimension contained both in $[n_{i_3}-1]$ and in $[n_{i_4}-1]$ is $(n_{i_3}+n_{i_4}-n-1)$-dimensional.

(GP_3) : the $[n_{i_1}+n_{i_2}-1]$ and the $[n_{i_3}+n_{i_4}-n-1]$ introduced just now have no common points.

Hereat we remark that two spaces have a (-1)-dimensional space as intersection if they have no points in common. Besides we notice that the first supposition is equivalent with

(GP_1^*) : the projective spaces $[n_{i_1}-1]$ and $[n_{i_2}-1]$ have no common points.

The validity of (GP) can be expressed by a number of three inequalities, in keeping with the fact that the four spaces can be divided on three manners into pairs of two.

Putting $i_1=1$, $i_2=2$, $i_3=3$, $i_4=4$ we arrive at

(GPA): $$(d^{n_4}c^{n-n_4})(c^{n_3+n_4-n}a^{n_1}b^{n_2})\neq 0.$$

Putting $i_1=1$, $i_2=3$, $i_2=2$, $i_4=4$ we obtain

(GPB): $$(d^{n_4}b^{n-n_4})(b^{n_2+n_4-n}a^{n_1}c^{n_3})\neq 0.$$

Finally we have

if $T=n_1<n-n_4$ for $i_1=1$, $i_2=4$, $i_3=2$, $i_3=3$

$$(c^{n_3}b^{n-n_3})(b^{n_2+n_3-n}a^{n_1}d^{n_4})\neq 0,$$

if $T = n - n_4 < n_1$ for $i_1 = 2,\ i_2 = 3,\ i_3 = 1,\ i_4 = 4$

$$(d^{n_4}\, a^{n-n_4})\,(a^{n_1+n_4-n}\, b^{n_3}\, c^{n_3}) \neq 0,$$

if $T = n_1 = n - n_4$ on account of (GP_1^*)

$$(d^{n_4}\, a^{n_1})\,(b^{n_3}\, c^{n_3}) \neq 0,$$

and thus irrespective the value of T

$$(GPC): \qquad I_T = (d^{n_4}\, a^T\, b^{n-n_4-T})\,(c^{n_3}\, a^{n_1-T}\, b^{n-n_1-n_3+T}) \neq 0.$$

Since all points of a line transversal of A, B, C and D belong to the $[n_{i_1} + n_{i_2} - 1]$ that joins $[n_{i_1} - 1]$ and $[n_{i_2} - 1]$, from (GP_3) it follows that such a line (if present) has no common point with the $[n_{i_3} + n_{i_4} - n - 1]$ defined by $[n_{i_3} - 1]$ and $[n_{i_4} - 1]$. Hence we are entitled to pronounce the

Lemma. *If l is a line tranversal of A, B, C and D, then the intersection points of l with these four spaces are mutual different.*

12. Let now be given an arbitrary point P_a

$$a_1\{y^{(1)}\} + a_2\{y^{(2)}\} + \ldots + a_{n_1}\{y^{(n_1)}\} = \sum_{i=1}^{n_1} a_i\{y^{(i)}\}$$

of A and likewise an arbitrary point P_b

$$\beta_1\{z^{(1)}\} + \beta_2\{z^{(2)}\} + \ldots + \beta_{n_2}\{z^{(n_2)}\} = \sum_{j=1}^{n_2} \beta_j\{z^{(j)}\}$$

of B (according to (GP_1^*) necessarily distinct from P_a). Then the straight line $P_a P_b$

$$\{x\} = \lambda \sum_i a_i\{y^{(i)}\} + \mu \sum_j \beta_j\{z^{(j)}\}$$

joining P_a and P_b meets C in a point P_c if and only if the equations

$$\lambda \sum_i a_i\,(v'_{(k)}\, y^{(i)}) + \mu \sum_j \beta_j\,(v'_{(k)}\, z^{(j)}) = 0 \qquad (k = 1, 2, \ldots, n - n_3)$$

have a common solution $(\lambda_c,\ \mu_c)$. In the same manner there exists an intersection point P_d of $P_a P_b$ with D if and only if the equations

$$\lambda \sum_i a_i\,(w'_{(l)}\, y^{(i)}) + \mu \sum_j \beta_j\,(w'_{(l)}\, z^{(j)}) = 0 \qquad (l = 1, 2, \ldots, n - n_4)$$

have a common solution $(\lambda_d,\ \mu_d)$. Thus a necessary and sufficient condition for the presence of a line transversal of A, B, C and D is the resolvability of the equations

$$\left.\begin{aligned} &\lambda_d \sum_i a_i\,(w'_{(l)}\, y^{(i)}) + \mu_d \sum_j \beta_j\,(w'_{(l)}\, z^{(j)}) = 0 \\ &\lambda_c \sum_i a_i\,(v'_{(k)}\, y^{(i)}) + \mu_c \sum_j \beta_j\,(v'_{(k)}\, z^{(j)}) = 0 \\ &(l = 1, 2, \ldots, n - n_4;\ k = 1, 2, \ldots, n - n_3) \end{aligned}\right\} \quad \ldots \quad (9)$$

with indeterminates $a_1, a_2, \ldots, a_{n_1};\ \beta_1, \beta_2, \ldots, \beta_{n_2};\ \lambda_c, \mu_c;\ \lambda_d, \mu_d$.

On account of the Lemma proved in the preceding section an arbitrary

solution of (9) will deliver two points P_c and P_d, which neither coincide with P_a or P_b. This means that there exist no solutions of (9) with λ_c, μ_c, λ_d or $\mu_d = 0$ and so we may put $\lambda_c = \mu_c = 1$; $\lambda_d = S\,\mu_d$, where S is a new indeterminate that supersedes (λ_d, μ_d).

Now we consider

$$\left.\begin{aligned} S\sum_i a_i\,(w'_{(l)}\,y^{(i)}) + \sum_j \beta_j\,(w'_{(l)}\,z^{(j)}) &= 0 \\ \sum_i a_i\,(v'_{(k)}\,y^{(i)}) + \sum_j \beta_j\,(v'_{(k)}\,z^{(j)}) &= 0 \\ (l = 1, 2, \ldots, n-n_4;\ k = 1, 2, \ldots, n-n_3) & \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (9^*)$$

as a system of $2n-n_3-n_4 = n_1 + n_2$ linear homogeneous equations in the $n_1 + n_2$ variables $(\alpha_1, \alpha_2, \ldots, \alpha_{n_1}, \beta_1, \beta_2, \ldots, \beta_{n_2})$ and observe that a solution as desired exists if and only if the determinant on the coefficients of (9*) vanishes. This imposes on S the condition

$$\begin{matrix} n-n_4 \\ n-n_3 \end{matrix}\left\{\begin{array}{c|c} \overbrace{S\,(w'_{(l)}\,y^{(i)})}^{n_1} & \overbrace{(w'_{(l)}\,z^{(j)})}^{n_2} \\ \hline (v'_{(k)}\,y^{(i)}) & (v'_{(k)}\,z^{(j)}) \end{array}\right| = 0, \quad . \quad . \quad . \quad . \quad (10)$$

or if we introduce complex-symbols (as allowed!)

$$\begin{matrix} n-n_4 \\ n-n_3 \end{matrix}\left\{\begin{array}{c|c} \overbrace{S\,(d'\,a)}^{n_1} & \overbrace{(d'\,b)}^{n_2} \\ \hline (c'\,a) & (c'\,b) \end{array}\right| = 0.$$

Simultaneous expansion with respect to the first $n-n_4$ rows gives

$$\sum_{p=0}^{T} (-1)^{(n_1-p)(n-n_4-p)} \binom{n_1}{p} \binom{n_2}{n-n_4-p} \times \left|\begin{array}{c|c} \overbrace{S\,(d'\,a)}^{p} & \overbrace{(d'\,b)}^{n-n_4-p} \\ \hline (c'\,a) & (c'\,b) \end{array}\right| \times$$

$$\times \left|\begin{array}{c|c} \overbrace{(c'\,a)}^{n_1-p} & \overbrace{(c'\,b)}^{n-n_1-n_3+p} \end{array}\right| = \sum_{p=0}^{T} (-1)^{(n_1-p)(n-n_4-p)} \binom{n_1}{p} \binom{n_2}{n-n_4-p} \times$$

$$\times (n-n_4)!\,(n-n_3)!\,(d'\,a)^p\,(d'\,b)^{n-n_4-p}\,(c'\,a)^{n_1-p}\,(c'\,b)^{n-n_1-n_3+p}\,S^p = 0,$$

where T is the integer introduced in section 10.

We simplify the coefficients putting

$$g^{n_3\,n_4}_{n_1\,n_2,\,p} = (-1)^{(n_1-p)(n-n_4-p)} \binom{n_1}{p} \binom{n_2}{n-n_4-p}$$

and then after multiplication with $\dfrac{n_4!}{(n-n_4)!} \cdot \dfrac{n_3!}{(n-n_3)!}$ we obtain

$$\sum_{p=0}^{T} g^{n_3\,n_4}_{n_1\,n_2,\,p}\,(d^{n_4}\,a^p\,b^{n-n_4-p})\,(c^{n_3}\,a^{n_1-p}\,b^{n-n_1-n_3+p})\,S^p = 0. \quad . \quad . \quad (11)$$

According to (*GPC*) this represents a non vanishing equation in S of

degree T, the roots of which we call $S_1, S_2, \ldots, S_T$. Since from (GP_1^*) it follows that there exists no solution of (9*) different from $(0, 0, \ldots, 0)$ with $\beta_1 = \beta_2 = \ldots = \beta_{n_3} = 0$ and since (GP_3) shows that a solution with $\alpha_1 = \alpha_2 = \ldots = \alpha_{n_1} = 0$ is not present, all T roots S_i of (11) are useful for us.

13. We suppose in this and the following sections that the T roots S_i of (11) are mutual different. Then putting in (9*) $S = S_i$ we obtain a system of equations with a single solution $(\alpha_1, \alpha_2, \ldots, \alpha_{n_1}, \beta_1, \beta_2, \ldots, \beta_{n_2})_{(i}$ that furnishes two points $P_a^{(i)}$ and $P_b^{(i)}$ which inversely determine S_i. Hence with every root S_i there corresponds a single line transversal l_i and in this case A, B, C and D possess T different line transversals [7]).

As the introduction of S in section 12 shows, the value of S_i is just equal to the cross ratio $(P_d^{(i)} P_c^{(i)} P_b^{(i)} P_a^{(i)})$ of the four intersection points on l_i. Now the considerations that led to (11) remain valid if we replace the assumption (2) by the fainter supposition

$$n_1 + n_2 \leqq n \quad ; \quad n_2 + n_4 \geqq n.$$

Thus on account of

$$n_1 + n_3 \leqq n \quad ; \quad n_3 + n_4 \geqq n$$

after changing the spaces B and C we can follow the same way. Consequently the cross ratios

$$(P_d^{(i)} P_b^{(i)} P_c^{(i)} P_a^{(i)}) = 1 - S_i \qquad (i = 1, 2, \ldots, T)$$

shall be roots of the equation

$$\sum_{q=0}^{T} g_{n_1 n_3, q}^{n_3 n_4} (d^{n_4} a^q c^{n-n_4-q}) (b^{n_2} a^{n_1-q} c^{n-n_1-n_3+q}) x^q = 0. \quad . \quad . \quad (12)$$

If we substitute in (11) $S = 1 - x$ we must arrive at an equation

$$\sum_{p=0}^{T} g_{n_1 n_3, p}^{n_3 n_4} (d^{n_4} a^p b^{n-n_4-p}) (c^{n_3} a^{n_1-p} b^{n-n_1-n_3+p}) (1-x)^p =$$

$$= \sum_{p=0}^{T} \sum_{q=0}^{p} (-1)^q \binom{p}{q} g_{n_1 n_3, p}^{n_3 n_4} (d^{n_4} a^p b^{n-n_4-p}) (c^{n_3} a^{n_1-p} b^{n-n_1-n_3+p}) x^q = 0$$

with the same roots as (12). Since the coefficients of x^T

$$(-1)^{(n_1-T)(n-n_4-T)} \binom{n_1}{T} \binom{n_3}{n-n_4-T} (d^{n_4} a^T c^{n-n_4-T}) (b^{n_2} a^{n_1-T} c^{n-n_1-n_3+T})$$

and

$$(-1)^{T+(n_1-T)(n-n_4-T)} \binom{T}{T} \binom{n_1}{T} \binom{n_2}{n-n_4-T} (d^{n_4} a^T b^{n-n_4-T}) (c^{n_3} a^{n_1-T} b^{n-n_1-n_3+T})$$

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7) Compare Math. Enc. III 2, 2 A, p. 815.

respectively, for $T = n - n_4 \leqq n_1$ are equal, apart from a factor

$$\varepsilon = (-1)^{n-n_4+n_2(n-n_2)+n_3(n_1+n_4-n)}$$

in this case the corresponding coefficients in both equations differ only by a factor ε. As $\sum_{p=0}^{T}\sum_{q=0}^{p} = \sum_{q=0}^{T}\sum_{p=q}^{T}$ we can deduce from this fact the identities

$$\left.\begin{aligned} g^{n_2 n_4}_{n_1 n_3, q}\,(d^{n_4} a^q c^{n-n_4-q})\,(b^{n_2} a^{n_1-q} c^{n-n_1-n_2+q}) \equiv \\ \equiv \varepsilon\,(-1)^q \sum_{p=q}^{n-n_4} \binom{p}{q}\, g^{n_3 n_4}_{n_1 n_2, p}\,(d^{n_4} a^p b^{n-n_4-p})\,(c^{n_3} a^{n_1-p} b^{n-n_1-n_3+p}) \\ (q = 0, 1, 2, \ldots, n - n_4 \leqq n_1). \end{aligned}\right\} \quad (13)$$

Since $n_1 + n_3 \leqq n$; $n_3 + n_4 \geqq n$ by means of considerations similar with those of section 12 we obtain

$$\left.\begin{aligned} \Delta = \frac{n_2!}{(n-n_2)!}\,\frac{n_3!}{(n-n_3)!} \begin{vmatrix} x\,(c'a) & (c'd) \\ (b'a) & (b'd) \end{vmatrix} = \\ = \sum_{q=0}^{n_1} g^{n_2 n_3}_{n_1 n_4, q}\,(c^{n_3} a^q d^{n-n_3-q})\,(b^{n_2} a^{n_1-q} d^{n-n_1-n_2+q})\, x^q \end{aligned}\right\} \quad . \quad . \quad (14)$$

This formula is true independent from the values of n_1 and n_4 and thus among others for $T = n - n_4 < n_1$. Then, however, after the substitution $x = 1$, Δ has become a determinant of rank $\leqq n$ and thus $x = 1$ is at least a $(n_1 + n_4 - n)$-fold root of $\Delta = 0$. Therefore in this case there exist also the identities

$$\left.\begin{aligned} \sum_{q=i}^{n_1} \binom{q}{i}\, g^{n_2 n_3}_{n_1 n_4, q}\,(c^{n_3} a^q d^{n-n_3-q})\,(b^{n_2} a^{n_1-q} d^{n-n_1-n_2+q}) \equiv 0 \\ (i = 0, 1, \ldots, n_1 + n_4 - n - 1). \end{aligned}\right\} \quad . \quad . \quad (15)$$

In the interest of an application in section 16 we notice that the results of section 12 hold irrespective the value of $n_2 + n_4$, if we replace in (11) the zero that indicates the minimum value of p by L and add

$$L = 0 \qquad \text{for} \quad n_2 + n_4 \geqq n$$
$$L = n - n_2 - n_4 \quad \text{for} \quad n_2 + n_4 \leqq n.$$

14. If we choose a $[n_3 - 1]$ through the T points

$$\{y^{(i)}\} + \{z^{(i)}\} \qquad (i = 1, 2, \ldots, T)$$

as C and a $[n_4 - 1]$ through the points

$$\omega_i \{y^{(i)}\} + \{z^{(i)}\} \qquad (i = 1, 2, \ldots, T)$$

as D, where ω_i are arbitrary real or complex numbers, then the equation in S corresponding to the four spaces A, B, C and D has the roots ω_i. Hence it is impossible that there exists a relation between the absolute invariants that appear as coefficients in the S-equation after dividing the

left side of (11) through $I_0 = (d^{n_4}\, b^{n-n_4})\,(c^{n_3}\, a^{n_1}\, b^{n-n_1-n_3})$. Therefore an integrity basis composed of less than $T+1$ invariants as ours cannot be present. In the general case (IV) the invariants

$$I_p = (d^{n_4}\, a^p\, b^{n-n_4-p})\,(c^{n_3}\, a^{n_1-p}\, b^{n-n_1-n_3+p}) \qquad (p = 0, 1, \ldots, T)$$

form a *smallest integrity basis*. In the third case (III) I_{n_1} becomes the product of $(d^{n_4}\, a^{n_1})$ and $(c^{n_3}\, b^{n_2})$ and so a smallest integrity basis consists of $I_0, I_1, \ldots, I_{n_1-1}$ and these invariants.

In the second case (II) likewise I_0 becomes reducible and a smallest basis is composed by

$$I_1, I_2, \ldots, I_{n_1-1}, (d^{n_4}\, a^{n_1}), (d^{n_4}\, b^{n_2}), (c^{n_3}\, a^{n_1}), (c^{n_3}\, b^{n_2}).$$

In the case of four medials there appear further the invariants $(a^{n_1}\, b^{n_2})$ and $(c^{n_3}\, d^{n_4})$. After adding them to the basis one of the invariants $I_1, I_2, \ldots, I_{n_1-1}$ can be omitted [8]). This follows from (13) for $n_1 = n_2 = n_3 = n_4$; $q = 0$ or from

$$(a^{n_1}\, b^{n_1})\,(c^{n_1}\, d^{n_1}) \equiv \sum_{p=0}^{n_1} (-1)^p \binom{n_1}{p}^2 I_p.$$

15. The points $P_a^{(i)}$ $(i = 1, 2, \ldots, T)$ are not contained in a $(T-2)$-dimensional subspace A' of A. For if this happens, we can choose the points $\{y^{(i)}\}$ so that the first $T-1$ of them occur in A', and then the $n_1 + n_2$ linear homogeneous equations

$$\left.\begin{aligned} S\sum_{i=1}^{T-1} \alpha_i\,(w'_{(l)}\, y^{(i)}) + \sum_{j=1}^{n_2} \beta_j\,(w'_{(l)}\, z^{(j)}) &= 0 \\ \sum_{i=1}^{T-1} \alpha_i\,(v'_{(k)}\, y^{(i)}) + \sum_{j=1}^{n_2} \beta_j\,(v'_{(k)}\, z^{(j)}) &= 0 \\ (l = 1, 2, \ldots, n-n_4;\ k = 1, 2, \ldots, n-n_3) & \end{aligned}\right\} \quad \ldots \quad (16)$$

in the $T-1+n_2 < n_1+n_2$ indeterminates $(\alpha_1, \alpha_2, \ldots, \alpha_{T-1}, \beta_1, \beta_2, \ldots, \beta_{n_2})$ shall become a common solution different from $(0, 0, \ldots, 0)$ through each substitution $S = S_i$ $(i = 1, 2, \ldots, T)$. This requires that the determinant on the coefficients of every number of $T-1+n_2$ of the equations (16) which is a polynomial in S of a degree lower than T, vanishes for T different values S_i of S, and thus is equal to zero irrespective the value of S. However for $S = 1$ at least one of these determinants differs from zero, since on account of (GP_3) the space joining A and B has no common points with the intersection of C and D. Hence there exists no $[T-2]$ containing $P_a^{(i)}$ and likewise no $[T-2]$ containing $P_b^{(i)}$ $(i = 1, 2, \ldots, T)$.

Now we pass to a new system of coordinates, the symplex of which we

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[8]) TURNBULL II, p. 61; BRUINS, p. 444—445.

indicate by $Q_1 Q_2 \dots Q_n$ and the unit point of which we call E. The n points Q_i are defined in this manner

T	points Q_i coincide with the points $P_a^{(i)}$
T	points Q_i coincide with the points $P_b^{(i)}$
$n_3 + n_4$—n	points Q_i are contained both in C and in D
$n_2 + n_4$—n	points Q_i are contained both in B and in D
n_1—T	points Q_i are contained both in A and in D
n—n_4—T	points Q_i are contained both in B and in C.

In order to determine E we agree that the projection of this point from the opposite $Q_{i_1} Q_{i_2} \dots Q_{i_{n-2}}$ on $P_a^{(j)} P_b^{(j)}$ coincides with $P_c^{(j)}$ for each value of j $(= 1, 2, \dots, T)$. Then the coordinates of the T points $P_d^{(j)}$ and thus the equations of A, B, C and D are defined unequivocally. Nevertheless there are yet ∞^{n-T-1} points, that can perform the duties of the unit point E.

From this we deduce that there exist ∞^{n-T-1} collineations for which the figure consisting of the four spaces A, B, C and D remains invariant.

16. The considerations of the preceding section show that if the roots S_i are mutual different, their values define unequivocally the projective geometrical type of the figure. A second number of four spaces in general position $[n_1$—$1]^*$, $[n_2$—$1]^*$, $[n_3$—$1]^*$ and $[n_4$—$1]^*$ for which the roots S_i^* have the same values as the roots S_i can be carried over in the original system by a projective transformation. This is not always possible when two or more roots S_i are equal. For after the substitution $S = S_j$, where S_j is a multiple root of (11) the equations (9*) may be both once and more than once dependent. In this case, that we don't exclude further, we must consider the minors of the determinant (10). Now the minor that appears after dropping the rows containing

$$\{w'_{(l_1)}\}, \{w'_{(l_2)}\}, \dots, \{w'_{(l_{p_0})}\}; \{v'_{(k_1)}\}, \{v'_{(k_2)}\}, \dots, \{v'_{(k_{q_0})}\}$$

and the columns containing

$$\{y^{(i_1)}\}, \{y^{(i_2)}\}, \dots, \{y^{(i_{r_0})}\}; \{z^{(j_1)}\}, \{z^{(j_2)}\}, \dots, \{z^{(j_{s_0})}\}$$

where $p_0 + q_0 = r_0 + s_0 = N \leqq T-1$, is just the S-determinant corresponding with the spaces

A': $(y^{(i\,r_0+1)} \dots y^{(i\,n_1)}\, \pi^{n-n_1+r_0}) = 0,$

B': $(z^{(j\,s_0+1)} \dots z^{(j\,n_2)}\, \pi^{n-n_2+s_0}) = 0,$

C': $(v'_{(k\,q_0+1)} \dots v'_{(k\,n-n_3)}\, \pi'^{n_3+q_0}) = 0,$

D': $(w'_{(l\,p_0+1)} \dots w'_{(l\,n-n_4)}\, \pi'^{n_4+p_0}) = 0.$

We suppose that A' is the intersection of A with the $[n-r_0-1]$: $(\alpha^{n-r_0}\pi^{r_0}) = 0$, that B' is the intersection of B with the $[n-s_0-1]$: $(\beta^{n-s_0}\pi^{s_0}) = 0$, that C' arises from joining C with the $[q_0-1]$: $(\gamma^{q_0}\pi^{n-q_0}) = 0$ and that D' arises from joining D with the $[p_0-1]$: $(\delta^{p_0}\pi^{n-p_0}) = 0$. Then the equations of our four spaces are

$$\begin{aligned}
&A': && (a^{n-r_0}a^{r_0})(a^{n_1-r_0}\pi^{n-n_1+r_0}) = 0 && (n_1' = n_1 - r_0),\\
&B': && (\beta^{n-s_0}b^{s_0})(b^{n_2-s_0}\pi^{n-n_2+s_0}) = 0 && (n_2' = n_2 - s_0),\\
&C': && (\gamma^{q_0}c^{n_3}\pi^{n-n_3-q_0}) = 0 && (n_3' = n_3 + q_0),\\
&D': && (\delta^{p_0}d^{n_4}\pi^{n-n_4-p_0}) = 0 && (n_4' = n_4 + p_0),
\end{aligned}$$

and apart from a constant number the value of the considered minor becomes

$$\left.\begin{aligned}\sum_{t=L'}^{T'} g^{n_1' n_2'}_{n_3' n_4', t}\,(a^{n-r_0}a^{r_0})(\beta^{n-s_0}b^{s_0})(\delta^{p_0}d^{n_4}a^t b^{n-n_4-p_0-t})\times\\ \times(\gamma^{q_0}c^{n_3}a^{n_1-t-r_0}b^{n-n_1-n_3+t+r_0-q_0})\,S^t\end{aligned}\right\} \quad . \quad . \quad (17)$$

where
$$\begin{aligned}
&T' = n_1 - r_0 && \text{for} && p_0 - r_0 \leqq n - n_1 - n_4\\
&T' = n - n_4 - p_0 && \text{for} && p_0 - r_0 \geqq n - n_1 - n_4\\
&L' = 0 && \text{for} && s_0 - p_0 \leqq n_2 + n_4 - n\\
&L' = n - n_2 - n_4 + s_0 - p_0 && \text{for} && s_0 - p_0 \geqq n_2 + n_4 - n.
\end{aligned}$$

If S_t is a common root of the equations

$$\left.\begin{aligned}\sum_t g^{n_3+q,\,n_4+p}_{n_1-r,\,n_2-s,\,t}\,(\pi^{n-r}_{(1)}a^r)(\pi^{n-s}_{(2)}b^s)(\pi^{p}_{(3)}d^{n_4}a^t b^{n-n_4-p-t})\times\\ \times(\pi^{q}_{(4)}c^{n_3}a^{n_1-t-r}b^{n-n_1-n_3-q+t+r})\,S^t \equiv 0 \begin{Bmatrix}\pi_{(1)},\pi_{(2)}\\ \pi_{(3)},\pi_{(4)}\end{Bmatrix}\\ (p+q=r+s=N;\ p,q,r,s=0,1,2,\ldots,N\leqq T-1)\end{aligned}\right\} \quad . \quad . \quad (18)$$

then after the substitution $S = S_t$ in (17) we obtain zero.

Evidently the resolvability of (18) is a sufficient condition for the existence of a root S_t that causes a diminishing of the rank of (10) to a number $< n_1 + n_2 - N$. Moreover this condition is necessary, since it is always possible to choose the points $\{y^{(1)}\}$, $\{y^{(2)}\}$, ..., $\{y^{(n_1-r)}\}$ in such a manner that they are contained in an arbitrary space $[n-r-1]:(a^{n-r}_{(1)}\pi^r)=0$, and so on. We conclude that a complete projective classification of the numbers of four spaces, for which the S-equation has one or more multiple roots is obtained, if besides the absolute invariants are taken in consideration the concomitants

$$\begin{aligned}&(\pi^{n-r}_{(1)}a^r)(\pi^{n-s}_{(2)}b^s)(\pi^{p}_{(3)}d^{n_4}a^t b^{n-n_4-p-t})(\pi^{q}_{(4)}c^{n_3}a^{n_1-t-r}b^{n-n_1-n_3-q+t+r})\\ &(0\leqq p+q=r+s=N\leqq T-1;\ p,q,r,s=0,1,2,\ldots,N).\end{aligned}$$

With every number of four spaces A, B, C, D is connected an expression as introduced by SEGRE in the theory of the elementary divisors and inversely such an expression from projective geometrical standpoint gives a complete summary of the properties of the figure [9]).

[9]) For $n_1 = n_2 = n_3 = n_4 = 3$ compare BOTTEMA p. 34.

Geology. — *A pseudotachylytic rock from Eastern Borneo.* By W. P. DE ROEVER. (Communicated by Prof. H. A. BROUWER.)

(Communicated at the meeting of November 29, 1947.)

A schistose pseudotachylyte-breccia (U 2285) was collected by Ir. J. G. H. UBAGHS in the S. Kajan somewhat downstream of the confluence with the S. Kat. The surrounding region is mainly constituted by a dynamometamorphic formation of slates, phyllitic slates, arkoses, and sandstones, which is covered and veined by younger volcanic rocks.

The sample of the rock described here has the appearance of a volcanic breccia. Microscopic examination reveals that it is essentially constituted by a monomikt pseudotachylyte-breccia of tectonic origin. One of the thin sections studied shows a local polymikt zone, in which there are also psammitic rock-fragments of the dynamometamorphic formation, crystals of hornblende and plagioclase, and larger fragments of the younger volcanic rocks.

Besides sporadic crystals of zircon the pseudotachylyte-fragments contain irregularly bounded crystal-fragments of quartz, of small to extremely small sizes, amidst a groundmass of varying appearance. The crystal-fragments of quartz often show a strongly or very strongly undulose extinction. Especially in the polymikt zone mentioned above the groundmass of the pseudotachylyte-fragments may show all characteristics of glass: the refractive index is considerably lower than that of the Canada balsam. In some of the other fragments the groundmass has the patchy appearance of the devitrified glassy mesostasis of some types of effusive rocks. In many other fragments again there has been a later formation of chlorite-like material, sometimes showing a distinct schistosity.

The monomikt parts of the thin sections studied show a distinct schistosity owing to the parallel arrangement of newly formed chlorite-like and sericitic material in the matrix of the breccia. The fragments may also show a schistose structure, but here there is a random orientation of the schistosity-planes, which may be explained by the assumption of two phases of dynamometamorphic recrystallization.

The rock is veined by quartz and zeolites.

The formation of this pseudotachylytic rock has been caused by intensive and rapid movements along a fault-zone. The adjoining rocks have been molten due to the development of frictional heat as a consequence of these movements; that the rocks have actually been vitrified and not only crushed to an isotropic aggregate of infinitesimal fragments, is illustrated by the low refractive index of the glass. Younger phases of movement caused the

brecciation and the development of the schistose structure. The components of the polymikt zone have probably been admixed during one of the youngest phases of movement. Though most fault-zones in this region are characterized by a phyllitization of the dynamometamorphic formation in their vicinity, the rock described here illustrates that movements have also occurred after the effusion of the younger, post-metamorphic volcanic rocks.

Bandoeng, Dienst van de Mijnbouw.

Geology. — *Upper Triassic fossiliferous limestones in the island of Bangka.* By G. A. DE NEVE and W. P. DE ROEVER. (Communicated by Prof. H. A. BROUWER.)

(Communicated at the meeting of November 29, 1947.)

The fossiliferous limestones described in this paper have been collected by W. P. DE ROEVER during recent geological investigations in NE. Bangka; the determination of the fossils was done by G. A. DE NEVE, who identified the following fossils, indicating a Norian (Upper Triassic) age:

Entrochus spec.
Encrinus spec.
Montlivaultia molukkana J. WANNER
Peronidella moluccana O. WILCKENS

These are the first fossils of determinable age found in the bedrock ("kong") of the islands of Bangka and Billiton. They indicate a post-Norian age of the tin-granites, which are younger than the metamorphic sedimentary formations of these islands.

The fossiliferous limestones have been collected in the Loemoet tin-mine (nr. 7 Belinjoe), situated SE. of the Klabat Bay. This mine is already indicated as a fossil locality on the geological map of VERBEEK [1]), because of the occurrence of Quaternary or Pliocene plant-remains.

The limestone is grey in colour and may be strongly crushed; it occurs as a faulted layer or lens amidst an intricately folded formation of light-coloured, strongly altered sericite-phyllites and schistose, fine-crystalline quartzites. Near the limestone there is a small outcrop of very strongly altered effusive rocks, the relations of which have not yet been elucidated. The light-coloured phyllitic formation may show local intercalations of very strongly altered phyllitic arkoses or greywackes (blastopsammitic phyllites) and phyllitic conglomerates (blastopsephitic phyllites). The latter rock-type contains pebbles of effusive rocks. The southern wall of the mine shows outcrops of red and greyish green chert-like schistose quartzites, which are separated from the light-coloured phyllitic formation of the main part of the mine by an important fault-zone. This fault-zone is of considerable age, as indicated by the occurrence of phyllitic fault-breccias (blastomylonitic phyllites). In the fault-zone the chert-like quartzites are accompanied by dark-coloured sericite-phyllites and schistose quartzites, which may show veins of cassiterite and quartz.

As a consequence of solution of the limestone and accompanying subsidence of the neighbouring rocks a depression with a maximal depth of about 100 m has been formed in the surface of the bedrock. The

[1]) R. D. M. VERBEEK, Geologische beschrijving van Bangka en Billiton. Jaarb. Mijnw. Ned. Oost-Indië **26** (1897).

presence of this depression gave rise to the formation of one of the richest detrital cassiterite-deposits in the island of Bangka that are known.

The limestones were discovered in 1920 by Ir. J. VELDKAMP; just before the last war a collection of undeterminable crinoid-stems was made by Ir. E. DE WILDE on behalf of Dr. Ir. C. P. A. ZEYLMANS VAN EMMICHOVEN.

Elsewhere in the island of Bangka limestones have hitherto only been found as pebbles, at two localities.

The fossils embedded in the limestones of the Loemoet tin-mine have been seriously attacked by percolating waters, and the worn condition of the specimens together with their fragmentary nature have rendered the task of interpreting these remains one of great difficulty [2]).

The crinoids are represented only by portions of the stem, with which probably a few scattered brachials are occasionally associated. Apparently they belong to the family *Encrinidae*; the fragments of cylindrical stem, mostly from the root end, probably all belong to a single species of the genus *Entrochus*. Some stem-fragments show alternating dimensions of the columnals; they are usually assigned to *Encrinus* species. Neither to the stem-fragments of *Entrochus* nor to those of *Encrinus* it seems advisable to attach specific names either new or old. The crinoid-remains show much resemblance to those figured by BATHER, JAWORSKI, KRUMBECK, and WANNER, which were found in the Upper Triassic deposits of the islands of Timor, Buru, and Misol [3]).

The fossil coral was determined as *Montlivaultia molukkana* J. WANNER, a species occurring in the Upper Triassic *Pharetrones*-limestone of Seran (Moluccas) and described by WANNER and WILCKENS, while a probable find from Timor is mentioned by VINASSA DE REGNY [4]).

Finally some cylindrical fossils embedded in the limestone, with cavities extending to their bases, show characteristic identity with one of the *Calcispongiae* described by WILCKENS as *Peronidella moluccana* n.sp. The

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[2]) The palaeontological description with the figures will be published in the periodical "Geologie en Mijnbouw".

[3]) F. A. BATHER, Triassic Echinoderms of Timor. Palaeontologie von Timor, XVI. Lieferung, Abschn. XXX, p. 215—251 (1929).

E. JAWORSKI, Die Fauna der obertriadischen Nucula-Mergel von Misol. Palaeontologie von Timor, II. Lieferung, Abschn. V, p. 1—174, Pl. 43—45 (1915).

L. KRUMBECK, Obere Trias von Buru und Misol. Palaeontographica, Supplement-Band IV, Abt. II, Abschn. I, p. 1—162, Pl. I—XI (1913).

J. WANNER, Beiträge zur Geologie der Insel Buru. Palaeontographica, Supplement-Band IV, Abt. III, Abschn. III, p. 59—112, Pl. IX, X (1919—1922).

[4]) J. WANNER, Triaspetrefakten der Molukken und des Timorarchipels. Neues Jahrb. f. Min. etc. Beil. Bd. **24**, p. 161—220, Pl. 7—12 (1907).

O. WILCKENS, Korallen und Kalkschwämme aus dem obertriadischen Pharetronenkalk von Seran (Molukken). Neues Jahrb. f. Min. etc. Beil.-Bd. **77** B, p. 171—208, Pl. 6—13 (1937).

P. VINASSA DE REGNY, Triadische Algen, Spongien, Anthozoen und Bryozoen aus Timor. Palaeontologie von Timor, IV. Lieferung, Abschn. VIII, p. 75—118, Pl. 63—72 (1915).

almost equal dimensions together with the same microscopic texture in oblique sections as given in WILCKENS' figure have led to this conclusion [5]).

It is to be hoped that future investigations will provide more and better preserved fossil remains, especially of the crinoids.

In his synopsis on the pretertiary history of the Malay Archipelago UMBGROVE has given a review of the deposits of Triassic age found in many islands and occurring in a number of different facies, while he also mentions the probable occurrence of Triassic sediments in the islands of Bangka and Billiton [6]).

In Northern Sumatra, in the highlands of Padang, North of Lake Singkarah, an important section with Upper Triassic fossiliferous deposits was found by MUSPER [7]). But also in other parts of Sumatra clay-shales and sandstones with *Halobia* and *Monotis salinaria* have been found, indicating Carnian and Norian ages.

SCRIVENOR [8]) mentions the occurrence of Upper Triassic fossiliferous shales, quartzites, and sandstones in Malacca. In the neighbouring Riau-Archipelago *Halobia*-bearing shales were found by BOTHÉ [9]), whilst MARTIN determined *Daonella* in brown-grey shales from the shore between Laboean Dadong and Sei. Keloemoe, Island of Lingga [10]), and ROGGEVEEN described *Protocupressinoxylon malayense*, a probably Triassic conifer, from the island of Soegi [11]).

Bandoeng, Dienst van de Mijnbouw.

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5) O. WILCKENS, see footnote 4.

6) J. H. F. UMBGROVE, De pretertiaire historie van den Indischen Archipel. Leidsche Geol. Med. **7**, 125, 128—134 (1935).

7) K. A. F. R. MUSPER, Beknopt verslag over de uitkomsten van nieuwe geologische onderzoekingen in de Padangsche Bovenlanden. Jaarb. Mijnw. Ned. Oost-Indië. Verh., **58**, 265—329 (1929).

8) J. B. SCRIVENOR, The geology of Malaya (1931).

9) A. CHR. D. BOTHÉ, Het voorkomen van tinerts in den Riau-Archipel en op de eilandengroep van Poelau Toedjoe (Anambas- en Natoena-eilanden). Verslagen en Mededeelingen betreffende Indische delfstoffen en hare toepassingen, **18**, 5 (1925).

10) A. CHR. D. BOTHÉ, Geologische verkenningen in den Riouw-Lingga Archipel en de eilandengroep der Poelau Toedjoeh (Anambas- en Natoena-eilanden). Jaarb. Mijnw. Ned. Oost-Indië **54**, Verh. II, 98, 143 (1925).

11) P. M. ROGGEVEEN, Mesozoisches Koniferenholz (Protocupressinoxylon malayense n.s.) von der Insel Soegi im Riouw-Archipel, Niederländisch Ost Indien. Proc. Kon. Akad. v. Wetensch., Amsterdam, **35**, 580—584 (1932).

Petrology. — *Anorthoclase-bearing granogabbroid to granonoritic rocks from Boeloengan (Eastern Borneo).* By W. P. DE ROEVER and A. KRAËFF. (Communicated by Prof. H. A. BROUWER.)

(Communicated at the meeting of November 29, 1947.)

In the years 1924 and 1931 several rock-samples of granogabbroid affinities have been collected by Ir. A. HARTING and Ir. J. G. H. UBAGHS in the vicinity of the confluence of the S. Kajan and the S. Bahau. Some of these rocks have been found as veins in a dynamometamorphic formation of slates, phyllitic slates, arkoses, and sandstones, which will be described in a later paper. Most of the samples, however, have been taken from boulders, some of which are idolized by the natives.

Two of the samples, H 120 and H 123 — both found as boulders in the S. Kajan, at a distance of ± 9 and ± 16 km, respectively, above the confluence with the S. Bahau — are fresher and less fine-crystalline than the others, and are adapted for a more detailed microscopic study of this comparatively rare rock-type.

The sample H 120 is medium-grained and of a brown-grey colour; the largest crystals of pyroxene are 5 mm in length. The other sample (H 123) shows a porphyritic development owing to the presence of many larger crystals of pyroxene — with sizes of up to 1 cm — amidst a finer crystalline mesostasis.

Under the microscope the main constituents of these rocks appear to be plagioclase, augite, hypersthene, quartz, and anorthoclase, with subordinate or accessory biotite, amphibole, ilmenite, and apatite. Volumetric analyses of one ordinary size thin section of each of these rocks gave the following results:

	H 120	H 123
Plagioclase	43 %	49 %
Augite	19 %	9 %
Hypersthene	15 %	10 %
Quartz	10 %	16 %
Anorthoclase	10 %	14 %
Other constituents	3 %	2 %

Under the microscope both rocks appear to show a rather feebly developed porphyritic structure owing to the presence of larger crystals of pyroxene, especially of augite. The crystals of plagioclase may reach sizes of 2 mm. The anorthoclase and quartz are often in granophyric intergrowth; these minerals are found in the interstices between the crystals of the other

constituents, with maximal crystal-sizes of slightly more than 1 mm (anorthoclase) and 2 mm (quartz). The anorthoclase is generally in subhedral to anhedral crystals.

The crystals of plagioclase generally show large cores of bytownite and lime-rich labradorite with recurrent zones; there are rather narrow rims of andesine. The average anorthite content is clearly more than 50 %.

The augite and hypersthene are rather slightly altered. The augite is twinned on (100); a minute striation according to the basal plane combined with this orthopinacoidal twinning produces the characteristic "herring-bone" appearance.

The anorthoclase has a well-developed murchisonite cleavage. The refractive indices are distinctly lower than that of the Canada balsam. This alkalifelspar shows decidedly oblique extinctions in sections perpendicular to n_β and n_γ. For the optic axial angle $2V_\alpha$ of the anorthoclase in the rock H 123 values of $\pm$ 35° were obtained with the aid of the universal rotation stage. In the other rock (H 120) the anorthoclase shows a zonary variation of the optic axial angle $2V_\alpha$ from $\pm$ 40° in the cores to $\pm$ 30° in the rims.

Red-brown biotite is of subordinate importance. A very small amount of original light greenish amphibole may also be mentioned.

According to the quantitative mineralogical classification of JOHANNSEN these rocks could be named anorthoclase-granogabbro-porphyry to -granonorite-porphyry.

Bandoeng, Dienst van de Mijnbouw.

Botany. — *Researches on plant growth regulators.* XIII. *Leaf growth factors.* II. By W. KRUYT and H. VELDSTRA. (Communicated by Prof. V. J. KONINGSBERGER.)

(Communicated at the meeting of September 27, 1947.)

4. *Experiment with Cosmos bipinnatus Cav. "Sensation Innocence" at Boskoop.*

BONNER and HAAGEN–SMIT (5) had already pointed out that the effect of adenine as a leaf growth factor in Cosmos was greater under short than under long exposure to daylight. Therefore we thought it might be worth while to ascertain whether during the natural short day-length in winter-months the effect of adenine under our conditions would perhaps be greater than we had observed up to this time.

As our previous experiments had showed clearly how important it is with such an investigation to eliminate a possible influence of the position beforehand all the pots irrespective of their groups were randomized.

In this experiment, which lasted from Nov. 13th 1943 till Febr. 1st 1944, we examined the influence of adenine 0.1 and 0.5 mg/l and of α-naphthylamine in the concentrations 0.1, 0.5 and 1.0 mg/l. Alternately every two or three days 50 ml of solution or distilled water was administered to every pot.

After the experiment was finished the 8 best specimens were selected from the 12 plants. After the measuring the weight was determined before and after drying. Owing to the slight development which is considerably less than it is during the months of summer this time both leaves of the second real leaf-pair of each plant, counting from the bottom, were weighed fresh and dry. The results are to be found in table IV.

TABLE IV.

Number of plants	Solution applied (Nov. 3rd 1943— Febr. 1st 1944)	Average length in mm	Number of plants with flower-bud	Fresh weight (g) of:			Dry weight (g) of:		
				second real leaf pair	shoots	roots	second real leaf pair	shoots	roots
8	Shive (= S)	153	5	0.85	5.85	0.91	0.055	0.368	0.085
8	S + adenine 0.1 mg/l	145	6	0.77	4.76	0.67	0.053	0.325	0.063
8	S + adenine 0.5 „	147	4	1.10	6.55	1.10	0.068	0.417	0.119
8	S + α–N.A. *) 0.1 „	145	4	0.88	5.78	0.96	0.057	0.370	0.082
8	S + α–N.A. 0.5 „	134	3	0.74	4.65	0.82	0.047	0.295	0.068
8	S + α–N.A. 1.0 „	147	6	0.80	5.18	0.73	0.050	0.321	0.064

*) N.A. = naphthylamine.

We may conclude that there is a slight influence of adenine noticeable, 0.5 mg/l not yet being toxic, as established by the American investigators. This effect of adenine, however, is not clearly visible in the plants. It is remarkable that in this experiment 0.5 mg/l α-naphthylamine does not have a favourable influence, whereas this was the case in our previous tests.

5. Experiment with Cosmos bipinnatus Cav. "Sensation Innocence" at Boskoop.

In this experiment an attempt was made to apply as much as possible the methods practised by BONNER and HAAGEN–SMIT (5), by watering the plants daily (excepting Sundays) alternately with nutrient-solution and distilled water; usually with 25 ml at a time. On Saturdays or when the weather was very hot 50 ml were administered, in which case care was taken that the next time again 50 ml were given so that the amount of nutrient solution and distilled water always remained equal.

Beside adenine in two concentrations (0.1 and 0.5 mg/l) we examined the influence of α-naphthylamine 0.1, 0.5, 1.0 mg/l added to Shive's solution so that in general the experiment corresponded with the preceding one. The p_H of the solutions was 4.3. The sowing was done on April 17th 1944 whilst on April 25th forty plants of each group were transferred into pots after the length of the shoot of all specimens had been determined (table V).

All the pots were dug into a layer of sand at random. The first real leaf-pair began to unfold on May 1st. On May 8th (1944) it was already noticeable that the plants in the most northern position were the largest. So evidently an influence of the southern side-wall could be noticed. Against this wall the hothouse shed, not covered with glass, was situated. This influence is comprehensible because in sunny weather a shadow continually fell on the southern part of the sand bed. This had already been stated in our first two experiments. Because this time all the pots were placed at random the unfavourable influence of the southern part of the bed was distributed equally over all the groups. The synoptical photo (Fig. 1) shows the position on May 15th 1944.

On May 22nd, 5 weeks after the sowing, the experiment was broken off. Flower-buds were then not yet visible. After determining the growth in length (table V) the fresh and dry weights of the different groups were determined for the aerial parts and the roots separately. This was done for the 10 best and the 25 next best specimens.

TABLE V.

Group	Solution applied (April 17th 1944— May 22nd 1944)	Length of the seedlings on April 25th 1944		Length of the shoot (mm) on May 22nd 1944			
		number	average length (mm)	number	average length of the 10 best specimens	average length of the 25 next specimens	average length of the whole group of 35 plants
1	Shive (= S)	58	22	35	177	143	153
2	S + adenine 0.1 mg/l	63	22	35	186	150	160
3	S + adenine 0.5 „	60	23	35	179	152	160
4	S + α–N.A. *) 0.1 „	61	24	35	187	147	159
5	S + α–N.A. 0.5 „	57	25	35	189	149	160
6	S + α–N.A. 1.0 „	50	22	35	184	140	153

*) N.A. = naphthylamine

The p_H of the sand was 7.2 at the start and after conclusion it was 6.8 in practically all groups; therefore as a result of the watering with nutrient-solution the medium has become slightly more acid. The results are summarized in table VI.

From this we may deduce the following. There is no clear influence of adenine or naphthylamine on the longitudinal growth of 4 days old seedlings (table V). The average length of 5 week old plants owing to the administration of adenine 0.1 mg/l and 0.5 mg/l and of α-naphthylamine

TABLE VI.

The first number in each column refers to the 10 best plants; the second number refers to the whole group of 35 plants.

Group	Solution applied (April 17th 1944— May 22nd 1944)	Fresh weight (g) of:					Dry weight (g) of:				
		the lowest leaf-pair	the second leaf-pair	shoots	roots	whole plants	the lowest leaf-pair	the second leaf-pair	shoots	roots	whole plants
1	Shive (= S)	1.90/6.30	3.90/13.95	33.20/104.9	13.60/43.12	46.80/148.02	0.15 /0.475	0.33/1.135	2.62 / 8.595	0.88 /3.005	3.50 /11.60
2	S + adenine 0.1 mg/l	1.80/6.05	4.35/13.80	39.95/115.25	15.87/46.02	55.82/161.27	0.14 /0.46	0.37/1.18	3.255/ 9.435	1.18 /3.365	4.435/12.80
3	S + adenine 0.5 ,,	1.90/7.05	4.15/16.10	34.65/123.3	13.32/48.62	47.97/171.92	0.14 /0.535	0.36/1.35	2.815/11.55	0.99 /3.625	3.805/15.175
4	S + α–N.A. *) 0.1 ,,	1.80/6.5	4.40/14.80	34.50/113.75	11.89/42.19	46.39/155.94	0.14 /0.49	0.38/1.225	2.79 / 8.88	0.89 /3.140	3.68 /12.02
5	S + α–N.A. 0.5 ,,	2.10/6.55	4.30/14.40	40.75/118.1	14.02/45.87	54.77/163.97	0.16 /0.51	0.38/1.24	3.33 / 9.61	1.055/3.450	4.385/13.06
6	S + α–N.A. 1.0 ,,	1.85/5.75	4.30/13.55	34.05/101.95	12.70/41.805	46.75/143.75	0.145/0.455	0.38/1.18	2.835/ 8.42	0.99 /3.310	3.825/11.73

*) N.A. = naphthylamine

TABLE VII.

The first number in each column refers to the 10 best plants; the second number refers to the whole group of 30 plants.

Group	Solution applied (May 30th 1944— July 14th 1944)	Average length (cm)	Fresh weight (g) of:				Dry weight (g) of:		
			the third leaf	shoots	roots	whole plants	shoots	roots	whole plants
1	Shive (= S)	44.5/43.1	3.7/ 9.7	76.0/194.5	15.5/37.5	91.5/232.0	5.5 /14.25	1.1 /2.6	6.6 /16.85
2	S + adenine 0.1 mg/l	45.9/43.8	5.0/11.6	92.1/226.1	17.9/41.5	110.0/267.6	6.7 /15.90	1.2 /2.9	7.9 /18.8
3	S + adenine 0.5 ,,	45.1/41.7	3.9/ 9.5	81.5/188.5	16.9/38.0	98.4/226.5	5.9 /13.05	1.1 /2.5	7.0 /15.55
4	S + α– N.A. *) 0.1 ,,	48.1/44.6	4.7/11.2	93.0/229.0	18.0/45.1	111.0/274.1	6.8 /16.90	1.35/3.25	8.15/20.15
5	S + ,, 0.5 ,,	44.7/40.9	4.0/10.3	81.1/119.0	14.3/38.4	95.4/237.4	5.75/14.10	0.9 /2.5	6.65/16.6
6	S + ,, 1.0 ,,	48.5/43.3	3.8/10.4	87.6/215.9	16.6/42.5	104.2/258.4	6.3 /15.75	1.0 /2.7	7.3 /18.45
7	S + β– N.A. 0.1 ,,	45.2/43.4	4.4/11.6	89.1/230.9	16.4/46.1	105.5/277.0	6.55/17.10	1.05/2.9	7.6 /20.0
8	S + ,, 0.5 ,,	47.2/43.6	4.8/12.3	93.7/229.2	16.9/45.7	110.6/274.9	6.8 /16.85	1.1 /2.8	7.9 /19.65
9	S + ,, 1.0 ,,	46.9/43.9	4.1/10.9	85.8/217.2	14.8/40.7	100.6/257.9	6.2 /15.85	1.0 /2.75	7.2 /18.6
10	S + α–A.M.N. **) 0.1 ,,	45.3/45.4	5.0/12.0	95.3/231.3	18.6/43.5	113.9/274.8	7.25/17.65	1.25/2.95	8.5 /20.6
11	S + ,, 0.5 ,,	46.6/42.8	3.6/ 9.6	80.6/195.3	15.8/39.2	96.4/234.5	6.05/14.80	1.1 /2.8	7.15/17.6
12	S + ,, 1.0 ,,	44.4/42.2	4.4/10.7	82.6/203.1	15.0/41.5	97.6/244.6	6.1 /15.10	1.05/2.7	7.15/17.8

*) N.A. = naphthylamine.

**) A.M.N. = (aminomethyl-)naphthalene.

0.5 mg/l is slightly more than that of the control plants. Considering the whole group the maximum of the fresh and dry weights of the aerial parts as well as those of the picked leaves (fig. 2 and 3) is observed with adenine 0.5 mg/l (group 3, table VI), after which α-naphthylamine 0.5 mg/l (group 5) usually follows. The increase of the dry weight of the aerial part is 34 %; of the leaves 12—18 %.

Also the fresh and dry weight of the roots was influenced most by adenine 0.5 mg/l. These results run practically parallel to those obtained with the aerial parts.

The increase of weight of the whole plant (aerial part and roots) is maximal in the group receiving adenine 0.5 mg/l and amounts to 31 %. α-Naphthylamine had the most favourable influence at a concentration of 0.5 mg/l; the effect lay between that of adenine 0.1 mg/l and 0.5 mg/l.

6. *Experiment with Cosmos bipinnatus Cav. "Sensation Innocence" at Lunteren.*

The rather extensive experiment made at Lunteren in 1943 (Nr: 3) gave results which on account of the big differences in position were not altogether trustworthy. Therefore this experiment was repeated in 1944 in the rebuilt hothouse equipped with glass walls on all sides. All pots of the 12 different groups were placed at random in a sand bedding in the centre of the hothouse, so that position influences were equally distributed. As was done in the previous experiment, the pots were watered daily, excluding Sundays, with alternately nutrient solution and distilled water. Beside adenine, we examined the influence of α- and β-naphthylamine and α-(aminomethyl-)naphthalene in three concentrations. The sowing was done on May 30th 1944 (the p_H of the sand being 6.3) and on June 5th the 40 best plants of each group were transferred into pots. On July 14th the experiment came to an end and of each group the 30 best plants were selected; of these the 10 very best specimens were always judged too. Table VII states the results.

As a result of cloudy weather during the last period of the experiment, combined with the fact that the hothouse was continually covered with hurdles, the plants considerably stretched in length. In comparing the dry weight figures of the aerial parts of all the 30 plants the lowest concentration (0.1 mg/l) here always proves to be optimal. In this case adenine gives an increase in weight of 12 %, α-naphthylamine 19 %, β-naphthylamine 20 % and α-(aminomethyl-)naphthalene 24 % in comparison with the control group. The influence of the added substances is also noticeable in the determination of the dry weights of the roots and again the lowest concentration is optimal. Adenine gives an increase in weight of 12 %, α-naphthylamine 25 %, β-naphthylamine 12 % and α-(aminomethyl-) naphthalene 13 % in comparison with the control plants. In this case α-naphthylamine evidently has the greatest effect.

On the total weight of the plants (sum of the aerial parts and roots) the influence of α-(aminomethyl-)naphthalene dominates. The concentration of 0.1 mg/l here gives an increase of weight of 22 %, adenine 12 %, α-naphthylamine 29 % and β-naphthylamine 19 %.

It is peculiar that in this experiment the lowest concentrations (0.1 mg/l) are always optimal, while in the previous ones it was usually the case with



Fig. 1. Arrangement of the groups, mixed at random, of experiment 5 at Boskoop (April 17th 1944—May 22nd 1944). Look for the coloured sticks showing to which group the plant belongs.



Fig. 2. *Cosmos bipinnatus* Cav. "Sensation Innocence", experiment 5 (April 17th 1944–May 22nd 1944). Leaves of the first real leaf-pair of the 10 best plants.
1. Shive (control); 2 and 3 Shive + adenine 0.1 and 0.5 mg/l; 4, 5 and



Fig. 3. *Cosmos bipinnatus* Cav. "Sensation Innocence" experiment 5 (April 17th 1944–May 22nd 1944). Leaves of the second real leaf-pair of the 10 best plants.
1 Shive (control); 2 and 3 Shive + adenine 0.1 and 0.5 mg/l; 4, 5 and

0.5 mg/l. There is a possibility that this shows a relation to the stage of development at the time the experiment is ended. The fact that in previous experiments adenine showed optimal activity, whilst here this is the case with α-naphthylamine or with α-(aminomethyl-)naphthalene, would indicate that the action of the substances examined is not a very specific one.

It has been clearly proved that the influence of the position in the hothouse on the development of the test plants in such experiments should not be underestimated. We would therefore only consider the results of our last 3 experiments (Nrs: 4, 5 and 6) as being quite trustworthy, as in these cases the plants of the whole experiment series were placed at random. (See the summary of the results in table VIII.)

As BONNER and HAAGEN–SMIT (5) did not mention any details as to the manner in which their plants were placed in the hothouse and as they probably worked with only a few pots (in which were several plants), that is to say, on a small scale (see p. 187 of their article), the possibility must be kept in mind that the great differences found are not trustworthy. In this connection it is worth noting that DE ROPP (11) in a study on the growth of stem-tips (isolated from rye-embryos) on a synthetic nutrient medium, observed that the purine derivatives adenine, guanine, uric acid and caffein had no influence whatever on the leaf growth.

Summary.

With reference to the investigations made by BONNER and HAAGEN–SMIT (5) and by D. M. BONNER and J. BONNER (6) concerning the influence of adenine on the leaf growth of Cosmos plants, with which, in certain cases an increase of dry weight up to three times that of the control was obtained, the influence of adenine was examined in an elaborate series of experiments on the growth of *Cosmos bipinnatus,* using more material for each group. Besides this the influence of the structurally somewhat related substances α-naphthylamine, its isomer β-naphthylamine and its homologue α-(aminomethyl-)naphthalene were examined on *Cosmos bipinnatus*.

The strong effects of adenine on the leaf growth observed by the above-mentioned investigators, could not be confirmed with this material. The differences of the best groups in this respect with the control had an average of + 15 %. As compared with this, the figures for α-naphthylamine are 13 %, β-naphthylamine 11 %, α-(aminomethyl-)naphthalene 11 %.

The effect of the three naphthalene derivatives, evidently of the same order of magnitude, and, similar to the effect of adenine, do not seem to be very specific.

Attention is drawn to the great influence which may be exerted by the arrangement of the experimental material if not all places are equivalent and the special measures that must be taken on this account.

We thank Messrs. H. Bosch and P. de Vogel, chief assistants of the "Tuinbouwvoorlichtingsdienst" at Boskoop for their cooperation in our experiments.

TABLE VIII.

Dry weight of all plants as compared with controls (= 100).
(Experiment 1—6).

Group	Treatment (concentration in mg/l)		Aerial part							Roots							Whole plants						
			1	2	3	4	5	6	Average of 4, 5 and 6	1	2	3	4	5	6	Average of 4, 5 and 6	1	2	3	4	5	6	Average of 4, 5 and 6
1	Shive (= S)		100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100
2	S + adenine	0.1	111	111	111	88	110	112	103	105	106	110	74	112	112	99	109	110	111	86	110	112	103
3	S + „	0.5	127	111	111	113	134	92	113	126	106	126	140	121	96	119	127	110	115	118	131	92	114
4	S + α–N.A. *)	0.1	116	119	100	101	103	119	108	124	119	83	97	105	125	109	119	119	96	100	104	120	108
5	S + „	0.5	136	129	109	80	112	99	97	156	119	90	80	115	96	97	143	127	104	80	113	99	97
6	S + „	1.0	129	157	94	87	98	111	99	124	150	83	75	110	104	96	127	155	91	85	101	110	99
7	S + β–N.A.	0.1			67			120	120			49			112	112			62			119	119
8	S + „	0.5			102			118	118			91			108	108			99			117	117
9	S + „	1.0			110			111	111			94			106	106			106			110	110
10	S + α–A.M.N. **)	0.1			98			124	124			86			113	113			94			122	122
11	S + „	0.5			91			104	104			78			108	108			87			104	104
12	S + „	1.0			68			106	106			51			104	104			63			106	106

*) N.A. = naphthylamine.
**) A.M.N. = (aminomethyl-)naphthalene.

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Anatomy. — *Opposite developmental tendencies in human denture.* Third Communication: *Tuberculum paramolare, mesiobuccal edge-prominency or both of them.* By TH. E. DE JONGE. (Communicated by Prof. M. W. WOERDEMAN.)

(Communicated at the meeting of November 29, 1947.)

Introduction.

After having read our *First Communication* on this subject [1]) our colleague BLANKEVOORT was so kind as to place the plaster cast of the lower jaw, discussed by us, at our disposal, in the buccal interstice of which another paramolar II was situated between the second and third right molars, which was extracted by him afterwards (c.f. picture in this and in our first communication).

On that account we claim —be it under quite another aspect — once more the attention, in the following, for the case, that constituted point de départ of our speculations under the title of *"Multiple hyperodontia in upper and lower jaw"* in our first communication. In the present communication we might as well speak of a "missing link".

* * *

The occurrence of superfluous elements, laterally off the row of the molars — either in the shape of free paramolars or of tubercula paramolaria, fused with the molars — was already known by the elder anatomists (e.g. ZUCKERKANDL (I)). BOLK, however, was the first to describe and explain them in their systematic connection.

As typical localization of the paramolars, alternating with the row of normal molars, he described the vestibular interstice between first and second molar (*paramolar I*), resp. second and third molar (*paramolar II*); the tubercula paramolaria, identic with these paramolars, turned out to be linked up, as a rule, with the mesiobuccal crown-cusp of second resp. third molar, as regards their localization and they were somewhat distally displaced. As superfluous elements they no more constitute an essential component of the well-known cusp-pattern of the permanent molars than e.g. the tuberculum Carabelli does: when ignoring them, a normal relief of the occlusal surface remains!

BOLK explained them as being rudiments of a third and fourth deciduous molar, that had got lost [2]).

The fact that he did not find even one single case where the paramolar had been implanted in the buccal interstice between second premolar and first molar, neither in his exceedingly rich material of skulls, nor in casuistic literature, — the same holds good for the tuberculum paramolare

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[1]) Proc. Ned. Akad. v. Wetensch., Amsterdam, Vol. XLIX, No. 7 (1946).

[2]) The distomolar resp. tuberculum distomolare likewise described by BOLK in this connection, are left out of the question in this communication.

mutatis mutandis — reveals the lactal character, already accepted by him before, of our first permanent molar [3]).

Shortly afterwards GREVE (IV) not being aware as yet of BOLK's researches on this subject — and undoubtedly this increases the objectivity of his observations — differentiated in the superfluous buccal crown-cusps between two forms, in a brief casuistic communication, that remained almost unnoted in odontologic literature. Of his speculations, pertaining to two molars of the lower jaw a.o. we quote the passage, that is of most interest for our exposition: "The two other cases both concern a lower second molar, one of which being a right one and the other a left one. Not the same holds good for their shape that holds good for the upper ones, but one right additional tuberculum conveys obviously the impression of having arisen by division resp. splitting from the mesio-buccal cusp of the tooth. Both parts are separated by a groove, distal end of which opens into the buccal side-groove. *If one were to resect the outer part, the entire tooth would not have the well-known rectangular shape of the lower second molars anymore* [4]). The left lower molar, on the contrary, has exactly the same shape, the upper one has. The tuberculum looks entirely like the tuberculum Carabelli *and if it should be removed, the tooth would not be different in any respect from a normally shaped second lower molar* [5]).

The picture of the unquestionable difference in appearance in the occurrence of the buccal tubercula faces us with the question as to the meaning of these facts [4])."

Referring to a number of examples we were able to demonstrate that in the lower molars we should definitely distinguish between the paramolar tubercles proper and those edge-tubercles that originate in the formation of grooves on the buccal surface of the mesiobuccal crown-cusp and which we have designated on that account as *mesio-buccal edge-prominencies.*

For further particulars we refer to our publications on that subject (V and VI), we wish to stress however, that the manifestation of this edge-prominency in the first lower molar may undoubtedly be regarded as one of the very greatest rarities; this is not absolutely true anymore of its distal synergist. In the third molar on the contrary the presence of the anterolateral prominency constitutes a phenomenon coming fairly near to the limits of a normal variability. It is evident therefore that, besides differentiation into a tubercle of a clearly defined individuality, we shall find transition forms in many variegation.

A systematic description of these would lie beyond the scope of this communication. Therefore we will only stress two peculiarities: In normal development of the prominency the groove that is separating it from the

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[3]) We are well aware that without some further explanation this thesis is no doubt debatable. Nevertheless it would carry us too far to go into it further here.

[4]) The italics are ours.

[5]) l.c. pag. 394—395.

mesiobuccal cusp proper, forms from basally to occlusally "a line that is convex at first and concave afterwards and ends near the mesial surface" [JANZER (VII)] [6]: like this we once described in the oral crown-cusps of the molars of the lower jaw the outlines of their occlusal edges (VIII) [7].

On the other hand its developmental tendency may prove so weak, that only a foramen coecum in the centre of the buccal surface of the anterior cusp marks the line of separation between mesiobuccal edge-prominency and the homonymous crown-cusp: it is this very foramen that we do not infrequently come across on the third molar!

For the sake of completeness we may add, that identical differences are to be observed in the formation of the roots of the lower molars. Summarizing these we may state that we know the radix paramolaris, occurring besides the vestibular aspect of the real mesial root — however, the buccal segment of the mesial root itself may gradually free itself and grow into an independent element, a mesiobuccal root, *which has nothing in common with the real radix paramolaris, however, but, at most, its corresponding mesiobuccal localization.*

In view of his own researches our co-operator VISSER (IX) was able to confirm our findings to this effect, afterwards.

* * *

In the foregoing we have briefly formulated our point of view with regard to the tuberculum paramolare as well as to the mesio-buccal edge-prominency and we are granted a really unique confirmation of our conception by the structure of the denture for more extensive description of which we refer to our *First Communication* on this subject (X).

The site of the paramolar (*Pa II inf. dext.*) claims the attention for two reasons. Firstly because its buccal crown-cusp, although situated slightly outside of the curve of the upper molars when the rows of the teeth are occluded, nevertheless articulates with the two analogous cusps of the third upper molar in an exact interdigitation.

Moreover the second and third lower molars on the right are characterized by an, assumedly, consecutive lingual displacement: for on the left, where no paramolar has developed, the relation of lower molars with their antagonists proves to be entirely normal — at least in this respect [8].

When premising that tuberculum paramolare and anterolateral edge-prominency have nothing to do with each other from genetical point of view, there must exist a possibility of both occurring together in one denture. On the other hand, when bucally from the edge-prominency a paramolar has forced its way, either in the shape of a free paramolar or

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[6]) Op. cit. page 411, last paragraph.

[7]) Op. cit. page 57, fig. 47 and 48.

[8]) In our *First Communication* we stated already, that in upper and lower jaw the first molar was not present anymore.

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Fig. 1.

as a tuberculum, then at one stroke the truth of the premise is above all doubt!

One should bear in mind that both paramolar and mesiobuccal edge-tuberculum fall, from teratologic point of view, into the category of rarely occurring dysmorphoses of human denture. In our case, however, for detailed structure of which we refer to the picture, the bilaterally-symmetrical development of mesio-buccal edge-prominency of the right molars is associated with that of a free paramolar II.

Thus the truth of our conception, that paramolar and mesio-buccal edge-prominency have a very individual and different character, not only from anatomical but especially from genetical point of view, is confirmed in an irrefutable way, quod erat demonstrandum.

Finally the objections raised by ADLOFF (XI) to our speculations, are, in our opinion, conclusively refuted here.

Samenvatting.

Aan de hand van eene uitzonderlijk zeldzame gebitformatie wordt het genetisch en morphologisch geheel verschillend caracter van paramolaris en mesiobuccale randprominentie in het licht gesteld.

Résumé.

On démontre à l'aide d'une formation de denture extraordinairement rare le caractère génétiquement et morphologiquement tout à fait différent d'une paramolaire et de la prominence latérale mésiobuccale.

Zusammenfassung.

Mittels einer überaus seltenen Gebissformation wird der genetisch und morphologisch ganz verschiedene Charakter des paramolaris und der mesiobuccalen Randprominenz erklärt.

Summary.

In connection with an exceptionally rare teeth-formation the genetically and morphologically entirely different character of paramolar and mesiobuccal edge-prominency is shown.

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Neurology. — *Carcinoma ovarii and cerebellar degeneration.* By B. BROUWER and F. G. SCHLESINGER. (From the neurological clinic and laboratory of the Wilhelmina Hospital and the Dutch Central institute for Brain Research at Amsterdam.)

(Communicated at the meeting of November 29, 1947.)

In 1919 one of us (B.) described the histopathological findings in a case of progressive diffuse cerebellar degeneration, which was combined with a sarcoma in the pelvis. The alterations were limited to the cerebellar cortex in which the Purkinje cells were selectively destroyed. The cells in the cerebellar nuclei were normal but the surrounding fibres, originating in the Purkinje cells, showed secondary degeneration. The pons Varoli, the inferior olives and the nuclei laterales medullae oblongatae were normal. The degeneration of the Purkinje cells was considered to be the consequence of a toxic influence from the sarcoma. In 1935 Dr. M. KENNARD described from our neurological laboratory a similar case of pure cerebellar degeneration in a woman who had suffered from a carcinoma of the ovarium. She also assumed this cerebellar degeneration to be the consequence of the malign tumour. In their report on the parenchymatous affections of the cerebellum BROUWER and BIEMOND mentioned 4 other similar observations from the literature and united these cases in a special group of cerebellar toxicosis, which they opposed to the cerebellar degenerations of MARIE, FOIX and ALAJOUANINE. In the first group the alterations are diffuse, in the second they are localised. In both groups an endogenous factor (abiotrophy) had to be assumed. In the following pages we report on a new case of cerebellar toxicosis.

On the first of May 1944 Mrs. T., aged 51, was admitted at the psychiatric-neurological clinic of the Wilhelmina Gasthuis (Director Prof. Dr. K. H. BOUMAN) with psychical disturbances, vomiting and difficulty in walking. She had complaints since six weeks, after having lived in heavily bombed cities in Germany. She had never been ill before and belonged to a healthy family. In the clinic the psychical disturbances disappeared after some time, but several symptoms of a severe affection of the central nervous system were found. Dysarthria, nystagmus horizontalis and verticalis, general ataxia were present. The tendonreflexes were normal and the optic nerves did not show alterations. The lumbar punction showed a normal pressure of the spinal fluid, but an increase of protein in the spinal fluid. A process in the posterior fossa was suspected. After consultation with the neurosurgeon Dr. C. H. LENSHOEK the patient was brought to the neurological clinic (3 July 1944). Here also many symptoms of cerebellar disorder were found (dysarthria, ataxia, dysmetria, dysdiadochokinesis, nystagmus, hypotonia). Again the pressure of the spinal fluid was normal, but the reactions of NONNE and PANDY were positive. The colloidal reaction of LANGE was 444333100 and the mastix reaction

532100. There was no pleocytosis and no increase of glucose. The optic disks remained normal and the Röntgenological examination of the skull did not show alterations. The chief of the clinic Dr. A. BIEMOND did not believe that a tumour in the brain was present but accepted the possibility of a diffuse cerebellar degeneration as described by Dr. KENNARD, although he could not find a tumour in the abdominal cavity. Operation on the posterior fossa seemed not to be indicated and the patient was brought back to the psychiatric clinic. Here the patient complained of dizziness and diplopia. She vomited repeatedly, the severe cerebellar ataxia and dysmetria persisted. In the history of the case it was noted at one time that the kneereflexes could not be evoked. Now and then the patient was incontinent for urine and defecation. The optic disks remained normal till the end, but the general condition was deteriorating. In the last phase of her life symptoms of ileus appeared. She died emaciated on the 25th June 1945, ten months after the beginning of her illness.

The post-mortem examination was done in the Institute of Prof. Dr. H. T. DEELMAN (number of the section 45020). A tumour of the left ovarium and adnexa was found with metastases in the bordering parts of the abdominal cavity. Microscopical examination of the tumour taught that this was an alveolar carcinoma. The spinal cord was not removed. Prof. H. T. DEELMAN gave the brain to the Central Institute for Brainresearch for further examination.

The surface of the brain did not show alterations and the cerebellum was not reduced in size. After cutting the brain in frontal direction no metastatic tumours were found. The ventricles were not enlarged. Pieces from various parts of the cerebral cortex were removed for staining after the method of NISSL and with haematoxylin-eosine. Of the cerebellar cortex several sections were stained after the methods of NISSL, BIELSCHOWSKY, CAJAL, HOLZER and with haematoxylin-eosine. Some sections through the midbrain with the nucleus ruber were stained after NISSL and with haematoxylin-eosine. Serial sections were made through the whole cerebellum, medulla oblongata and pons Varoli. The oral part of this series was alternatively stained after WEIGERT–PAL and VAN GIESON, the middle and caudal part after NISSL and here and there with haematoxylin-eosine.

Description of the microscopical conditions.

In the preparations of the cerebral cortex no alterations are visible. In the cerebellar cortex the Purkinje cells have completely disappeared (figure I) and in the sections, stained with BIELSCHOWSKY's silver method, basket fibres and empty baskets can be seen in several places (figure II). In nearly all of the cerebellar foliae the molecular and granular layers are normal; only in some foliae situated in ventral parts of the left neocerebellum these layers are shrunken. There is some increase of microglial nuclei in the molecular layer but there is no sclerosis and in the sections, stained after HOLZER and CAJAL, the number of BERGMANN fibres is not enlarged. The left dentate nucleus has lost almost all its cells and a denser gliosis is seen around this nucleus. On the right side cells have suffered only in the ventral part of this nucleus and several cells are missing. On the controlateral side there is a considerable increase of microglia surrounding the dentate nucleus, caused by secondary degeneration of the descending



Figure I. Cerebellar cortex (praeparation after NISSL).
Absence of Purkinje cells.

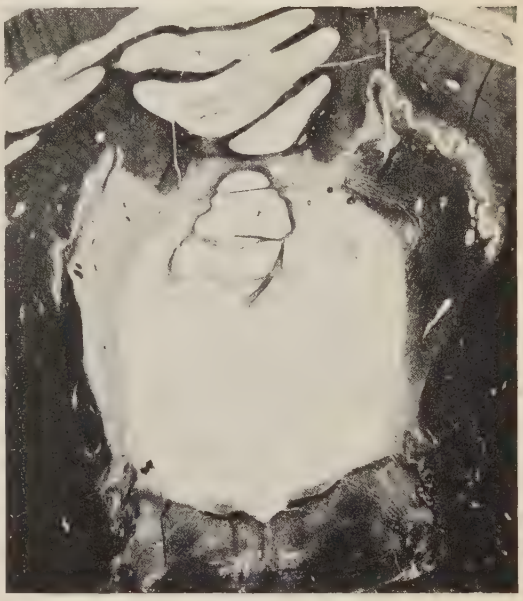


Figure III. Cerebellar nuclei (praeparation after WEIGERT–PAL).
Diminution of myelinated fibres.

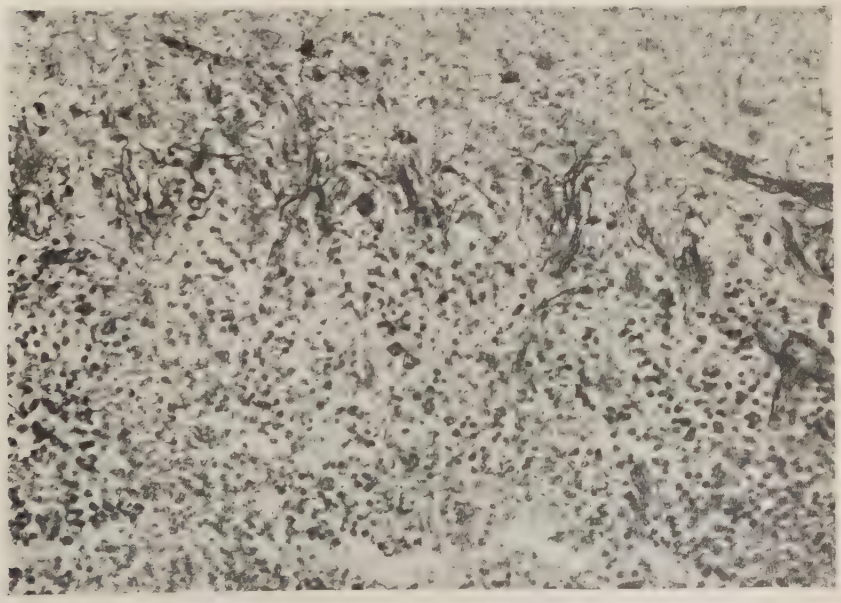


Figure II. Cerebellar cortex (praeparation after BIELSCHOWSKY).
Empty baskets in the layer of Purkinje cells.

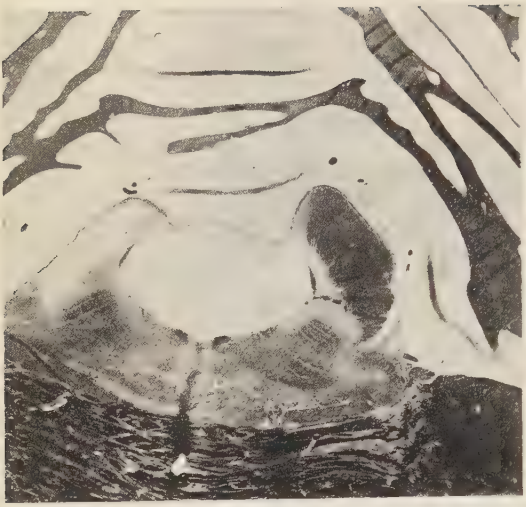


Figure IV. Brachia conjunctiva (praeparation after WEIGERT–PAL).
Secondary degeneration of the left superior cerebellar peduncle.

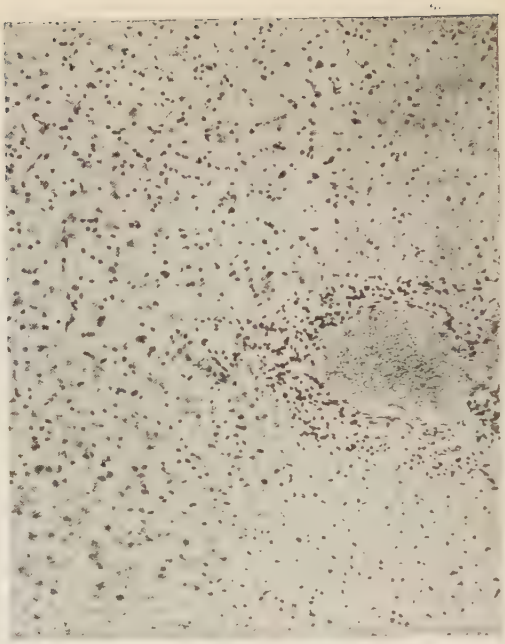


Figure VI. Deeper area of the cerebellum (praeparation after NISSL).
Perivascular accumulation of lymphocytes.



Figure V. Level of the oculomotor nerves (praeparation after WEIGERT–PAL).
Secondary degeneration on the right side of the fibres, originating in the left dentate nucleus.

fibres of the Purkinje cells. Also the nucleus emboliformis, globosus and tecti on both sides have lost many cells and are too pale in the Weigert–Pal sections owing to loss of myelinated fibres (Figure III). In consequence of the loss of cells in the left dentate nuclus the brachium conjunctivum is degenerated (Figure IV), while the ascending mesencephalic root of the trigeminal nerve and the spino-cerebellar tract of Gowers remain normally stained in the sections of this level. On the right side no secondary degeneration in the brachium conjunctivum is visible. The degeneration of the left brachium conjunctivum continues in the crossing of WERNEKINK. Figure V shows this secondary degeneration chiefly on the right side, because the fibres originating in the left dentate nuclei have already crossed the midline. The fourth ventricle and the aquaeductus Sylvii are not enlarged and the ependyma is unchanged. The cells of the inferior olives, the nuclei laterales oblongatae with the nuclei reticulo-tegmentales and the nuclei pontis are normal. There is however loss of many cells in the nuclei of Deiters and Bechterew and in the nuclei vestibulares mediales. The number of microglia cells in these regions is increased. The descending vestibular systems and the other nuclei of the cranial nerves do not show alteration. There is some loss of cells in the superior olives, but not in the nuclei of the trapezoid body. The fasciculus longitudinalis posterior and the lateral lemniscus are normal in the Nissl-sections and the cells of the nucleus ruber do not show changes. The spinal cord was not available, but there is no secondary degeneration in the lowest level of the nuclei of Goll and Burdach.

There is lymphocytic infiltration in the meninges of the cerebellar foliae and also here and there at the base of the medulla oblongata. A few plasma cells could be identified. A perivascular accumulation of lymphocytes is also formd in the deeper situated areas of the cerebellum and in the dentate nuclei (Figure VI). They are also seen in various levels of the medulla oblongata and the tegmentum pontis. The walls of the blood-vessels are not thickened and there are no haemorrhages. Nowhere in this series of sections metastatic tumour cells have been found.

Discussion.

Summarising the pathological facts, a diffuse parenchymatous degeneration of the cerebellar cortex is found, in which the Purkinje cells diappeared almost exclusively. On the other hand the inferior olives, the nuclei laterales medullae oblongatae and the pons Varoli are normal. So far this observation is in agreement with the findings in the cases of BROUWER (1919) and KENNARD (1935) and could be regarded as a new example of cerebellar toxicosis, caused by a malign tumour outside the central nervous system. However there are also differences. In the first place we found severe degeneration of the left dentate nucleus, loss of cells in both nuclei emboliformes, globosi and tecti and a partial degeneration of the cells in the right dentate nucleus. As a consequence of the disappearance of the cells

in the left dentate nucleus the corresponding fibres of the superior cerebellar peduncle were degenerated secondarily. The process had also attacked the medulla oblongata and the tegmentum pontis in which cells in the nuclei of Deiters, Bechterew and the medial vestibular centres and in the superior olives were degenerated. In the second place symptoms of inflammation are found in the meninges of the cerebellum and in the rhombencephalon itself. Such symptoms of inflammation were not present in the above mentioned cases of BROUWER, KENNARD and some others. If we did not know of the carcinoma in the abdominal cavity the conclusion could be drawn that a meningo-encephalitis localised in the rhombencephalon was present, eventually caused by a microbic virus. If so, we had to accept that the patient suffered from two diseases, which developed independently from each other. We are not inclined to this conclusion. In his article on subacute spino-cerebellar degeneration occurring in elderly patients (1934) GREENFIELD described a case, which showed several points of resemblance with our observation. Here a carcinoma of the lung was accompanied by a degeneration of the columns in the spinal cord and of the cerebellar cortex, in which especially the Purkinje cells had suffered. The dentate nuclei were degenerated, especially on the left side, where the majority of the cells had disappeared and both superior cerebellar peduncles were degenerated. Furthermore there were also changes in the cells and fibres of the oblongata, the pons Varoli and the corpus Luysii. GREENFIELD found also perivascular infiltrations of lymphocytes in both cases. These infiltrations occurred in areas where degeneration was proceeding and had to be regarded as "symptomatic" in SPIELMEYER's sence. They were caused by the degeneration of tracts and cells and needed not to be evidence of infection by a microbe or virus.

In our case the relation between the degenerated parenchymatous tissue and the perivascular infiltrations is not so schematic as in GREENFIELD's observations but on the other hand it is also clear that the number of perivascular infiltrates is especially pronounced in these areas in which the loss of cells and fibres is the most intense, i.e. in the region of the left dentate nucleus. In this respect the difference between the left superior cerebellar peduncle, which is degenerated and the right one, which is normal, is very striking: in the former many perivascular infiltrations are seen, in the latter these are almost absent.

In making a decision as to the question wether the cerebellar degeneration has to be brought in causal relation with the malign tumour in the abdominal cavity, one has to take into account that the number of such observations is still small, chiefly because the attention of the investigators is not focussed in this direction. Some authors who saw the combination of a malign tumour with cerebellar degeneration are not inclined to relate these two affections to each other. PARKER and KERNOHAN f.e. described in detail the clinico-pathological findings in a woman, aged 58, showing the typical picture of a parenchymatous cortical cerebellar atrophy, while at

the post-mortem examination a bilateral carcinoma of the ovary was found, which has metastasized into the liver and the lungs. With regard to the influence of the carcinoma on the cerebellar disease the authors only assume that in the main it constituted a terminal event. This may be so in cases, where the cerebellar degeneration had existed already many years before but in their observation the whole process lasted about 27 months, so that a causal relation seems not impossible. BIELSCHOWSKY, BOUMAN and SILLEVIS SMITT published the pathological findings in a case of familiar cerebellar atrophy, where death occurred from carcinoma in the pancreas. In this observation it has not been possible to relate this tumour in the abdominal cavity with the alterations in the central nervous system because the first symptoms of ataxia appeared already 17 years before death. In LHERMITTE's case the patient died also from a malign tumour in the pancreas, but the first symptoms of the cerebellar degeneration were already seen 12 years earlier. GARCIN, BERTRAND and Mrs. GODET–GUILLAIN gave the description of a man, aged 71, with a typical syndrome, caused by a degeneration of the cerebellar cortex combined with alterations in the dentate nuclei. The patient died from a cancer in the stomach, but the first symptoms of ataxia were already stated several years before. The degeneration was not diffuse, but limited to a part of the cerebellar cortex. We agree with the conclusion of these authors that their observation does not belong to our group of toxic degeneration. On the other hand the cases of CASPER and ZÜLCH are examples of such a cerebellar degeneration, caused by a malign tumour outside the central nervous system. CASPER examined furthermore the cerebellar cortex in 20 cases of such tumours and found in the majority of these alterations in the Purkinje cells and in the granular layer. In his above mentioned publication of spino-cerebellar degeneration occurring in elderly patients GREENFIELD concluded: "that in general the cerebellar degenerations appear to be too well systematized to be the result of virus infection and in the present state of our knowledge we can do no more than place them among the system diseases of unknown aetiology". From a recent personal communication to one of us (B.) we learned that Prof. GREENFIELD has a new case of spino-cerebellar atrophy in a woman with a silent cancer of the ovary. The spino-cerebellar tracts in the cord were degenerated and there was almost total loss of Purkinje cells and some thinning of the granule cells. Hence it may be concluded that the number of investigated cases of cerebellar or spino-cerebellar degeneration combined with tumours outside the central nervous system is gradually increasing. It may be expected that in further investigations there will be variety in the histopathological picture of the central nervous system. In this respect it is interesting to note the observation of MEYER in a recent article on the Wernicke syndrome. MEYER saw the combination of a carcinoma ventriculi with alterations in the hypothalamus and the cerebellum. Here the Purkinje cells had suffered less, but the internal granular layer of the cerebellar cortex had lost many cells.

In his report on the cerebellar atrophies (1936) HALLERVORDEN stressed the endogenous origin of these affections, but admits the cooperating influence of toxic factors on these diseases. HALLERVORDEN mentions also the significance of tumours outside the central nervous system. This is the same conclusion, which BIEMOND and BROUWER have drawn from their investigations on the primary parenchymatous degenerations of the cerebellum. The disturbance of the metabolism of the body, caused by a malign tumour, affects those parts of the central nervous system weakened from the beginning (abiotrophy in GOWER's sence). Especially in cases where the cerebellar symptoms are gradually developing in elderly patients and where an increase of protein with positive colloidal reactions are found in the spinal fluid, eventually with pleocytosis, one has to consider the presence of a malign tumour outside the central nervous system.

Summary.

In this article the result of a clinico-anatomical investigation is given, in which a malign tumour in the abdominal cavity was combined with degeneration in the cerebellum and in some regions of the medulla oblongata. The conclusion is reached that this observation is a new example of the toxic degeneration of the cerebellum, caused by a malign tumour outside the central nervous system.

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Zoology. — *The influence of lithium chloride and calcium chloride on viscosity and tension at the surface of uncleaved eggs of Limnaea stagnalis L.* By G. A. DE VRIES. (From the Zoological Laboratory, University of Utrecht). (Communicated by Prof. CHR. P. RAVEN.)

(Communicated at the meeting of November 29, 1947.)

Introduction.

Investigations on the tension at the surface and the viscosity of *Limnaea stagnalis* eggs in the uncleaved stage gave evidence of variations in both physical properties from the moment of oviposition till first cleavage (RAVEN 1945). Since Ca- and Li-salts have a great influence on the development of *Limnaea* eggs (RAVEN 1942; RAVEN & MIGHORST 1946), it was of great importance to investigate their direct effects on viscosity and tension at the surface of the eggs of this snail.

Material and methods.

Limnaea stagnalis eggs in the uncleaved stage were used for the experiments. Egg masses were obtained in the manner described by RAVEN & BRETSCHNEIDER (1942). For each experiment one egg mass, as large as possible, was used; it was ascertained by means of a binocular microscope that all eggs had a normal appearance. Only eggs taken from the middle of the egg mass were used, because the stage of development of the eggs is nearly equal there. The eggs were delivered from the adhering jelly as nearly as possible, but remained within their capsules. About 10 eggs were used to determine exactly the stage of development; 4 other samples, each of about 15 eggs, were transferred to the centrifuge tubes. Then distilled water was poured out on one of these samples, serving as a control, salt solutions on the other ones. Care was taken that there was plenty of fluid over the eggs. By shaking from time to time, adhering of the eggs was precluded and the contact with the surrounding medium rendered as intense as possible.

After a lapse of 20 minutes the eggs were centrifuged for 15 minutes in the same tubes at a velocity of ± 2750 revolutions per minute, giving a centrifugal pressure of about 1000 × gravity. Immediately after centrifugation the eggs were transferred to small glass containers and examined in transmitted light by means of a horizontal microscope. The eggs remained in the medium up to the end of the experiment. Measurings were performed by means of an ocular micrometer. Centrifuged eggs almost immediately orientate with their heavy yolk pole down. For each egg sample no more than 4 to 6 measurings were performed, to keep the error resulting from recovery to the original form as low as possible.

A total of 100 experiments, each consisting of 4 samples, have been carried out. The solutions used were: 3.0 %, 1.5 %, 0.5 %, 0.1 %, 0.08 %, 0.06 %, 0.05 %, 0.03 %, 0.02 % and 0.008 % $CaCl_2$; 1.5 %, 1.0 %, 0.6 %, 0.1 % 0.08 %, 0.06 %, 0.05 %, 0.03 %, 0.02 %, 0.01 %, 0.008 % and 0.006 % LiCl.

An attempt was made to express the relative viscosity in a quantitative manner. To this end, the centrifuged eggs were compared with a standard series of 11 photographs, numbered 0 to 10, and representing different degrees of stratification brought about by centrifuging (fig. 1). The number of this series to which the stratification of the eggs corresponded was called their "viscosity number" and considered as a measure of their viscosity. Viscosity number 0 means: stratification very distinct, viscosity low; viscosity number 10: stratification indistinct, viscosity high. I want to emphasize that my evaluations have been based primarily upon the occurrence of vacuoles or granules in the hyaloplasm zone, contrary to RAVEN's paper (1945), where the height of this zone has also been taken into account. Though, as a rule, both phenomena change concurrently, in some cases the two methods of evaluation may lead to slightly different results. As, however, the height of the hyaloplasm zone does not only depend on the viscosity, but also on the degree of stretching, i.e. on the tension at the surface, I thought it preferable to use only the first of the above-mentioned criteria.

The tension at the surface is expressed by the quotient $\frac{B}{L}$, i.e. the relation between the largest diameter B, perpendicular to the direction of the centrifugal force, and the largest diameter L, in the direction of this force (RAVEN 1945). For the sake of brevity, this quotient will be called the "tension index". The recovery to spherical form is very slow. We therefore may consider the value of this index at $\pm$ 5 minutes after centrifugation to be a measure of the tension at the surface at the moment, at which centrifugation ended.

The time was related to the developmental processes, as observed in untreated eggs of the same batch. In all instances, the time is given, at which centrifugation ended.

The experiments were performed at room temperature of $\pm$ 20° C. No attempt was made to keep it constant. Two periods of very hot weather caused a temporary rise to 32° C. This renders it difficult to compare different experiments; firstly, because there may be a direct influence of temperature on viscosity and tension at the surface; in the second place, because the rate of development increases with temperature. For the construction of the graphs experiments have been put together, performed at different temperatures. The elaboration of my results showed, however, that no regular influence of temperature upon viscosity and tension at the surface could be deduced from them; obviously, the effect of this factor is only small as compared with the individual differences between

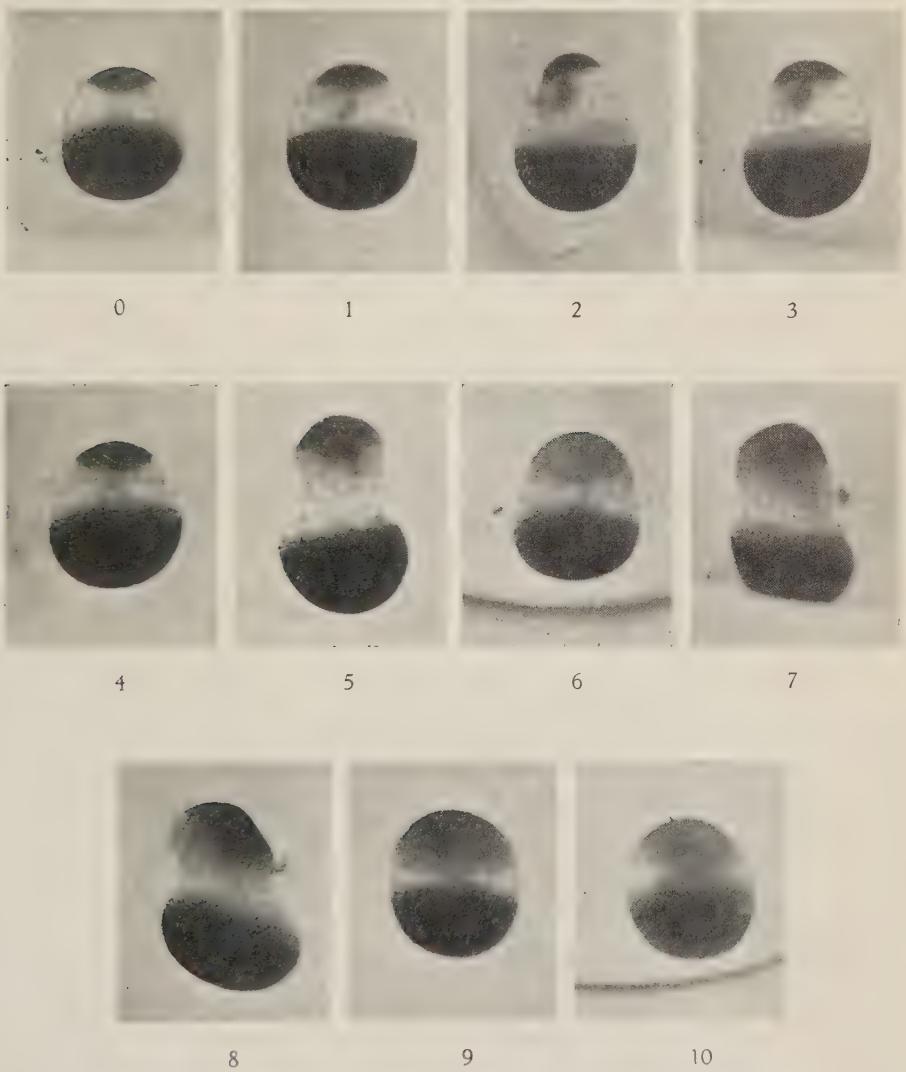


Fig. 1. Standard series of photographs of centrifuged eggs, representing viscosity numbers 0 to 10. Cf. text.

the batches. So, we may neglect this source of error in considering only the major trend of the curves. In the single experiments, the influence of this factor does not play a part, as the samples of one experiment were all at the same temperature.

In the graphs, the time between the two maturation divisions was taken as 60 minutes, the time between second polar body formation and first cleavage as 100 minutes.

In order to have a measure of the influence of the salt solutions on viscosity and tension at the surface, each sample was compared with its control. The difference between the viscosity numbers and tension indices of both will be called the "deviation"; it is taken positive, when the value of the salt-treated sample is higher, negative when it is lower than that of the control. From the deviations of all samples, treated with a same solution, an average deviation for this solution can be computed.

Viscosity.

In fig. 2, the viscosity of control samples has been plotted against time. When one compares this viscosity curve with the curve, given by RAVEN (1945, fig. 8), there is a rather good agreement. Before the first maturation

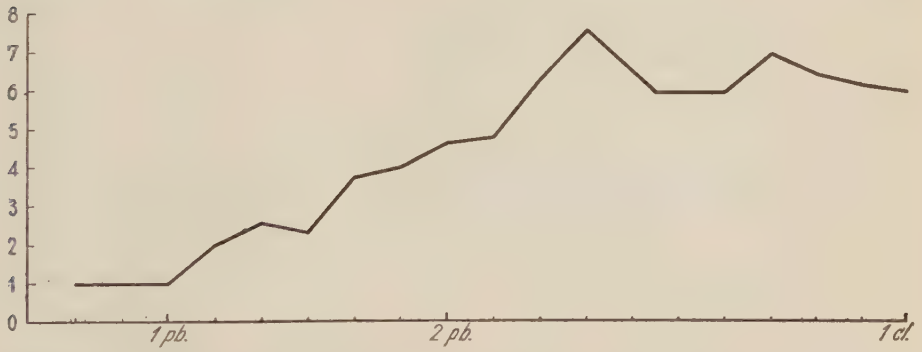


Fig. 2. Graph showing change of viscosity of normal eggs from oviposition till 1st cleavage. Abscissae: time. Ordinates: viscosity number.

division viscosity in both curves is low. Between the two maturation divisions viscosity rises earlier and to a greater height in my curve than in RAVEN's. Viscosity at the moment of the extrusion of the second polar body is "medium" in my curve, "low" in RAVEN's. This is due to a difference in evaluation of viscosity. RAVEN e.g. indicates the viscosity of his photographic pictures 8 and 9 as "rather low", whereas it is called "medium" by me (viscosity number 4). Evidently, this difference is due to the different methods of evaluating the viscosity, referred to above, RAVEN laying more stress on the fact that the height of the hyaloplasm zone has not yet diminished at this moment. After the extrusion of the second polar body there is in both curves a rapid rise to a maximum, after which there is a decrease. This drop of viscosity shortly before first

cleavage in much more pronounced in RAVEN's curve, than in mine. Again, this may be due partly to differences in evaluation. However, this does not suffice to explain the difference. The eggs of RAVEN (1945, fig. 7, egg 17 and 18) centrifuged 36 and 20 minutes before 1st cleavage, respectively, show a stratification corresponding to my viscosity numbers 1—3. The same holds true for the eggs pictured by Miss HEIKENS (1947, fig. 1, egg 1 and 2), centrifuged about 30 and 10 minutes before 1st cleavage. In my experiments, during this period an average viscosity of 6—7 was observed. In RAVEN's and Miss HEIKENS' experiments, the eggs had been centrifuged for 5 minutes at a velocity of 3800 revolutions per minute (1860 × gravity), whereas I centrifuged for 15 minutes at 2750 revolutions per minute (1000 × gravity). Probably, this difference in velocity and duration of centrifugation is responsible for the difference of the results. Furthermore, RAVEN's eggs were kept in tap water, whereas I centrifuged them in distilled water; perhaps, the difference in pH between these media has some influence on the viscosity of the eggs.

Table I shows the average deviations of viscosity number in the salt solutions. 1.0 %, 0.6 % and 0.1 % LiCl and 1.5 %, 0.5 % and 0.1 % $CaCl_2$ solutions all caused a significant decrease of viscosity. Greater dilutions of both salts had no effect, but with still greater dilutions there is a tendency to increase the viscosity; this increase is significant in the case of 0.01 % and 0.006 % LiCl and 0.05 % $CaCl_2$. The number of experiments for each

TABLE I.

Average deviations of viscosity number and tension index in various salt solutions.

		Number of experim.	Deviations of	
			Viscosity number	Tension index
LiCl	1.0 %	9	**— 2.3 ± 0.9**	**+ 0.15 ± 0.03**
	0.6 %	24	**— 2.3 ± 0.6**	**+ 0.09 ± 0.03**
	0.1 %	23	**— 1.4 ± 0.3**	**— 0.07 ± 0.02**
	0.08 %	14	— 0.1 ± 0.3	— 0.04 ± 0.04
	0.06 %	5	0 ± 0	— 0.01 ± 0.04
	0.05 %	5	0 ± 0.3	— 0.03 ± 0.02
	0.03 %	5	+ 0.2 ± 0.2	— 0.09 ± 0.07
	0.02 %	5	+ 1.0 ± 0.8	+ 0.03 ± 0.03
	0.01 %	5	**+ 0.6 ± 0.2**	+ 0.07 ± 0.05
	0.008 %	5	+ 0.2 ± 0.2	— 0.05 ± 0.03
	0.006 %	5	**+ 1.0 ± 0.3**	+ 0.09 ± 0.05
$CaCl_2$	1.5 %	27	**— 3.2 ± 0.5**	**+ 0.12 ± 0.02**
	0.5 %	23	**— 1.2 ± 0.3**	+ 0.02 ± 0.03
	0.1 %	20	**— 1.1 ± 0.2**	— 0.02 ± 0.03
	0,08 %	14	— 0.3 ± 0.4	+ 0.01 ± 0.03
	0.06 %	11	+ 0.5 ± 0.4	+ 0.02 ± 0.04
	0.05 %	13	**+ 1.0 ± 0.3**	— 0.06 ± 0.05
	0.03 %	8	— 0.4 ± 0.3	— 0.01 ± 0.02
	0.02 %	20	+ 0.3 ± 0.3	+ 0.02 ± 0.03
	0.008 %	5	+ 0.2 ± 0.2	— 0.03 ± 0.02

of these solutions is only small; however, when one takes together the experiments with 0.03 % to 0.006 % LiCl (25 experiments), there is a significant increase (0.6 ± 0.2). It seems, therefore, that this increase of viscosity at great dilutions is real.

With 1.5 % LiCl and 3 % $CaCl_2$, no stratification appeared; the eggs are dark and flattened perpendicularly to the centrifugal force. Apparently, these highly hypertonic solutions withdraw too much water from the eggs. Although the osmotic value of a 3.0 % $CaCl_2$ solution is higher than that of a 1.5 % LiCl-solution, the effect of the latter is much stronger. Variations of permeability may have played a part. One could try to explain the observation by assuming that $CaCl_2$ renders the egg more impermeable to water.

According to HARVEY (1945), stratification of *Arbacia punctulata* eggs by centrifugation is more rapid in $CaCl_2$ than in isosmotic NaCl or KCl solutions. The viscosity increases in the order: $CaCl_2 < MgCl_2 <$ seawater $<$ NaCl $<$ KCl. In my experiments, isosmotic solutions of $CaCl_2$ and LiCl did not differ significantly in their effect upon the viscosity of the egg.

Tension at the surface.

The curve in fig. 3, obtained by plotting the tension indices of the controls against time, shows a periodical variation of the tension at the surface of normal eggs in the uncleaved stage. It agrees rather well with RAVEN's curve (1945, fig. 8); the tension at the surface drops at both maturation divisions and before the first cleavage in both cases. However, in my curve the variations are somewhat less pronounced. This may be due to the difference in centrifugal force employed.

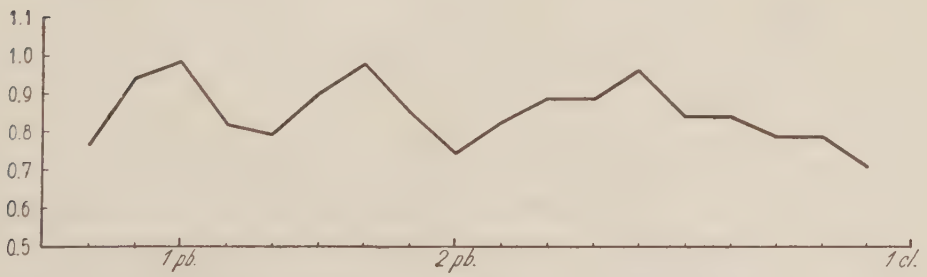


Fig. 3. Graph showing variations of tension at the surface of normal eggs from oviposition till 1st cleavage. Abscissae: time. Ordinates: tension index.

In table I, the average deviations of tension index in the salt solutions are given. There is a significant increase of tension at the surface in 1.0 % and 0.6 % LiCl and in 1.5 % $CaCl_2$; on the contrary, there is decrease of this tension in 0.1 % LiCl. Lower concentrations gave no significant effects.

According to HARVEY (1945), *Arbacia punctulata* eggs break with increasing rate in the following order: $CaCl_2 <$ seawater $<$ NaCl $<$ KCl.

Hence, the tension at the surface is increased in isosmotic $CaCl_2$ solution, decreased in isosmotic NaCl and KCl solutions. In my experiments, $CaCl_2$ increases the tension only at a rather high concentration (1.5 %), whereas LiCl gives an increase or a decrease according to concentration.

There is a typical influence of certain concentrations on the shape of the second polar body. The polar body is drawn out during centrifugation and gets the form of a carrot, the broad end turned towards the egg. This malformation took place in 1.5 % $CaCl_2$ and 1.0 % and 0.6 % LiCl, but not always. Control eggs in distilled water did never show a similar malformation of the second polar body. This phenomenon may be due to a decrease in tension at the surface of the polar body, though viscosity may play a part as well.

Discussion.

As mentioned above, the degree of stratification of the egg after centrifugation, especially the abundance of granules and vacuoles in the hyaloplasm zone, has been taken as a measure of viscosity.

The first question we have to answer is, whether a change in this respect, caused by the action of the salt solutions, really points to a change of viscosity. The altered degree of stratification in salt solutions might be due e.g. to a change in the specific gravity of the inclusions of the egg.

The vacuoles in the hyaloplasm zone, according to RAVEN (1945), arise from γ-granules, which have a high specific gravity immediately after oviposition, but become lighter by swelling and, then, remain in the hyaloplasm zone at centrifugation.

The swelling of the granules is due to an absorption of water by the eggs from the surrounding egg capsule fluid. By hypertonic solutions, water might be withdrawn from the vacuoles; in this way, their specific gravity would increase and at centrifugation they would accumulate in the centrifugal part of the egg. This would give rise to a decrease of the number of inclusions of the hyaloplasm zone and, therefore, to an apparent decrease of viscosity.

However, we have seen that a decrease of viscosity number is not only caused by hypertonic solutions, but also by 0.1 % LiCl and $CaCl_2$ solutions, which are considerably hypotonic to the eggs. Furthermore, the increase of viscosity at still greater dilutions cannot be explained in this way, either. We must conclude, therefore, that besides possible osmotic effects, real changes of viscosity under the influence of LiCl and $CaCl_2$ solutions have taken place.

In colloid chemistry, the effect of a decrease of viscosity by adding electrolytes to hydrophilous colloids is well known. The viscosity is increased by an electric charge. This phenomenon is called "the effect of electro-viscosity". The ζ-potential, or kinetically active difference in potentials, of the proteins is lowered by the first millimoles. When the ζ-potential is zero, the effect of electro-viscosity has disappeared. A further adding

of electrolyte may cause a reverse of charge, and consequently a rise of viscosity.

The egg protoplasm probably contains hydrophilous colloids for a greater part. Therefore, this factor may be of great importance. Other factors may play a part in lowering the viscosity as well, e.g. the influence of kations on the combination of the protein molecules.

In this connection, the experiments carried out by RANZI (1943—46) on the influence of certain salt solutions, among which LiCl, on the eggs of *Rana virescens* should be mentioned. He prepared an extract from *Rana* gastrulae and separated from it different protein fractions. In all experiments, the action of LiCl took place during 12 hours at 2° C; after that, relative viscosity was determined at 14.6° C by means of an Ostwald-viscosimeter. He found a rise of viscosity in concentrations from 0.42 % up to 2.1 % LiCl. Lower concentrations did not change viscosity. This fact certainly deserves attention in connection with the above-mentioned effect of electro-viscosity. The difference in results probably is not due to a reverse in charge, but perhaps to a process of denaturation during the 12 hours' stay in the solution at low temperature.

In my experiments, the *Limnaea* eggs remained undamaged, contrary to RANZI's *Rana* eggs. The egg surface plays an important part and can cause a limitation of the amount of ions passing. My experiments are, therefore, not comparable with RANZI's.

According to HARVEY and SHAPIRO (1941), the eggs of *Arbacia punctulata* and *Asterias forbesii* have a cortical layer with considerably higher viscosity than the interior plasm. By the presence of this layer the egg is inhomogeneous. It is supposed that the amount of $Ca^{\cdot\cdot}$-ions combined with proteins at the cell surface depends upon the $Ca^{\cdot\cdot}$-ion concentration in the medium and that it determines the firmness of the cell membrane (SHAPIRO 1941).

BUNGENBERG DE JONG (1935) considers the protoplasmic membranes as phosphatide autocomplex systems and can give an explanation of the tightening action of $Ca^{\cdot\cdot}$-ions in certain concentrations. Although tension at the surface is not mentioned, we could imagine that this property becomes higher by the increase of electric attractions. From his experiments with phosphatide coacervates it is evident that the tightening influence rapidly decreases in the order: $CaCl_2 > LiCl > NaCl > KCl$.

As already mentioned above, in *Limnaea* the egg surface probably is rendered more impermeable to water by Ca, while the tension at the surface increases.

The rate of elongation certainly will depend upon the degree of viscosity (RAVEN 1945), although in my experiments no relation can be traced. On the other hand, viscosity may depend upon the character of the membrane, e.g. tension at the surface and permeability.

HARVEY (1945) centrifuged unfertilized *Arbacia punctulata* eggs in isosmotic salt solutions and found a decrease of rate of stratification in

the following order: $CaCl_2 > MgCl_2 >$ seawater $> NaCl > KCl$. This would point to an increasing viscosity. The eggs break in two parts more easily in the same order. In $CaCl_2$, they stratify best and break with the greatest difficulty. The effect of breaking is due to an alteration of the surface.

These experiments are of course not quite comparable with mine. Anyhow, in sea urchins $CaCl_2$ causes a more distinct stratification and a rise of the tension at the surface, results, which I found also for *Limnaea* eggs with higher $CaCl_2$ concentrations. Whereas, however, in HARVEY's experiments monovalent kations had the opposite effect, in *Limnaea* the effect of the Li^{+}-ion on the viscosity is entirely comparable to that of the Ca^{++}-ion, while its effect on the tension at the surface agrees with that of Ca in hypertonic solutions, but is reversed in a slightly hypotonic solution.

Summary.

1. The influence of LiCl and $CaCl_2$ solutions on viscosity and tension at the surface of eggs of *Limnaea stagnalis* was studied by centrifugation.

2. The viscosity of the eggs is decreased by 0.1 %—1.0 % LiCl and 0.1 %—1.5 % $CaCl_2$ solutions; it is increased by 0.006 %—0.03 % LiCl and 0.05 % $CaCl_2$ solutions.

3. The tension at the surface is increased by 0.6 %—1.0 % LiCl and 1.5 % $CaCl_2$ solutions; it is decreased in 0.1 % LiCl solutions. Lower concentrations of both salts gave no significant effects.

4. The solutions which increased the tension at the surface of the egg apparently decrease this tension of the second polar body.

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Mineralogy. — *On cassiterite and garnet from the Kaokoveld, S.W. Africa.* By L. P. G. KONING. (Communicated by Prof. H. A. BROUWER.)

(Communicated at the meeting of November 29, 1947.)

Introduction.

In the present paper the results of the crystallographic and optical investigation of cassiterite and garnet from the Kaokoveld, S.W. Africa will be given.

These minerals have been sent to me for investigation by Mr. G. W. BRANDT from the West Africa Company, Grootfontein, Southwest Africa, for which I wish to express my gratitude.

The crystallographic measurements have been carried out with a two-circle goniometer. The collection of minerals contained no complete crystals. The size varied from 4—20 mm for the cassiterite and from 4—10 mm for the garnet.

The cassiterite is mainly yellowish and greenish brown coloured and somewhat transparent; sometimes, however, the mineral is black coloured. The prismatic faces are mainly greenish and yellowish brown and slightly transparent; the pyramidal faces are black. Mainly the black faces are lustreous.

The garnet is black coloured and opaque. The rhombendodecahedron {110} is highly lustreous.

a. *Cassiterite.*

1. Crystallographic investigation of the cassiterite.

Several crystals have been subjected to crystallographic measurements. The mineral is ditetragonal dipyramidal. The prisms are highly striated parallel to the *c*-axis.

Forms: *b*(010), *a*(100), *m*(110), *h*(120), *s*(111), *e*(011), see fig. 1. The forms *m*(110) and *s*(111) have the principle development.

The results of the crystallographic measurements are tabulated in table I.

TABLE I.

Face	Symbol	φ	ϱ
1 *b*	(010)	0°	90°
2 *a*	(100)	90	90
3 *m*	(110)	45	90
4 *h*	(120)	26 30′	90
5 *e*	(011)	0	33 55′
6 *s*	(111)	45	43 32

From calculations it followed:

$$p'_0 = 0{,}6721$$
$$a : c = 1 : 0{,}6721$$
$$f = d = 56°\ 06'$$

In the crystallographic system of BARKER [1]) the mineral cassiterite is characterized by the angle

$$\mathbf{cr = 33°\ 55'}$$



Fig. 1.

2. Optical investigation of the cassiterite.

As the mineral cassiterite did not lend itself to optical measurements I studied thin sections and found the mineral optically uniaxial and positive with very high birefringence. The colour in the sections showed an irregular zonal and segmental distribution. The brownish grey central parts are darker than the surrounding parties, while reddish brown bands and segments are irregularly distributed within them. Cleavage parallel to $a(100)$ and parallel to $m(110)$ is present but not perfect. A distinct dichroism in reddish brown colours is observed: $\varepsilon > \omega$.

b. Garnet.

1. Crystallographic investigation of the garnet.

Several fragments of the black and highly lustreous, isometric garnet crystals have been subjected to crystallographic measurements.

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[1]) T. V. BARKER, Systematic crystallography, an essay on crystal description, classification and identification, London (1930). — P. TERPSTRA, Kristallometrie, Groningen (1946).

The following crystal forms have been observed: $d(110)$, $n(211)$, $s(321)$, $t(431)$ see fig. 2.

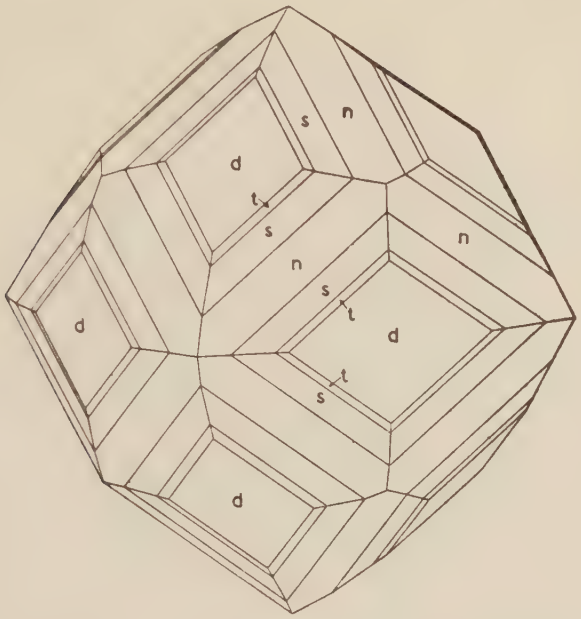


Fig. 2.

From these forms the rhombendodecahedron $d(110)$ has not only the principle development, but shows also the highest reflection capacity, whereas the form $t(431)$ has the minor development.

2. Optical investigation of the garnet.

As the mineral garnet was not suitable for measuring optical data, a thin section has been prepared for microscopical investigation. From this microscopical investigation it followed that the central parts of the garnet are composed of a mineral association, a paragenesis, about which more details shall be given here.

The mineral association showed little or no alteration phenomena, so that all the minerals present could be determined microscopically.

Brief description of the minerals.

Garnet.

The garnet is light reddish brown coloured and appeared wholy isotropic. The garnet grains present in a large number and belonging to the same original individual are in general irregularly shaped; sometimes crystallographic boundaries or a tendency to idiomorphism are present (see fig. 3).

The index of refraction could not be exactly determined. With the Becke test an index of refraction of n between 1,74 and 1,78 was found.



Fig. 3. Garnet and epidote crystals, showing a tendency to idiomorphism, surrounded by calcite, feldspars and quartz. Magnification about 30 ×.

Epidote.

A large part of the mineral association is composed of epidote. In several cases well-developed elongated prismatic crystals could be observed. The refractive indices and birefringence are high, while the mineral appeared to be optically biaxial negative. The epidote showed distinct cleavage lines parallel to $c(001)$ and parallel to $a(100)$ and many cracks in different directions.

In general the spaces between the garnet grains are filled up by epidote crystals, these crystals belonging to the same epidote individual. The remaining spaces between the garnet grains are occupied by either calcite, or quartz or feldspar, also belonging to the same individual of respectively calcite, quartz or feldspar.

The epidote is light yellowish green coloured and showed distinct pleochroism. Sometimes twinning is observed.

Feldspar.

The mineral association is also characterized by the presence of two feldspars: a potash-feldspar and a plagioclase. The potash-feldspar, which dominates over the plagioclase, appeared to be microcline. In some places the characteristic grating-structure could be observed. The plagioclase, only present in a small quantity was determined as albite; this mineral

showed twinning according to the albite-law, giving rise to a fine striation. In parallel polarized light the feldspars appear turbid and brownish from the presence of very minute scales of kaoline and flakes of sericite, due to alteration. The feldspar grains are irregularly shaped and show distinct cleavage lines parallel to $c(001)$ and to $b(010)$.

Calcite.

In the mineral association calcite is one of the principle constituents. This mineral, characterized by its large birefringence and its optically uniaxial negative sign, showed in a few cases twinning lamellae according to $(01\bar{1}2)$. The crystal grains are irregularly shaped.

Quartz.

Some grains of quartz, filling the remaining spaces between the above described minerals are present.

Apatite, chlorite and orthite.

Minor constituents of the mineral association are apatite, chlorite and orthite. Apatite occurs as small idiomorphic crystals, sometimes containing minute indeterminable, finely distributed particles, causing a dark grey colour and resulting in a distinct pseudopleochroism from light to dark grey.

Some small grains of brown orthite are present as inclusions in epidote. Chlorite occurs as small green flakes, showing distinct pleochroism.

3. General discussion of the paragenesis.

As mentioned above the interior of the garnet contains a definite mineral association. The garnet may be regarded as a shell, so-called perimorph, surrounding a nucleus of other mineral species.

The mode of occurrence of garnet and epidote seems to suggest that they belong to the first crystallization products of the solidifying substance. In some places a replacement of the garnet by epidote may be observed.

After the crystallization of the principle constituents of the paragenesis — garnet and epidote — had reached a certain degree, the other minerals originated from the solidifying substance. These minerals — feldspars, calcite and quartz — are present as irregularly shaped grains.

The thin section of the paragenesis described in the present paper gives the impression that a primary mineral association is dealt with: in the calcite the small extent of twinning lamellae may point to the fact, that this mineral has hardly been subjected to mechanical influences. This and the mutual relations between the calcite and the other constituents of the mineral association seems to justify considering the calcite as a primary crystallization product.

Geological Institute of the University of Amsterdam.

Mineralogy. — *On manganite.* By L. P. G. KONING. (Communicated by Prof. H. A. BROUWER.)

(Communicated at the meeting of November 29, 1947.)

In a collection of minerals used for pyrochemical experiments in the Geological Institute of the University of Amsterdam I found a piece of ore wholy consisting of manganite.

Though the place of occurrence is not known the very well-developed crystals, among which also a beautiful penetration twin, containing a large number of forms, justified a detailed crystal description. The crystal measurements have been made with a two-circle goniometer.

Nine prismatic crystals elongated parallel to the *c*-axis and a penetration twin, with a size of 1—5 mm, have been subjected to crystallographic investigation, the results of which will be given in the present paper.

The crystals are characterized by the following prismatic forms:

$a(100)$, $b(010)$, $m(110)$, $l(120)$, $d(210)$, $q(310)$, $t(250)$ and the rare forms $r(1\bar{5}0)$, $w(520)$, $k(230)$, $i(430)$, $y(720)$, $h(410)$, $z(490)$, $\delta(\bar{5}80)$, $\varepsilon(5\bar{6}0)$, $\omega(470)$, $\mu(5\bar{1}0)$, $\eta(370)$, and by the following pyramidal forms: $u(101)$, $p(111)$, $n(121)$, $v(221)$, $g(313)$, $\alpha(515)$, and the rare forms: $s(212)$, $\varphi(323)$, $\beta(525)$, $\gamma(\bar{5}35)$, $x(4\bar{1}4)$, $\psi(7\bar{1}7)$, $\sigma(10.3.10)$, $\varrho(3\bar{4}2)$, $\vartheta(\bar{3}44)$, $\tau(\bar{1}2.5.15)$, $\theta(103)$ and $c(001)$, see fig. 1.

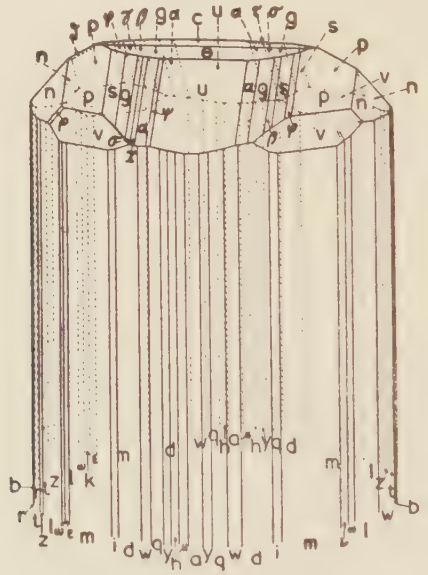


Fig. 1.

Terminal faces are only found at one side. Among these forms the prism $m(110)$ has the principle development, whereas $b(010)$ is present in very narrow faces. Among the pyramidal forms, $u(101)$, $p(111)$ and $n(121)$

TABLE I.

Nr. crystal / Face	1	2	3	4	5	6	7	8	9	Twin I	Twin II
m (110)	m	m	m	m	m	m	m	m	m	m	m
l (120)	l	l	l	l	l	l	—	l	l	l	l
q (310)	q	q	q	q	—	q	q	q	q	q	q
d (210)	d	d	d	d	d	—	d	—	—	d	d
k (230)	—	k	—	k	k	—	k	k	—	k	k
h (410)	h	h	h	h	—	h	—	—	—	—	h
ω (470) *	—	ω	ω	—	—	ω	ω	ω	ω	—	—
b (010)	b	b	b	—	b	—	—	b	—	—	—
t (250)	—	—	t	t	—	t	—	t	t	—	—
i (430)	i	—	i	—	—	—	i	—	—	i	i
w (520)	—	w	—	—	w	—	—	—	—	w	w
y (720) *	—	—	y	—	—	y	—	—	—	y	y
z (490) *	z	z	z	—	—	—	—	z	—	—	—
μ ($5\bar{1}0$) *	—	μ	—	—	—	—	—	μ	—	μ	μ
a (100)	—	a	—	—	—	—	—	—	—	—	a
δ ($\bar{5}80$) *	—	—	—	—	—	—	—	—	—	δ	δ
η (370) *	—	—	—	—	—	—	—	—	—	η	η
r ($1\bar{5}0$)	—	—	—	—	—	—	—	r	—	—	—
ε ($5\bar{6}0$)	—	—	ε	—	—	—	—	—	—	—	—
g (313)	g	g	g	g	g	g	g	g	g	g	g
p (111)	—	—	p	—	p	p	p	p	p	p	p
s (212)	—	s	—	—	s	s	s	s	s	s	s
u (101)	—	u	u	u	—	—	u	u	u	u	u
n (121)	—	—	n	—	n	n	—	n	n	n	n
α (515)	α	—	α	—	α	—	α	—	α	α	α
v (221)	—	—	v	—	—	v	v	v	—	v	—
θ (103)	—	—	—	—	—	—	θ	—	—	θ	θ
φ (323)	φ	—	φ	—	—	—	—	—	—	φ	—
β (525)	β	β	—	—	—	—	—	—	—	—	—
x ($4\bar{1}4$)	x	x	—	—	—	—	—	—	—	—	—
σ ($10.\bar{3}.10$) *	—	—	—	σ	—	—	—	—	σ	—	—
c (001)	—	—	—	—	—	—	—	—	—	c	—
γ ($\bar{5}\bar{3}5$) *	—	—	—	—	γ	—	—	—	—	—	—
ψ ($7\bar{1}7$) *	—	—	—	ψ	—	—	—	—	—	—	—
ϱ ($3\bar{4}2$) *	—	—	—	—	—	—	—	—	ϱ	—	—
ϑ ($\bar{3}44$) *	—	—	—	—	—	ϑ	—	—	—	—	—
τ ($\bar{1}2.5.15$) *	—	—	—	—	τ	—	—	—	—	—	—

are the principle ones, whereas the other faces belonging to the zone [101] are present in more or less narrow faces. In some cases the form $v(221)$ has a minor development. The distribution of the several forms observed in the investigated crystals have been tabulated in table I, the new forms being marked *.

Several faces of manganite show a distinct striation, particularly the faces belonging to the zone [001] and those faces of the zone [101] lying between $p(111)$ and $u(101)$.

Manganite has been determined morphologically as orthorhombic, though BUERGER [1]) has presented X-ray and polished section evidence showing that manganite is monoclinic.

The results of the crystallographic measurements have been tabulated in table II.

TABLE II.

Face	Symbol	φ	ϱ	Face	Symbol	φ	ϱ
1 c	(001)	—	0°	20 δ	$(\bar{5}80)$*	323° 24′	90°
2 b	(010)	0°	90	21 u	(101)	90	32 53′
3 t	(250)	25 22′	90	22 p	(111)	49 50	41 15
4 z	(490)*	27 55	90	23 n	(121)	30 43	51 45
5 l	(120)	30 43	90	24 s	(212)	67 12	35 10
6 ω	(470)*	34 06	90	25 g	(313)	74 31	33 56
7 k	(230)	38 29	90	26 v	(221)	49 50	59 32
8 m	(110)	49 50	90	27 α	(515)	80 27	33 10
9 i	(430)	57 36	90	28 φ	(323)	60 25	36 30
10 d	(210)	67 12	90	29 β	(525)	71 28	34 15
11 w	(520)	71 28	90	30 γ	$(\bar{5}\bar{3}5)$*	243 15	35 49
12 q	(310)	74 31	90	31 x	$(4\bar{1}4)$	102 08	33 25
13 y	(720)*	76 39	90	32 ψ	$(7\bar{1}7)$*	96 48	33 03
14 h	(410)	77 52	90	33 σ	$(10.\bar{3}.10)$*	103 59	33 28
15 a	(100)	90	90	34 τ	$(\bar{1}\bar{2}.5.15)$*	288 35	28 25
16 μ	$(5\bar{1}0)$*	99 33	90	35 ϱ	$(3\bar{4}2)$*	138 23	55 24
17 ε	$(5\bar{6}0)$	135 33	90	36 θ	(103)	90	11 53
18 r	$(1\bar{5}0)$	162 42	90	37 ϑ	$(\bar{3}44)$*	221 13	37 02
19 η	(370)*	26 46	90				

From these results the following numerical data could be calculated:

$p'_0 = 0{,}6455$ $\quad$ $a = 0{,}8431$

$b = 1$

$q'_0 = 0{,}5442$ $\quad$ $c = 0{,}5442$

$d = 61°\ 27'$

$f = 57°\ 09'$

[1]) J. M. BUERGER, The symmetry and crystal structure of manganite, Zeitschr. f. Krist., **95**, 163 (1936).

The calculations of the elements according to HAIDINGER [2]) differ slightly from te above-mentioned results:

$$p'_0 = 0{,}6454 \qquad a = 0{,}8441$$
$$b = 1$$
$$q'_0 = 0{,}5448 \qquad c = 0{,}5448$$

The calculation of the characteristic angles for orthorhombic manganite according to BARKER's [3]) systematic crystallography of crystals leads to the following results:

$$\mathbf{cr} = \mathbf{40^\circ\ 10'}$$
$$bq = 52^\circ\ 24\tfrac{1}{2}'$$
$$am = 42^\circ\ 21\tfrac{1}{2}'$$

The twin of manganite.

As has been mentioned in litterature on manganite this mineral may occur either as contact or as penetration twin. The twin plane is (011). The twin crystal investigated was a very nice-developed penetration form with twin plane ($0\bar{1}1$). The two individuals penetrate each other excentrically (see fig. 2).

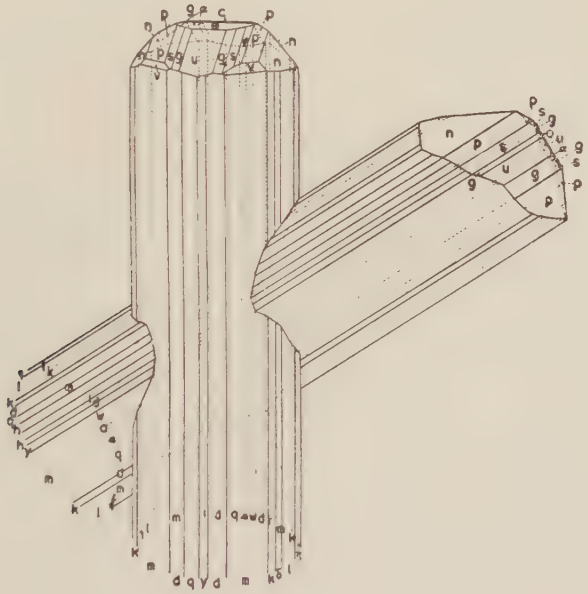


Fig. 2.

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[2]) As the publication of HAIDINGER (Edinburgh J. Sc., **4**, 41 (1826)) has not been accessible to me, reference is made to DANA, A system of Mineralogie, 249 (1892) and the new edition by PALACHE, BERMAN, FRONDEL, Vol. I, 646 (1946).

[3]) T. V. BARKER, Systematic crystallography, an essay on crystal description, classification and identification, London (1930). — P. TERPSTRA, Kristallometrie, Groningen (1946).

Investigation of the polished section.

Under the microscope the polished section of the manganite appeared to be composed of manganite crystals with veinlets and small particles of probably pyrolusite. The manganite showed distinct anisotropy between crossed nicols: parallel to the prismatic direction bluish grey and perpendicular to this direction dark violet grey. Distinct cleavage lines parallel to $b(010)$ or to $m(110)$ have been observed.

The pyrolusite, stronger reflecting than manganite is yellowish white and showed reflection pleochroism. Anisotropy between crossed nicols was also observed. The pyrolusite is only present as veinlets and small particles and may originate from manganite [4]). In many cases the pyrolusite veinlets follow the cleavage lines of the manganite.

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[4]) H. SCHNEIDERHÖHN und P. RAMDOHR, Lehrbuch der Erzmikroskopie, 2. Bd., 555 (1931).

Geological Institute of the University of Amsterdam.

Zoology. — *Remarks upon the species of the genus Lagenorhynchus.* I. By W. H. BIERMAN (Haarlem) and E. J. SLIJPER (Institute of Veterinary Anatomy, State University, Utrecht). (Communicated by Prof. G. KREDIET.)

(Communicated at the meeting of November 29, 1947.)

I. *Lagenorhynchus wilsoni* Lillie.

On board the Dutch floating factory "Willem Barendsz" we saw on the 14th of April 1947 between 11.30 and 11.45 at about 48° 59′ S. and 6° 36′ E. four small black and white Dolphins swimming and leaping around the bow of the ship. That same day between 15.00 and 16.00 and also the next day at about 46° 52′ S. and 8° 30′ E. a school of the same animals was seen by Mr H. VAN DER LEE from a whale catcher. The temperature of the sea-water on these two days was resp. 3,7° C. and 5,5° C. The animals moved very quickly, they came above water with a tumbling movement that reminded of the movements of penguins at the surface of the water, and they breathed in a fraction of a second. Sometimes they leaped completely out of the water. We often saw them swimming in pairs.

The length of the animals was estimated at 1.50—2.00 m. The short beak, well marked off from the moderately arched forehead, the high pointed back fin with concave caudal border and the medium sized, pointed flippers immediately showed that the animals belonged to the genus *Lagenorhynchus*. Fig. 1 shows that the cranial part of the head, the back fin, the tail fin and the flippers were black, as well as the whole back up to a point lying about 20—30 cm cranially of the tail-fin. The lateral side showed a black band, joining the black of the head with that of the tail-fin. At the level of the dorsal fin the ventral and dorsal black were joined by a black band extending in caudo-dorsal direction. In some animals, however, this band was very narrow. We could not see the colour of the ventral side of the head and trunk; the lower parts of the tail were white. The swimming animal is usually characterised by the two large lateral patches of white, separated just below the dorsal fin by a black band. It is also characterised by the fact that the right and left lateral patches join above the tail. Thus at the dorsal side there is an area of white between the black of the back and that of the tail-fin.

Now it is just by this area of white by which Wilson's Hourglass Dolphin (*Lagenorhynchus wilsoni* Lillie) is characterised and by which it can be distinguished from *Lagenorhynchus cruciger* (Quoi et Gaim.) [= *L. cruciger* (d'Orb. et Gerv.)]. FRASER (1937, p. 322) doubts whether the two animals belong to one single species or not and thinks that Wilson's Dolphin should be characterised by a more pronounced snout, a

somewhat larger size and by the fact that the lower black band would meet the one on the other side on the under side of the body in the region of the flipper insertion. Since, however, *Lagenorhynchus wilsoni* has always been observed in the open sea and never ashore or on the deck of a ship, the first two characteristics have little value and about the third one nothing appears in the text or the figures of WILSON (1907; p. 9, fig. 7). The only reliable difference between the two species (subspecies or varieties; see p. 1364) is the white above the tail. WILSON (1907) thought that the animals were 2.60—3.30 m long, but LILLIE (1915; p. 123) estimated their length at 1.80—2.20 m and this corresponds with our estimation.

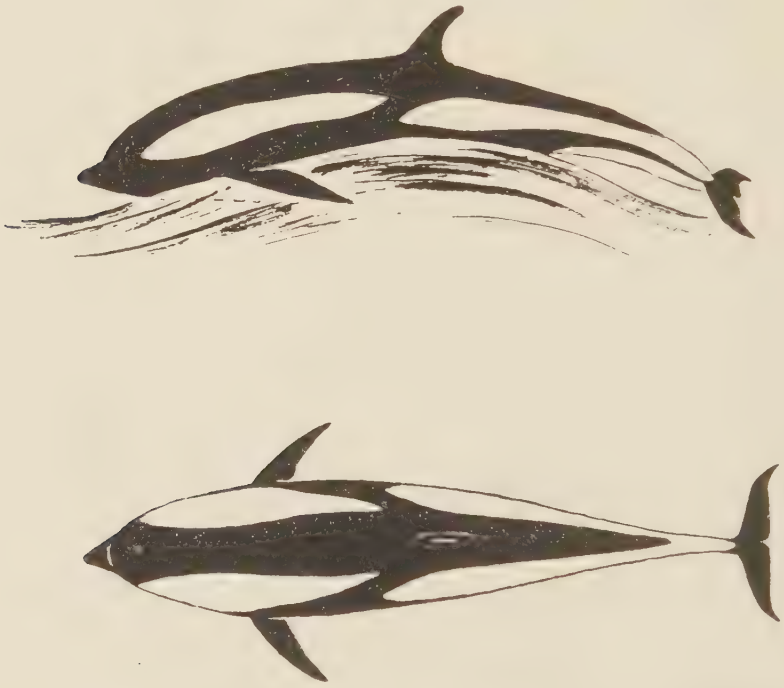


Fig. 1. *Lagenorhynchus wilsoni* Lillie. Drawing made by H. VAN DER LEE.

Lagenorhynchus wilsoni has been seen by WILSON (1907; p. 9) in November—January at the outer zone of the pack-ice at 55—60° S. and about 135° E. (fig. 3). According to LILLIE (1915; p. 123) they are confined to a comparatively narrow band just north of the pack-ice. He saw them from December—April at 54—65° S. and from 157° E.—88° W. LILLIE gives no description or figures of the animals, so that it is uncertain whether he saw *L. wilsoni* or *L. cruciger*. It seems, however, that up to the present Wilson's Dolphin has only been seen in the Australian and Pacific sectors (area V and VI) of the Antarctic Ocean and always close to the pack-ice below 54° S. So the occurrence in the Atlantic or South African sector (Bouvet sector; area III) and at 46° S. (the border of the pack-ice lying at 62° S.) may be regarded as a new contribution to the distribution of this species (resp. subspecies).

II. *Lagenorhynchus obscurus* (Gray).

During a short visit to the South African Museum in Cape Town one of us studied the external characters of a stuffed specimen of an undetermined dolphin of which the skull was also present. The Director of the Museum could give no definite particulars about this specimen except that it was probably caught in Table Bay some 40 years ago. The roughly made sketch reproduced in fig. 2 shows that the upper side of the animal is entirely black and that the lower side is white from snout to anus. The ventral side of the tail is black and also the flippers and the left lower jaw, the right one is white. On the neck and trunk there is an area of grey between the black and white, which shows a larger extension on the right side than on the left side. A narrow black band runs from the eye to the insertion of the flipper. In the lumbar region and the cranial part of the tail there is a white patch in the black; the right and left patch differ in shape. The total length of the animal (tip of snout — notch of flukes) is about 180 cm, the dental formula $\frac{26}{30}$.



Fig. 2. Roughly made sketch of a stuffed specimen of *Lagenorhynchus obscurus* (Gray) in the South African Museum, Cape Town.

This specimen definitely belongs to the Dusky Dolphin (*Lagenorhynchus obscurus* (Gray)) and corresponds almost completely with the specimen drawn and described by QUOY et GAIMARD (1830; p. 151, pl. 28, fig. 3; Cape of Good Hope) except that in that specimen the black band from the eye to the basis of the flipper is lacking. The band is, however, present in the specimens described and drawn by GRAY (1828; p. 37, pl. 2, fig. 2, 3; Cape of Good Hope) and CASSIN [1858; pl. 5, fig. 1;

coast of Patagonia; referred by KELLOGG (1941) as *Lag. australis* (Peale)]. The total length corresponds quite well with the statements of QUOY et GAIMARD (1830; p. 151; 155 cm), PEARSON (1935; p. 183; 180 cm) and GRAY (1828; p. 37; 185 cm); the specimen of CASSIN (1858) had a length of 213 cm. The dental formula is almost the same as the one referred to in literature, viz: QUOY et GAIMARD (1830; p. 151): $\frac{27}{26}$, GRAY (1846; p. 37): $\frac{24—26}{24—26}$, SLIJPER (1938; p. 26; specimen of unknown origin in the Brussels Museum): $\frac{29}{27}$, TRUE (1889; p. 104; 7 skulls): $\frac{31—34}{26—33}$ and PEARSON (1935; p. 183): $\frac{32}{32}$.

The colour pattern of *Lagenorhynchus obscurus* is very variable (LILLIE; 1915). The upper side may be entirely black (LILLIE; 1915, fig. 2; GRAY; 1828, pl. 2, fig. 3), the black reaching up to the ventro-median line caudal of the middle of the tail or being also restricted to the dorsal side in the tail. There may be a white patch at the latero-caudal side as described above and by CASSIN (1848) and GRAY (1828; fig. 2). This patch may or may not join with the ventral white (GRAY; 1828), it may show two cranial tongues (fig. 2; QUOY et GAIMARD; 1830, fig. 3) or there may be a broad or narrow caudo-ventral white or grey band in the black. The caudo-ventral black band at the eye may be absent or present, the colour of the dorsal fin is highly variable. In the drawings of LILLIE (1915; pl. 8) the beak is very long. KELLOGG (1941; p. 300) therefore thinks that LILLIE's animals do not belong to the genus *Lagenorhynchus*. But LILLIE saw the animals only in the water and thus could not observe the proportions quite accurately. FRASER (1937), PEARSON (1935) and OLIVER (1922) consider the animals of LILLIE as belonging to *Lag. obscurus*.

The Dusky Dolphin is very widely distributed in the southern seas (fig 3). Without further indication it has been reported by GRAY (1846) and WILSON (1907; p. 8) from the Southern Ocean. Specimens from the Cape of Good Hope and adjacent waters are described by GRAY (1828; p. 2), QUOY et GAIMARD (1830; p. 151), SCLATER (1901; Houtbay), LILLIE (1915; 5° 57′ S. – 18° 49′ E.), FITZSIMONS (1920; p. 225). The species is not mentioned in POLLEN's (1863) enumerations of Cetacea from Madagascar; according to LE BARBIER (1908; p. 4), however, it has been observed at the southern part of the west-coast of this island. From South America it has been reported by QUOY et GAIMARD [1824; *Delphinus albigena*, according to KELLOGG (1941) *Lag. obscurus*; between Cape Horn and Australia at 49° S.], PEALE (1848) and CASSIN (1858; Street le Maire, 55° S.), TRUE (1889; p. 105; Coquimbo, Chile, 30° S. – 72° W.), SCHNEIDER (1944; p. 188; Concepcion, Chile, 37° S. – 73° W.) and CARTER, HILL and TATE (1946; p. 134; Falkland Isles, 53° S.). In the seas south of Australia, Tasmania and N. Zealand and in the N. Zealand

sector of the Antarctic Ocean it occurs after LILLIE (1915; p. 122) from 38—52° S. and 125—178° E. especially in the vicinity of N. Zealand. In this sector it has also been reported by OLIVER (1922; p. 581; coast of N. Zealand), IREDALE and THROUGHTON (1934; N. Zealand, Tasmania), PEARSON (1935; p. 183; N. Zealand, Tasmania), CARTER, HILL and TATE (1946: p. 134; N. Zealand). It is not mentioned, however, in WOOD JONES' (1925) list of mammals of South Australia.

According to GRAY (1846), CASSIN (1858) and IREDALE and THROUGHTON (1934) the species is identical with *Lag. fitzroyi* (Waterhouse), *Lag. cruciger* (d'Orb. et Gerv.), *Lag. cruciger* (Quoy et Gaim.), *Lag. bivittatus* (Lesson et Garnot) (= *Lag. cruciger*), *Lag. superciliosus* (Lesson et Garnot) and *Lag. clanculus* (Gray 1849) (= *Lag. cruciger*). According to CASSIN (1858) it is identical with *Phocaena australis* (Peale 1848), but according to the opinion of KELLOGG (1941) this is a distinct species (*Lag. australis* (Peale)).

III. *Some remarks upon the species of the genus Lagenorhynchus.*

Up to the present the following species of the genus *Lagenorhynchus* have been regarded as valid: *Lag acutus* (Gray), *Lag. albirostris* Gray, *Lag. fitzroyi* (Waterhouse), *Lag. superciliosus* (Lesson et Garnot), *Lag. cruciger* (Quoy et Gaim.), *Lag. wilsoni* Lillie, *Lag. australis* (Peale), *Lag. obliquidens* Gill and *Lag. electra* Gray. HARMER (1922) has shown that *Lag. floweri* Moreno (MORENO; 1892) which has also been described by BRUCE (1915) under the name *Lag. cruciger* (d'Orb. et Gerv.)?, belongs to the genus *Cephalorhynchus*.

Because of their distribution as well as because of their very well known skeletal characteristics there is no doubt at all that *Lag. albirostris* and *acutus* represent true species. The other species, however, have been identified with one another in various different ways and their skeletal characteristics are only poorly known. Besides a good comparison of all the species has never been made. Unfortunately lack of material does not permit us to do this, but nevertheless it may be usefull to compare the external characteristics and distribution (fig. 3) as far as they are known from literature. *Lag. wilsoni* and *obscurus* have been dealt with in the preceding pages and now we shall discuss the other species:

1. *Lagenorhynchus fitzroyi* (Waterhouse).

The back of this dolphin (about 1/3 of the body) is black, the lateral and ventral parts white. A little cranial of the back fin and at the level of this fin two bands or stripes of black or grey run from the dorsal black into the lateral and ventral white in a caudo-ventral direction. The length and breadth of these stripes is very variable, the same applies to the other characteristics of the colour pattern. A narrow band from the angle of the mouth to the basis of the flipper is present in the specimens described

by WATERHOUSE (1839; p. 25, pl. 10) and GALLARDO (1912; p. 394, fig. 1). It is absent in those from PHILIPPI (1893; p. 2, fig. 1: *Phocaena posidonia* Philippi) and LAHILLE (1901). In the specimen of LAHILLE (1901) the caudal part of the dorsal fin is white, in the other specimens this fin is entirely black. Especially the external appearance of the animal described by LAHILLE (1901) corresponds almost completely with the specimens of *Lag. obscurus* (Gray) figured by LILLIE (1915; fig. 4 and 5), QUOY et GAIMARD (1830; pl. 28, fig. 2) and GRAY (1828; pl. 2, fig. 2 and 3).

The following statements on the total length of the animal can be found: WATERHOUSE (1839): 163 cm, LAHILLE (1901): 165 cm, GALLARDO (1912): 183 cm, PHILIPPI (1893): 185 cm. The dental formula is: $\frac{28}{27}$ (WATERHOUSE; 1839), $\frac{29}{27}$ (LAHILLE; 1901), $\frac{28}{30}$ (GALLARDO; 1912), or $\frac{28}{30}$ (PHILIPPI; 1893).

The animals have chiefly been observed at the southern part of the east coast of S. America, viz at: Rio de la Plata (35° S.; JACQUINOT et PUCHERAN; 1853), Mar del Plata (38° S.; GALLARDO; 1912), Bay of San Matias (41° S.; LAHILLE; 1901), Bay of St. Joseph (42° S.; WATERHOUSE; 1839), Bay of Santa Cruz (50° S.; MORENO; 1892; *Lag. burmeisteri* Moreno). The animal described as *Phocaena posidonia* Philippi by PHILIPPI (1893) was seen on the west coast of S. America at 48° 10′ S. and 77° 0′ W.

WATERHOUSE (1839) has already pointed out the great resemblance between this species and *Lag. superciliosus* (Lesson et Garnot). BEDDARD (1900) and LIOUVILLE (1913, 1913*a*) identify this species with *Lag. cruciger* (Quoy et Gaim.), *Lag. clanculus* Gray (= *Lag. cruciger*; see also TRUE; 1889) and *Lag. wilsoni* Lillie. KELLOGG (1941) on the contrary thinks that *Lag. fitzroyi* (Waterhouse) and *Lag. obscurus* (Gray) belong to the same species.

2. *Lagenorhynchus australis* (Peale).

Under the name *Phocaena australis* this species has first been described by PEALE (1848). CASSIN (1858) homologised the species with *Lag. obscurus* (Gray). According to the opinion of KELLOGG (1941), however, it should be a distinct species, identical with two skulls of unknown origin, but also collected during the U.S. Exploring Expedition and described by COPE (1866) as *Sagmatias amblodon* Cope. KELLOGG's description of the specimen from Chiloe Island is not based on the animal itself but on five photographs (pl. 8—9), showing some difference in the colour pattern that is obviously caused by differences in the lighting of the object. Nevertheless it may be stated that the species is characterised by a large white or grey patch in the black caudal part of the head, neck and thorax and by a similar patch in the lumbar and caudal region. Below the dorsal fin the patches

are separated by a black band running from cranio-dorsal to caudo-ventral. The length of the animals was 213 (CASSIN; 1858) and 210 (KELLOGG; 1941) cm, their dental formula resp. $\frac{31}{29}$ and $\frac{32—33}{29—33}$. The species has been described by PEALE (1848; see also CASSIN; 1858) from Street le Maire (55° S.) and by KELLOGG (1941) from Chiloe Island (Chile, 43° S.), so that it occurs on both sides of the southern part of S. America.

In its external appearance the species totally corresponds with the specimen of *Lag. cruciger* (Quoy et Gaim.) as depicted by NICHOLS (1908) and shows a very close resemblance with *Lag. fitzroyi* (Waterhouse) (specimen of GALLARDO; 1912, fig. 1) and *Lag. obscurus* (Gray) [specimens of GRAY (1828, pl. 2 fig. 2), QUOY et GAIMARD (1830; pl. 28, fig. 3) and LILLIE (1915; pl. 7)]. According to CASSIN (1858) the species should be identical with *Lag. obscurus* (Gray), *cruciger* (Quoy et Gaim.) and *fitzroyi* (Waterhouse).

3. *Lagenorhynchus superciliosus* (Lesson et Garnot).

This species has been described by LESSON et GARNOT (1826; p. 180, pl. 9, fig. 2) from the Tasmanian waters (43—45° S.; the figure is from a specimen harpooned off South Cape Tasmania) and by SCHLEGEL (1841; p. 22) from the Cape of Good Hope. According to the only known picture (LESSON et GARNOT; 1826, p. 9, fig. 2) of the external appearance of the animal, its shape and colour almost completely correspond with that of *Lag. obscurus* (Gray) as depicted by LILLIE (1915; pl. 8, fig. 2). The same may be said from the total length of the animals (resp. 127 and 142 cm; LESSON et GARNOT; 1826 and SCHLEGEL; 1841) and their dental formula (resp. $\frac{30}{29}$ and $\frac{30}{30}$). SCHLEGEL (1841) identified the species with *Lagenorhynchus obscurus* (Gray) and *Lag. fitzroyi* (Waterhouse). According to the opinion of KELLOGG (1941) it should be a distinct species because of the white ventral side of the tail and the absence of a black band between the angle of the mouth and the flipper. We have, however, already shown how very variable these characteristics within the limits of the species may be.

4. *Lagenorhynchus cruciger* (Quoy et Gaim.).

This species has been reported from the waters east of the southern part of S. America by LESSON et GARNOT (1826; between Cape Horn and Falkland Islands), D'ORBIGNY et GERVAIS (1847; 57—76° S., E. and S. of Cape Horn; according to PHILIPPI (1893; p. 9), however, 76° S. must be a misprint since the ship did not sail so far south) and NICHOLS (1908; 36—52° S. – 53—66° W.). South of Cape Horn it was seen by RACOVITZA (1903; 59° S. – 63° W., Drake passage) and the Stockholm Museum possesses a skeleton and stuffed specimen from Cape Horn (MALM; 1871, p. 68; *Electra clancula* Gray = *Lag. clanculus* Gray). From the waters west

of the southern part of S. America and from the Pacific sector of the Southern Ocean it has been reported by QUOY et GAIMARD (1824; 49° S. between Cape Horn and Australia), LIOUVILLE (1913, 1913*a*; 56° S.–95° W.), GRAY (1849, see also GRAY, 1866; *Lag. clanculus*; Pacific Ocean) and NICHOLS (1908; 49—53° S.–78° W.). TRUE (1889; p. 91) describes a skull in the Paris Museum sent from N. Zealand.

The animals are characterised by two large white patches in the dorsal and lateral black, separated below the back fin by an oblique or vertical band of black (see also p. 1353). This band is not present in the animal figured by D'ORBIGNY et GERVAIS (1847; pl. 21, fig. 1—4) and it is very narrow in the animal of NICHOLS (1908). The figures of LIOUVILLE (1913; fig. 5 and 6) show the variability in the colour pattern of the animals of one and the same school. Thus we think that it is perfectly justified to reckon the animal figured by QUOY et GAIMARD (1824; pl. 11, fig. 3 and 4) among the other representatives of this species, although it lacks the ventro-lateral longitudinal black band. It was, however, only seen from some distance at sea.

If the great variability of this species is taken into account, it may be said that there is a very close resemblance between *Lag. cruciger* as depicted by NICHOLS (1908) on the one side and *Lag. fitzroyi* (Waterhouse) as depicted by GALLARDO (1912; fig. 1) or *Lag. australis* (Peale) as depicted by KELLOGG (1941; pl. 8, fig. 2) on the other side. This applies also to *Lag. obscurus* (Gray) as depicted by GRAY (1828; pl. 2, fig. 3). The length of the animals [110 (estimation), 147, 160, 163 cm; see resp. LIOUVILLE (1913), MALM (1871), PHILIPPI (1893; spec. of D'ORBIGNY et GERVAIS, 1847) and NICHOLS (1908)] and their dental formulae [$\frac{31—32}{32—33}$, $\frac{33}{32}$; see resp. MALM (1871) and GRAY (1849)] correspond quite well with the above described species of the genus.

Lag. cruciger has been identified by LIOUVILLE (1913, 1913*a*) with *Lag. fitzroyi* (Waterhouse) and *Lag. wilsoni* Lillie (see also FLOWER; 1883). It includes *Lag. clanculus* Gray (TRUE; 1889) and *Phocaena d'Orbignyi* Philippi (TRUE; 1901).

5. *Lagenorhynchus obliquidens* Gill.

This animal is, contrary to the above described species, only known from the North-Pacific. It has been reported by GILL (1865) from San Francisco and by SCAMMON (1874), TRUE (1889) and ELLIOT (1901) from the coast of California [Puget Sound (48° N.) – Monterey Bay (37° N.); after BEDDARD (1900) herds of many hundreds]. KELLOGG (1940; p. 82) states that it seems to be most numerous along the coast of Japan, though he does not give any further reference to this statement, however.

ANTHONY (1928), ELLIOT (1901), TRUE (1889) and FLOWER (1883) all point out the close resemblance between this species and *Lag. acutus*

(Gray) from the North Atlantic. We cannot agree with that. The only known figure of the animal (Scammon; 1874; pl. 19, fig. 2) does not greatly correspond with the white sided Dolphin but is very much like the figures of *Lag. obscurus* (Gray) published by Gray (1828; pl. 2, fig. 3) and Lillie (1915; pl. 8, fig. 4 and 5) and that of *Lag. fitzroyi* (Waterhouse) published by Phillippi (1893; p. 2, fig. 1; *Phocaena posidonia*). The resemblance with *Lag. cruciger* (True; 1889; p. 96) is not so apparent. The dental formula is $\frac{32}{29}$ (True; 1889), the total length of the animal is 221 cm (Elliot; 1901). Therefore it is somewhat longer than the majority of the southern species although from *Lag. australis* a length of 210—213 cm is reported. The length of the adult animals of the North Atlantic species (*Lag. acutus* and *albirostris*) varies, however, between 250 and 300 cm.

Lag. longidens Cope may belong to this species (True; 1889). *Lag. thicolea* Gray (Gray, 1846; True, 1889; Elliot, 1901) from the W. coast of N. America differs from it in some respects (i.e. dental formula $\frac{36}{40}$), but is only known from a damaged skull.

6. *Lagenorhynchus electra* Gray.

This species, which includes *Lag. asia* Gray (Gray; 1846, p. 35), was founded by Gray (1846; p. 35) on two skulls of unknown origin collected at the voyage of the Erebus and Terror. Its geographic distribution is very remarkable; records are known from the Pacific and Indian Ocean as well as from the Atlantic Ocean at the Senegal and Guinea Coast of N. Africa. Peale (1848) and Cassin (1858; p. 28; *Phocaena pectoralis* Peale) report a herd of 60 animals from Hilo Bay, Hawai, Van Beneden et Gervais (1880; p. 597) also give a record from Hawai, Sclater (1891) from the Sandwich Islands. Owen (1865; p. 22; *Delphinus fusiformis* Owen) describes an animal from Waltair near Vizagapatam (Madras; 17° N.–83° E.) and Weber (1923; p. 11) bought four skulls and two lower jaws at the isle of Solor (8° S.–123° E.; Dutch East Indies). After de Rochebrune (1883; p. 194) it is infrequently seen at the whole coast of Senegal from Cape Blanco (21° N.) to Cape Verga (10° N.) and Van Benenden et Gervais (1880; p. 597) give a record from the Bissagos Islands (Portuguese Guinea; 12° N.–17° W.).

There are two figures known, showing the external appearance of the animal, viz, that of Peale (1848; pl. 6, fig. 1) and Cassin (1858; pl. 5, fig. 2) and that of Owen (1865; pl. 5, fig. 1). Both drawings show an animal that is almost entirely black. According to Owen (1865) the ventral side is ashy-grey, after Peale (1848) and Cassin (1858) there is a small grey patch at the neck and a light mottled area between the umbilicus and the anus. The length of the animals is 183 (Owen) and 264 (Peale, Cassin)

cm, their dental formulae resp. $\frac{22}{21}$ and $\frac{23}{23}$ and (after GRAY; 1846; pl. 13 and 14) $\frac{25}{24}$ and $\frac{24}{23}$. In OWEN's figure the beak is much too long for a *Lagenorhynchus* but in the figure of the skull (pl. 7) this characteristic corresponds perfectly with that of the other species of the genus, so that the drawing of pl. 6 may be not quite correct.

7. *Discussion.*

We are fully aware of the fact that a new division of the species of the genus *Lagenorhynchus* is only possible after a close examination and comparison of all the available stuffed and skeletal material. In our discussion of the external characters and geographic distribution we have, however, shown that it seems to be highly probable that the southern species of *Lagenorhynchus* up to this referred to as *Lag. obscurus, fitzroyi, cruciger, wilsoni, australis* and *superciliosus* all belong to the same species. In this case the species should bare the name of *Lagenorhynchus cruciger* (Quoy et Gaim.). The possibility may not be excluded that *Lag. obliquidens* also belongs to this species, although the area in which these north Pacific animals have been observed is far remote from that of their southern relatives. Because of its different colour pattern, dental formula and geographical distribution, it is quite obvious that *Lag. electra* Gray may be considered as a different species.

Our opinion that all six above mentioned black and white southern dolphins belong to the same species is based on the following facts: 1. They appear to be distributed over the whole Southern Ocean from about 25° S. up to the border of the pack-ice (fig. 3) and so their area of distribution is quite distinct from that of the North Atlantic species *Lag. albirostris* and *acutus* and from that of *Lag. electra* that has only been observed in the tropical seas north of 10° S. 2. The total length of all animals falls between 150 and 220 cm so that they are strikingly smaller than the other species. 3. They all have about the same dental formula (26—34 teeth in both jaws) which is quite different from the dental formulae of *Lag. electra* (21—25), *albirostris* (22—28; TRUE, 1889; TURNER, 1912; HARMER, 1927; SLIJPER, 1938) and *acutus* (33—40; same authors). 4. A vertebral formula is only known from *Lag. obscurus, superciliosus, cruciger* and *obliquidens* (FLOWER, 1896; TRUE, 1889; SLIJPER, 1936). They all have 12—14 thoracic, 16—24 lumbar and 29—33 caudal vertebrae, the total number of vertebrae varying between 70 and 74. The vertebral formulae greatly differ from those of *Lag. acutus* (Th. 14—15; L. 18—22; C. 33—41; total 77—82) and *Lag. albirostris* (Th. 14—16; L. 22—27; C. 41—46; total 88—93) (see TRUE, 1889; LÜTKEN, 1887; SLIJPER, 1936, 1939; MALM, 1871). 5. A comparison of the skulls of all species of *Lagenorhynchus* as depicted by VAN BENEDEN et GERVAIS (1880; pl. 36), TRUE (1889; pl. 23—29) and KELLOGG (1941; pl. 10—12) shows that there are no striking differences between the

species. Small differences may perhaps be found during a profound examination of a large amount of material about all species. 6. The external appearance of any of the six above mentioned southern species shows with regard to one or more of their figured representatives such a

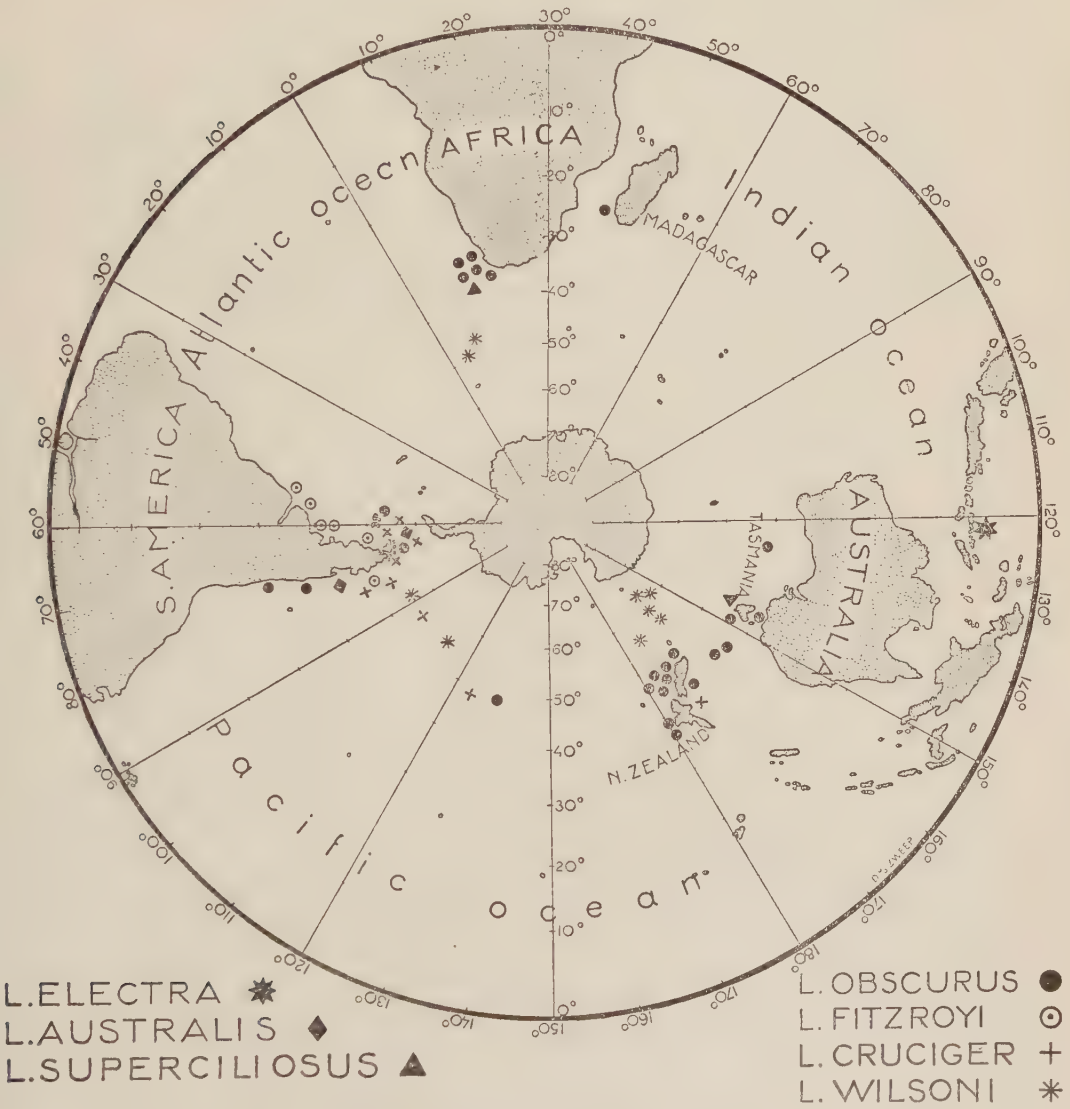


Fig. 3. Map of the southern hemisphere with the localities where representatives of the genus *Lagenorhynchus* have been observed. *L. electra* has also been observed north of the equator.

close resemblance to one, two, three or even four of the other species, that they all may be identified with one another. In literature these identifications have often been made between two or three of the species.

In nearly all the above mentioned characteristics *Lag. obliquidens* of the North Pacific corresponds with its southern relatives, but has a quite different geographic distribution so that we hesitate to incorporate these animals in the southern species until a good comparison between the different skeletons can be made.

It remains an open question whether within the limits of the new southern species *Lagenorhynchus cruciger* (Quoy et Gaim.) two or more of the old species may be regarded as subspecies, geographical races or not. The number of records about their distribution and colour variation is too small. With reference to this last information, however, some attention may in the future be paid to the statement of WILSON (1907; p. 9) that "the distribution of these two Dolphins (*Lag. obscurus* (Gray) and *Lag. wilsoni* Lillie) appears to overlap, and yet, though we had many schools of each from time to time around the ship, they never mingled".

(To be continued.)

Zoology. — *On the position of the Golgi-apparatus in the thyroid-cell under normal and experimental conditions.* (Preliminary note). By J. LEVER. (Zoological Laboratory, Dept. of Endocrinology, University of Utrecht) [1]). (Communicated by Prof. CHR. P. RAVEN.)

(Communicated at the meeting of November 29, 1947.)

It is generally accepted that the position of the Golgi-apparatus in a cell is related to its metabolic function. Consequently in secreting cells the Golgi-apparatus is mostly found between the nucleus and the pole of the cell where its secretion is discharged. Therefore one can easily understand that in cells in which the direction of secretion is always the same the position of the Golgi-apparatus is stable. It was also found by BEAMS and KING (1933) that the position of the Golgi-apparatus of the rat-ameloblast changes from the side of the cell, where the secretion takes place, to the other side when the inactive phase is initiated.

In case the direction of secretion changes, the position of the Golgi-apparatus is not always necessarily constant, and may vary according to the physiological state of the cell.

An example of this mode of secretion is offered by the thyroid-epitheliumcell. Many authors believe that the thyroidcell is not only able to pour its secretion into the follicular lumen, where it is stored in the so-called colloid, but also may resorb the hormone from the colloid in order to secrete it into the blood capillaries. According to COWDRY (1922) the differences in direction of secretion may be reflected by a reversal in the position of the Golgi-apparatus of the thyroidcell. In the latter the Golgi-apparatus is normally found at the apical side of the nucleus, which points to a secretion-activity towards the follicular lumen (fig. 1).

In some cases, however, COWDRY observed in the adult guinea-pig thyroid that the Golgi-apparatus had reversed to the other side of the nucleus, lying between the latter and the basal side of the thyroid cell. These observations seemed to prove that the Golgi-apparatus may migrate from one part of the cell to the other, according to the direction in which the hormone is secreted. On the other hand CRAMER and LUDFORD (1926) found that in thyroid glands, hyperactive under the influence of cold or of β-tetra-hydronaphthyl-amine, a reversed Golgi-apparatus is only very rarely observed. Moreover, these authors claim that COWDRY's observations relate to inactive thyroid cells. Reversals of the Golgi-apparatus were also found by LUDFORD and CRAMER (1928) in men and mice with exophthalmic goitre. Reversal of the Golgi-apparatus was stated by some other investigators too, e.g. by SHIRO ISHIMARU (1926) in the rabbit-

[1]) 24th Communication of the "Werkgemeenschap voor Endocrinologie", part of the "National Council for Agricultural Research T.N.O.".

thyroid, from which a large part had been resected; INGRAM (1930) and KROGH, LINDBERG and OKKELS (1932) induced in very rare cases reversal of the Golgi-apparatus after administering thyrotrophic hormone to guinea-pigs, but WAGSCHAL (1930), UHLENHUTH (1931), OKKELS (1932, 1934), SEVERINGHAUS (1933) and GILLMAN (1934) were not able to demonstrate it.

With regard to the functional state of the thyroid cell, GILLMAN (1934) distinguished 3 cellphases viz. 1. the resting stage, 2. the anabolic stage, in which secretion towards the lumen takes place, and 3. the katabolic stage, characterized by the secretion towards the interfollicular bloodsystem of the gland. After a detailed study of serial sections he concluded that the reversal of the Golgi-apparatus, observed by COWDRY and in rare cases by CRAMER and LUDFORD (1926), was only apparent, because in resting and in anabolic thyroid cells the Golgi-apparatus is often a perinuclear one. In katabolic cells, however, in which the nucleus is not so closely surrounded by the Golgi-apparatus, it is not possible to observe a basal fragment of the latter. Therefore a reversal of it has never been stated in katabolic cells. GILLMAN concludes that the phenomenon of reversal would be proved, if it could be observed in a katabolic cell stage, this being the only stage in which secretion towards the bloodvessels takes place.

Recent investigations with the aid of radioactive J^{131} and of antithyroid substances have strongly enlarged our knowledge of the functional processes in the thyroid cell. It is now generally accepted that especially thiourea-derivatives prevent the normal production of the active thyroid hormone (LARSON c.s., 1945a, 1945b; RAWSON c.s., 1944, 1946). Moreover, it is known that as a result of the hormone synthesis being prevented in the thyroid, a decrease of the hormone concentration in the blood and of the basal metabolic rate in the peripheral bodycells takes place, when the colloid supply in the follicles begins to diminish, which results in an increased production of thyrotrophic hormone by the anterior lobe of the hypophysis. In addition BAUMANN c.s. (1942) showed that this hormone induces the resorption of the thyroid hormone by the follicle cell and its secretion towards the blood capillaries; moreover, ASTWOOD c.s. (1944), DEMPSEY (1944) and GRASSO (1946) found, after administering antithyroid substances that the thyroid is able to secrete an inactive product in which thyroxine is lacking.

Taking these facts into consideration we thought it interesting to trace the influence of 4-methyl-2-thiouracil (Brocapharm) on the position of the Golgi-apparatus in the follicle cells of the thyroid gland. We might expect that then the anterior lobe of the hypophysis would respond by a thyrotrophic hormone output, which likely would increase gradually and would cause a more physiological process in the thyroid than the injection of thyrotrophic hormone can accomplish.

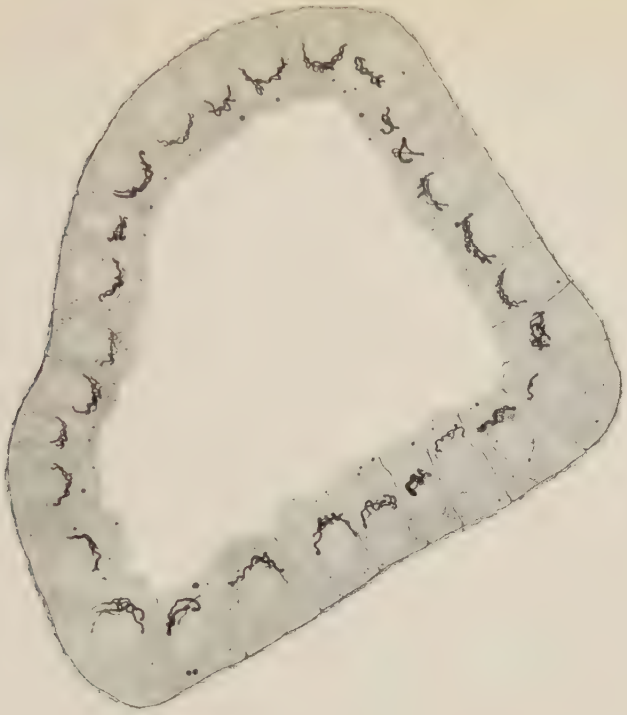


Fig. 1. Thyroid-follicle of a normal cockerel. Follicle cells more or less cubical; Golgi-apparatus exclusively at apical side of the nucleus. × 1200.

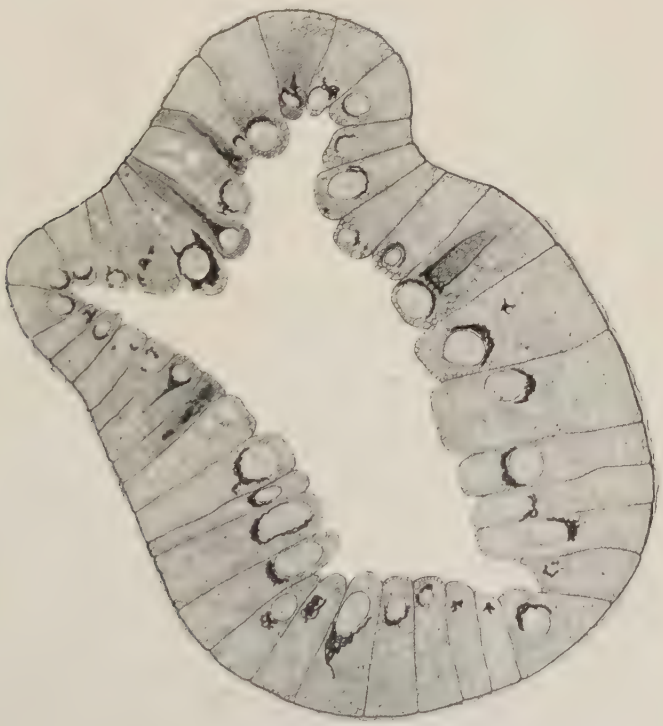


Fig. 2. Thyroid-follicle of a fattened cockerel, treated with 4-methyl-2-thiouracil. Follicle cells high columnar; in many cases reversal of the Golgi-apparatus to the basal side of the nucleus. × 1200.

As experimental animals cockerels, 10—14 weeks old, were used; they were kept in small cages in which their bodymovements were more or less hampered. Each cockerel received a daily dose of 50 mgr. 4-methyl-2-thiouracil pro 100 grams of food for 2 or 3 weeks; they were put on a diet, used normally in fattening, i.e. practically without proteins, but rich in carbohydrates. As controls birds on normal and fattening food were used.

The thyroid glands were fixed in Bouin's or Helly's fluid and stained with haematoxylin-eosin or azan, respectively. Moreover, Golgi-preparations were made after fixation in Champy's and Cajal's fluid.

It became evident that within 2 weeks the microscopical structure of the thyroids changed completely. In most of the follicles the colloid had disappeared totally. The epithelium cells which are cubical in normal animals, became high columnar in the experimentals, their height increasing from 5—10 μ to more than 20 μ. Their nuclei, lying normally at the basal pole of the cell, migrated towards the apical side.

In normal resting or anabolic thyroid cells the Golgi-apparatus was found lying as a small reticulum at the apical side of the nucleus (fig. 1). In methyl-thiouracil-treated birds it was found in about 60 % of the thyroid cells at the opposite side (fig. 2). As these cells have a high columnar shape and possess a migrated nucleus, they must be classified as katabolic cells.

In the thyroids of fattened birds, which had not been treated with methyl-thiouracil, the phenomena described above, i.e. increase in height of the epithelial cells, decrease of colloid content and reversal of the Golgi-apparatus, occurred in a less degree. In the food with which the cockerels had been fattened the iodine-content is much lower than normal, this obviously being the limiting factor for hormone synthesis, which finally results in an increase of the thyrotrophic hormone output.

From these observations it follows that under extreme conditions a reversal of the Golgi-apparatus can take place in the epithelium cells of the thyroid gland.

As in normal cockerels fed on a diet, containing iodine in sufficient quantities, this reversal does not take place and possibly the body's need of thyroid hormone is satisfied gradually, we presume that in this case a reversal of the Golgi-apparatus is not necessary. On the other hand a reversal in some cells may be thinkable under normal conditions as e.g. under the influence of severe cold, when the body needs much thyroid hormone. Also one can understand that an increased secretion of thyrotrophic hormone by the anterior lobe of the hypophysis may be the cause of reversal of the Golgi-apparatus after iodine deficiency. Therefore it is also comprehensible that under extreme pathological or pharmacological circumstances a general reversal will take place. Consequently it is understood that SHIRO ISHIMARU (1926) observed reversals in the thyroid of the rabbit after resection of a large part of it, as well as LUDFORD and

CRAMER (1928) in cases of exophthalmic goitre in men and mice. In such cases the thyroid cells must function as much as possible to provide the large quantity of thyroid hormone, needed by the body.

Summary.

4-methyl-2-thiouracil in normal and in fattening diets was fed in a dose of 50 mgr. pro 100 grams of food during 2—3 weeks to cockerels, 10—14 weeks old.

In normal fattened cockerels and, in a higher degree, in fattened cockerels, treated with methylthiouracil, many reversals of the Golgi-apparatus from the apical towards the basal side of the thyroid epithelium-cell were found.

As it is known that in the first case the iodine was deficient, in the second the antithyroid substance moreover acted on the thyroid by inhibiting the process of iodinization, resulting in hyperactivity of the thyroid gland, it is thought that both reactions are induced by thyrotrophic hormone, secreted in excess by the anterior lobe of the hypophysis.

Therefore all cases of reversal of the Golgi-apparatus, those found under pathological conditions [ISHIMARU (1926); LUDFORD and CRAMER (1928)] included, can be explained by an extreme activity of the thyroid epithelium-cell, being the result of an extreme output of thyrotrophic hormone.

Acknowledgements. I wish to thank Prof. G. J. VAN OORDT for his kind advice and encouragement during this investigation, Dr. J. J. DUYVENÉ DE WIT for suggesting histological studies on the influence of antithyroid substances on the fowl's thyroid and Ir. P. UBBELS for his valuable help in obtaining the experimental material. My thanks are also due to "Brocapharm" (Amsterdam) for the gift of methylthiouracil.

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Anatomy. — *On the Function of the Crico-arytaenoid Joints in the Movements of the Vocal Cords.* By CHR. A. R. D. SNELL. (From the anatomical Institution of the Medical College at Batavia.) (Communicated by Prof. J. BOEKE.)

(Communicated at the meeting of September 27, 1947.)

In describing the movements of the vocal cords it is generally assumed that the movements which take place at the crico-arytaenoid joints are of a twofold kind, viz. gliding and rotatory. The gliding movements are of such a character that the two arytaenoid cartilages approach and retreat from each other and from the median plane. In the rotatory movements the arytaenoid cartilage revolves around a vertical axis. A diagram elucidating these movements is to be found in TESTUT–JACOBS textbook of anatomy [1]) as well as in many of the later ones [2]). From the literature available at Batavia I could not make out by which author this diagram has been first introduced. Since 1914 however it is of common use in explaining the functions of the laryngeal muscles.

In consulting older literature it appears that quite other views regarding the function of the crico-arytaenoid joints have been expressed. HENLE [3]) for instance gave in 1866 a complete picture of the function of the laryngeal muscles without making use of rotatory movements in the crico-arytaenoid joints. But already 1902 MERKEL [4]) wrote: "Die Cart. arytaenoidea kann sich vorwärts und rückwärts neigen, sie kann sich medianwärts und lateralwärts verschieben und sie kann endlich um ihre Längsachse rotiert werden."

Not all modern authors admit the occurrence of rotatory movements of the arytaenoid cartilages as freely as those mentioned above. LANGER–TOLDT [5]) for instance acknowledged 1921 their occurrence, although they did not mention them in their description of the movements at the joint, which runs as follows: ".....infolgedessen sind den Gieszbeckenknorpeln Exkursionen in Ebenen angewiesen, welche lateral geneigt und nach vorn konvergieren".

TANDLER wrote 1929: "Das Gelenk ist ein Eigelenk, in welchem allerdings eine Rotationsmöglichkeit vorhanden ist."

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[1]) L. TESTUT et O. JACOB, Traité d'anatomie topographique. Tome I, 1914.

[2]) H. BRAUS, Anatomie des Menschen. Bd. 2. 1924. — A. J. P. VAN DEN BROEK, J. BOEKE and J. A. J. BARGE, Leerobek der beschrijvende ontleedkunde van den mensch Dl. 3, 1944. — J. TANDLER, Lehrbuch der systematischen Anatomie. Bd. 2. 1929.

[3]) J. HENLE, Handbuch der Eingeweidelehre des Menschen. 1866.

[4]) FR. MERKEL, Atmungsorgane. In K. VON BARDELEBENS Handbuch der Anatomie des Menschen. 1902.

[5]) C. LANGER and C. TOLDT, Lehrbuch der systematischen und topographischen Anatomie. 1921.

Some modern authors give an exact description of the position of the axis of rotation which is said to coincide with the vertical axis of the arytaenoid cartilage (TESTUT et JACOB l.c.), passing through the middle of the cartilage (VAN DEN BROEK l.c.). BRAUS (l.c.) describes the movements occurring at the crico-arytaenoid joints in the following way: "Die Stellknorpel sind in dem Gelenk mit dem Ringknorpel so beweglich, dass sie sich sowohl um die Höhenachse der Pyramide drehen können wie ein Rad um seine Achse, als auch auf dem Rand des Ringknorpels hin — und herrutschen können oder beides zugleich vermögen, wie das Rad eines fahrenden Wagens, das zugleich vorwärts und um seine Achse rollt."

The rotatory movements described remind of the rotation at the humeroradial and proximel radio-ulnar joints, but in the latter the strong annular ligament of the radius is present directing the rotatory movements. In the circo-arytaenoid joint a ligament of corresponding function is lacking. The lig. crico-arytaenoideum is described as a structure preventing the arytaenoid cartilages from gliding too far outwards. One gets the impression that the cartilages participating in the crico-arytaenoid joint are only loosely held together, thus permitting of a high degree of mobility.

Still it seems scarcely possible to explain some physiological observations on the movements of the vocal processes from the anatomical facts mentioned. LUCIANI [6]) in discussing the function of the Mm. crico-arytaenoidei dorsales states that they "draw the two muscular processes of the arytaenoid down and back and consequently further from the median line and at the same time raise the vocal processes". CURRY [7]) writes with regard to the M. thyreo-arytaenoideus: "The second is the inner oblique, ... It acts to draw the arytaenoids down, inward and forwards."

Raising and lowering the vocal processes require movements at the crico-arytaenoid joints which are not provided for by the current conception of their anatomical structure. Thus an anatomical reexamination of the joint and its functions seemed indicated. Since neither laryngoscopy nor X-ray photography is of any use in this case I had to rely on careful dissection of the joint and related structures. Special attention was paid to the form of the articular surfaces and to the topographic relations of the laryngeal muscles and their separate fibres to the axis of movement. The material available for this investigation consisted of 13 Javanese larynges which had belonged to inhabitants of Batavia and the environment of this town. The specimens were taken from the bodies after full preservation.

The results of my investigations are as follows.

1. The facies articularis of the cricoid cartilage is part of the surface of a cylinder the axis of which passes through the cricoid parallel to its upper border. This axis meets the median sagittal plane at an angle of 35°—40° and the horizontal plane through the lower cricoid margin at

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6) L. LUCIANI, Human Physiology. Vol. 3. 1915.

7) R. CURRY, The mechanism of the human voice. 1940.

an angle of about 60° (vide fig. 1 and 2). This result corroborates MERKELs statement: "Die verlängerten Achsen der beiden Gelenkflächen kreuzen sich hinten oben unter einem Winkel von 50° bis 60°; einen ebensolchen Winkel bilden sie mit der Horizontalen."



Fig. 1. Cranial View of cricoid and arytaenoid cartilages with *M*-interarytaenoideus transversus and longitudinal axes of the crico-arytaenoid joints (*s* and *d*).



Fig. 2. Dorsal View of arytaenoid and cricoid cartilages with longitudinal axes of the crico-arytaenoid joints (*s* and *d*).

It is not possible to fix the position of the axis more accurately. To the description given it can be added that the axis enters the cricoid cartilage at the slight projection marking the insertion of the crico-arytaenoid ligament (vid. fig. 2 and 3).

In the direction parallel to the said axis the articular surface is quite straight (fig. 3). Only in one case it was found to be slightly convex in this direction. No trace of concavity was even met with.

In sections perpendicular to the axis the articular surface is highly convex (vide fig. 4).

As will be explained sub 6 the axis described here is the one about which the arytaenoid cartilage moves. In the sequel it will be called the longitudinal axis of the crico-arytaenoid joint.

2. On the arytaenoid cartilage the articular surface is in all directions much smaller but of a shape exactly corresponding with the cricoid surface.

It should be emphasized that the articular surface is not to be found at the base of the arytaenoid; it lies distinctly further backwards, that is in the region of the processus muscularis (vide figure 4). In my specimens, that means when in a position of easy breathing with the living, the arytaenoids are in apposition with the outer halves of the articular surfaces of the cricoid cartilage.

3. The capsula articularis is very loose and is strengthened by one ligament only, the ligamentum crico-arytaenoideum, which for so small a joint is a very strong one. It arises from a slight projection at the lateral end of the horizontal middle part of the upper border of the cricoid lamina,

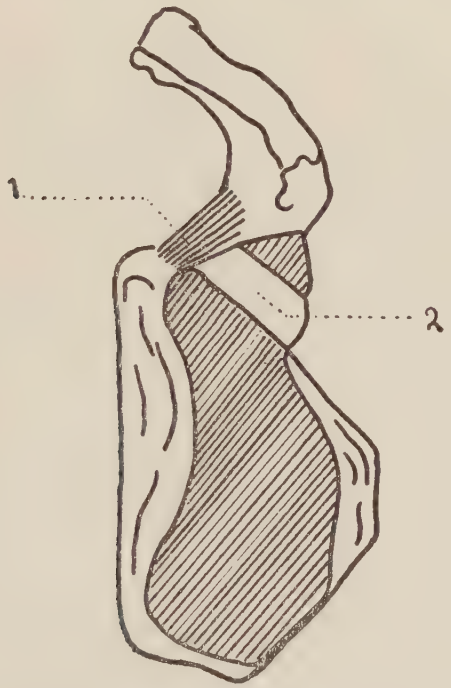


Fig. 3. Section of crico-arytaenoid joint parallel with its longitudinal axis.
1: crico-arytaenoid ligament;
2: cavum articulare.
Cut surfaces hatched (drawn after photograph).

situated immediately next to the articular surface at its medial side. The anterior fibres of the fanshaped ligament pass ventrally to be inserted into the posterior border of the medial surface of the arytaenoid, whereas its hindmost fibres take a nearly frontal course and are attached to the dorso-medial corner of the processus muscularis (fig. 5).

4. Since it is the processus muscularis of the arytaenoid cartilage which together with the upper border of the cricoid enters into the formation of the crico-arytaenoid joint, the greater ventral part of the base of the

arytaenoid in the position of easy breathing projects freely into space. Suspended by the lig. crico-arytaenoideum it is kept in place (vide fig. 1 and 4).

In this position the own axis of the arytaenoid cartilage passing through its apex and its base is more or less vertically directed, whereas the medial surface of the said cartilage is in a sagittal plane and parallel with the corresponding surface of its fellow of the opposite side. Consequently this vertical axis is identical with or parallel to BRAUS' "Höhenachse", MERKEL's "Längsachse", VAN DEN BROEK's, TESTUT–JACOBs and others "Vertical axis of rotation".

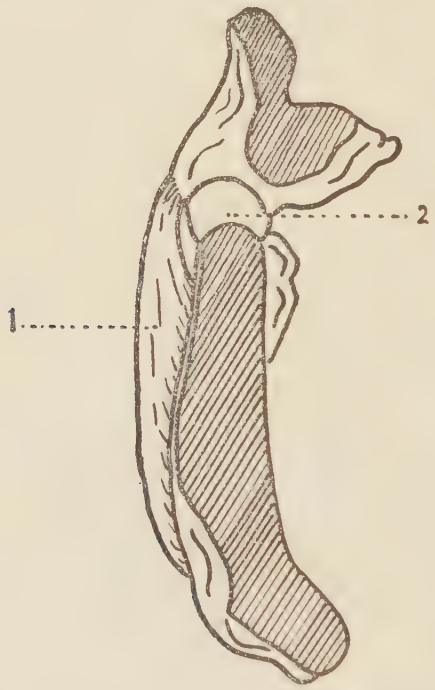


Fig. 4. Section of crico-arytaenoid joint perpendicular to its longitudinal axis.
1: *M.* crico-arytaenoideus dorsalis;
2: cavum articulare.
Cut surfaces hatched (drawn after photograph).

The latter axis is at an angle of about 30° with the longitudinal axis of the crico-arytaenoid joint.

5. The rima glottidis is surrounded by a mass of muscle-fibres from which a number of separate muscles can be isolated, each with its own function. The more important are a) the Mm. crico-arytaenoideus dorsalis, crico-arytaenoideus lateralis, interarytaenoideus transversus et obliquus and thyreo-arytaenoideus which all have developed from the inner layer of the primitive sphincter laryngis, and b) the crico-thyreoid muscle which

has sprung from the outer layer of the said sphincter (FRAZER [8]), NEGUS [9])).

The *M. crico-arytaenoideus dorsalis* arises from the dorsal surface of the lamina cricoidea far below the crico-arytaenoid joint. From this origin its fibres converge laterally and upwards to be inserted into the dorso-medial corner of the processus muscularis of the corresponding arytaenoid cartilage. The main axis of the muscle, i.e. the line connecting insertion and centre of origin, which indicates the direction of the traction exercised on the insertion in case all fibres contract with equal strength, is directed perpendicular to the longitudinal axis of the crico-arytaenoid joint.

Only in case of very strongly developed laryngeal muscles the ultimate fibres of the dorsal crico-arytaenoid muscle take a nearly horizontal resp. vertical course and even then these marginal fibres are at a rather large angle with the longitudinal axis of the joint.

The *M. crico-arytaenoideus lateralis* arises from the lateral surface of the arch of the cricoid cartilage; its fibres are inserted into the antero-lateral corner of the processus muscularis of the arytaenoid cartilage. The main axis of this muscle is at an angle of about 30° with the longitudinal axis of the joint.

The *M. interarytaenoideus transversus* is an unpaired muscle bridging across the interval between the two arytaenoid cartilages and presenting a parallel arrangement of its fibres. Its lower fibres are attached to the medio-dorsal corner of the processus muscularis, immediately above the M. crico-arytaenoideus dorsalis, whereas its upper fibres are attached to the dorsal surface of each arytaenoid cartilage. Since the longitudinal axes of the two crico-arytaenoid joints converge dorsally and upwards meeting at an angle of about 70°—80°, the main axis of the said muscle is at an angle of 50°—55° with the longitudinal axis of each joint (vide fig. 1 and 2).

The *M. interarytaenoideus obliquus* arises from the processus muscularis. Each muscle crosses the median line after which it divides into three fasciculi. The upper bundle is inserted into the lateral border of the dorsal surface of the arytaenoid cartilage of the opposite side immediately above the attachment of the transverse interarytaenoid muscle. The fibres of the middle fasciculus curve round the dorsal border of the arytaenoid cartilage without being inserted into this cartilage; then they extend ventrally and end radiating within the aryepiglottic fold. The fibres of the lower bundle pass the arytaenoid cartilage in a corresponding way and end by joining the fibres of the thyreo-arytaenoid muscle.

The *M. thyreo-arytaenoideus* arises from the lower half of the medial surface of the lamina cartilaginis thyreoideae, very close to its ventral

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[8]) J. E. FRAZER, The development of the larynx. Journ. of Anat. and Physiol. Vol. 44, 1910.

[9]) V. E. NEGUS, The mechanism of the larynx. 1929.

angle. The muscular fibres pass dorsally and laterally to be inserted into the lateral aspect of the vocal process and the anterolateral surface of the arytaenoid cartilage almost reaching the base of the processus muscularis. Separating this muscular mass into two separate muscular units can be performed by artificial means only. Still it is usual to describe the medial bundles underlying the plica vocalis and being inserted into the vocal process separately as M. vocalis. The main axis of the thyreo-arytaenoid muscle is directed dorsally and somewhat laterally and upwards and is consequently at a rather large angle with the longitudinal axis of the crico-arytaenoid joint.

6. After carefully removing all muscles the passive mobility of the crico-arytaenoid joint can be tested. It appears that without causing distortion or rupture of the capsula articularis only two kinds of movements of the arytaenoid cartilage can be brought about. In the first place the cartilage can be caused to glide along the articular surface of the cricoid in a direction parallel to the longitudinal axis of the joint, i.e. forwards, laterally and downwards resp. backwards, medially and upwards. The articular surface in this direction being straight-lined, the arytaenoids in moving keep their first vertical position.

In the second place the arytaenoid cartilage can be caused to perform movements around the longitudinal axis of the joint. In this case every point of the arytaenoid cartilage will describe part of a circle the centre of which is to be found where the longitudinal axis of the joint meets the plane perpendicular to this axis in which the point in question is situated. This rocking movement affects all parts of the arytaenoid and it is interesting to note that when the vocal process is turned downwards, medially and forwards the processus muscularis at the same time is turned upwards.

Already MERKEL (l.c.) has observed this "vorwärts und rückwärts Neigen" of the arytaenoid cartilages and LUCIANI (l.c.) wrote: "The two posterior crico-arytaenoid muscles are the chief, if not the only dilators of the glottis; owing to their attachment and the oblique course of their fibres they rotate the bases of the arytaenoids round their vertical axis and, therefore, draw the two muscular processes of the arytaenoids down and back and consequently further from the median line, and at the same time raise the vocal processes." Rotation of the arytaenoid cartilage around its vertical axis however is impossible. The fan-shaped lig. crico-arytaenoideum instantly checks and arrests rotation in both directions.

Combining the observations mentioned sub 1—6 leads to the following considerations. Fig. 5 in which the vertical projection of the crico-arytaenoid ligament has been sketched may be of use in elucidating the discussion.

Quite apart from the form of the articular surfaces it will be clear from fig. 5 that rotatory movements of the arytaenoid around a vertical axis passing through the centre of the cartilage (*A* vide fig. 5) or around

BRAUS' "Höhenachse" (H in fig. 5) cannot be performed since the fan-shaped lig. crico-arytaenoideum ($F_1 M F_2$ in fig. 5) will instantly check the rotation in either direction, its portions MF_1 and MF_2 respectively almost at once becoming tense.

Another possibility should be considered next, viz. the occurrence of a vertical axis of rotation passing through the cartilage not at a fixed point, but shifting its position in the course of the rotatory movement. In the case of endorotation the fan-shaped ligament would almost instantly cause the moving vertical axis to come to a standstill at F_1. And continuing endorotation of the arytaenoid around a vertical axis passing through this point should inevitably cause distortion of the joint. In the case of exorotation the axis would nearly at once move to F_2 and forcing the exorotation to be continued should yield the same damaging results. Consequently also this possibility has to be discarded the more so as it seems highly improbable that in case of exorotation and endorotation the moving axis of rotation should migrate in different directions.

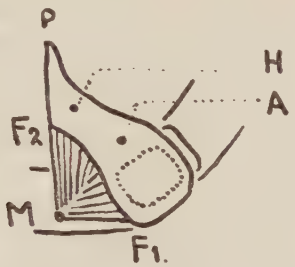


Fig. 5. Vertical projection of crico-arytaenoid joint on basal surface of cricoid cartilage.
P: processus vocalis;
F_1MF_2: crico-arytaenoid ligament.
Articular surface dotted.

A third more attractive possibility remains, viz. that of rotation of the arytaenoid cartilage around a vertical axis passing through the point where the crico-arytaenoid ligament is attached to the cricoid (point M, vide fig. 5). Since all the fibres of the fan-shaped crico-arytaenoid ligament converge to be attached to the cricoid cartilage at M none of them will offer resistance to movements of the arytaenoid about any axis whichsoever passing through M. Thus also a vertical axis passing through M might do. In movements of this kind the crico-arytaenoid ligament will act as a kind of leading-string ruling the course of the moving arytaenoid. In case this axis be a vertical one every point of the arytaenoid should describe a circle in a horizontal plane. The cylindric shape and the oblique direction of the articular surfaces of the cartilages participating in the formation of the crcio-arytaenoid joint interfere with movements of this kind since the latter should cause the facies articularis arytaenoideae to fit no longer to

the articular surface of the cricoid. Moreover just as any other part of the arytaenoid cartilages, the processus vocalis et muscularis should describe circles in a horizontal plane which means that the raising and lowering of the vocal processes described before cannot be explained in this way.

From the foregoing I must draw the conclusion that in the crico-arytaenoid joint movements of the arytaenoid around a vertical axis do not occur.

It has been explained that the strong crico-arytaenoid ligament allows movements of the arytaenoid cartilage around any axis passing through the slight projection at the upper border of the cricoid to which the fibres of this ligament are attached.

It will moreover be clear that the vocal process will be raised and depressed in movements of the arytaenoid cartilage around an axis the direction of which deviates from the vertical. The stronger this deviation the more marked these upward and downward motions of the vocal processes.

If it be assumed that the axis around which the arytaenoid cartilage moves coincides with "the longitudinal axis of the joint" which name I provisorily have given to the axis of the cylindrical surface of which the facies articularis cart. cricoideae forms part, the conditions laid down in the last two sentences are satisfied. Moreover in every stage of the movement the contiguous articular surfaces of the arytaenoid and cricoid cartilages will show a perfect fit.

Thus I draw the conclusion that the movements of the arytaenoid cartilage at the crico-arytaenoid joint are performed around the axis of the cylindrical surface from which the articular surface of the cricoid forms part. This rocking movement may be combined with a gliding movement, in which the arytaenoid moves in a direction parallel to the said axis, i.e. forwards, downwards and outwards or reversely. The strong crico-arytaenoid ligament not only is a structure preventing the arytaenoid cartilage from gliding too far outwards; its principal function rather consists in acting as a kind of leading-string ruling the course of the arytaenoid in its rocking movements.

It is a remarkable fact that the result of my investigation on the movements at the crico-arytaenoid joint although highly differing from the current recent views are in full accordance with the opinions expressed by HENLE as far back as 1866.

It is self-evident that the improved knowledge as to the position and the direction of the axis of the crico-arytaenoid joint necessitates alterations of the conception as to the functions of separate laryngeal muscles. A survey of their functions will be given now.

1. *M. crico-thyreoideus.* By the contraction of this muscle the thyreoid cartilage is caused to move at the two crico-thyreoid joints around a frontal axis. In this movement the lower border of the thyreoid approaches the

cricoid ring. Thus tension of the vocal cords is produced in case the arytaenoid cartilages are kept in place by tension of the crico-arytaenoid ligaments or actively by action of the dorsal crico-arytaenoid muscles. At the same time the fore-end of the vocal cord is lowered.

2. *M. crico-arytaenoideus dorsalis.* By the action of this muscle the arytaenoid cartilage is tilted laterally and backwards around the longitudinal axis of the crico-arytaenoid joint. This movement causes

a. tension of the vocal cords;
b. opening of the rima glottidis;
c. raising of the posterior end of the vocal cords.

The backward tilt of the arytaenoid cartilage is checked by the lig. crico-thyreoideum medium becoming tense and by the action of the thyreo-arytaenoid and crico-thyreoid muscles.

3. The *M. thyreo-arytaenoideus* acts in close collaboration with the former two (vide fgi. 6). By tilting the arytaenoid cartilage forwards it

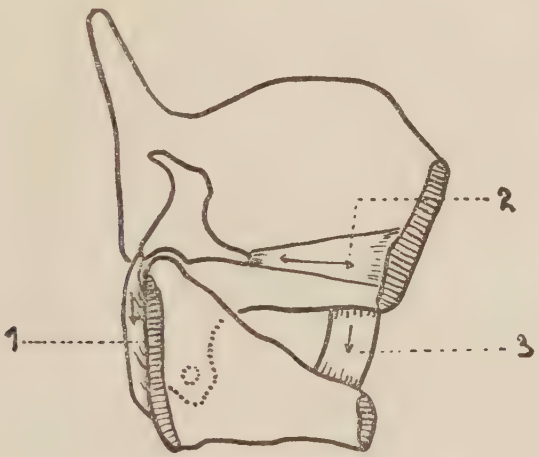


Fig. 6. Crico-arytaenoid joint with

1: *M.* crico-arytaenoideus dorsalis;
2: *M.* thyreo-arytaenoideus;
3: *M.* crico-thyreoideus.

Diagram modified after TANDLER.

causes relaxation of the vocal cords; in this movement the vocal process is lowered and brought nearer to the median plane so that the muscle together with its fellow of the opposite side closes the rima glottidis. In collaboration with the M. crico-arytaenoideus dorsalis the arytaenoid cartilage is balanced in different positions upon the upper border of the cricoid. Finally the muscle protects the vocal cord from being overstretched by the action of the Mm. crico-arytaenoideus dorsalis and crico-thyreoideus. In causing relaxation of the vocal cords the latter at the same time are

shortened and pressed one against the other; in this respect especially the M. vocales are important.

The length and the tension of the vocal cords and their elevation with regard to the horizontal plane depend for the main part on the coordinated action of the three muscles mentioned and on two ligaments, the lig. crico-arytaenoideum and crico-thyreoideum medium (fig. 6).

4. *M. crico-arytaenoideus lateralis.* This muscle by contracting drags the arytaenoid cartilage along the articular surface in a direction parallel to the axis of the joint ventro-laterally and downwards; thus the rima glottidis is widened and the posterior end of the vocal cord is depressed. Only a very small portion of the M. crico-arytaenoideus posterior consisting of its uppermost fibres can counteract this movement; in this it is supported by the M. interarytaenoideus transversus whose traction has a large component in a direction parallel to the axis of the joint. The crico-arytaenoid ligament however is the principal structure checking the gliding movement.

The traction exercised by the lateral crico-arytaenoid muscle has moreover a rather large second component at right angles to the axis of the joint which causes a tilting forwards of the arytaenoid cartilage, a movement opposite to the one caused by the M. crico-arytaenoideus dorsalis.

5. *M. inter-arytaenoideus transversus.* The fibres of this muscle taking an exactly frontal course its traction on the arytaenoid cartilage has a large component at right angles to the axis of the joint in a forward direction. Its component parallel to this axis causes the arytaenoid to slide dorsomedially and upwards, a gliding movement contrary to the one caused by the M. crico-arytaenoideus lateralis. Its other component causes the arytaenoid to tilt forwards; consequently complete closure of the rima glottidis may be effected.

6. *M. interarytaenoideus obliquus.* This muscle supports the function of the preceding one and moreover acts as a constrictor of the aditus laryngis.

Summary.

1. The cartilago arytaenoidea does not rotate around a vertical axis.

2. The cartilago arytaenoidea moves along the mantle of a cylinder of which the articular surface situated on the upper border of the cricoid cartilage forms a part. The axis of the cylinder is identical with the axis of this articular surface and it coincides with the axis of movement at the crico-arytaenoid joint. This axis passes through the upper part of the cricoid and it enters this cartilage at exactly the point of insertion of the crico-arytaenoid ligament. The axis of the two crico-arytaenoid joints converge backwards and upwards. Each axis is at an angle of about 40° with the medio-sagittal plane and at an angle of about 60° with the horizontal plane (passing through the base of the cricoid cartilage).

3. The ligamentum crico-arytaenoideum acts as the leading-string of the moving arytaenoid cartilage.

4. The movements of the arytaenoid cartilage result from two kinds of component movements, viz.

a. Gliding movements along the cylindrical articular surface of the cricoid cartilage parallel to the axis of the joint.

b. Rocking movements around the axis of the crico-arytaenoid joint. In the latter movements the arytaenoid falls over laterally and backwards or medally and forwards; the processus vocalis and muscularis are raised or depressed, but always the one process moves in a direction opposite to the other.

5. A short description of the functions of the laryngeal muscles is given based on the altered conceptions of the movements taking place at the crico-arytaenoid joints.

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p. 682, line 3 from foot, *for* ($\mu = 0.257$ and $\lambda = 0.313$ Å *read* ($\mu = 0.271$ and $\lambda = 0.332$ Å.

p. 683, fig. 8: the full curve should have been drawn slightly closer to the upper dotted line, in accordance with the following figures

x =	0	1	2	3	5	10
$\frac{J_x}{D_0}$ =	1.32	1.24	1.064	0.895	0.614	0.211

p. 683, text of fig. 8, line 3, *for* $\lambda = 0.363$ Å ($\mu = 0.3$) *read* $\lambda = 0.332$ Å ($\mu = 0.271$).

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